# THE DEFINITION IN GEOMETRY.[1]

By T. M. Smith,
*Lash High School, Zanesville, O.*

A critical examination of current texts in geometry reveals a variety of notions about the axioms, the postulates, and the assumptions, as well as the contents of Book I. This observation would seem to indicate that a careful study of these phases of the subject should receive attention.

When we consider the great expanse of territory addressed by the author of a school text, we may readily concede that he should not be responsible if the list of originals does not suit every locality into which the book happens to go. He cannot be expected to supply supplementary work for pupils of widely divergent interests which, by the nature of productive industries in different parts of the country, are as far apart as some of the localities themselves. Certainly here the instructor has a responsible duty in collecting or having his pupils collect original applied problems of local interest and application. But when we turn to the proved proposition, axiom, and definitions, no such diversity of demands upon the author is apparent. Hence it would seem that some sort of united effort should be made to reduce this modern "confusion of tongues" to a minimum.

How often do we read in current texts that "an axiom is a self-evident truth," while located due south of this statement is a list of facts purporting to be self-evident, but which may be demonstrated or derived from other facts of a genuine axiomatic nature.

Again, how clearly was pointed out to us, as we conned our first lessons in elementary physics, the important fact that a "great gulf" is fixed between zero and absolute zero; while in geometry where we are supposed to teach the art of thinking clearly and reasoning concisely, we often draw no dividing line between relatively self-evident and absolutely self-evident truths; nor is our course more commendable when, as Collins points out, we permit a class to call the statement that " 'Two geometric figures are equal or congruent if they can be made to coincide exactly,' a self-evident truth, when it is at best a definition of our notion of the terms equal or congruence."

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This is no religious question of infinite verities about which one expert's opinions may well be balanced by observations of another equally conscientious student of man's future destiny. But on the other hand it is a very human question of finite fact, having to do with the training of boys and girls to think to some purpose; and being such, it is essential to put first things first, and see that each is properly labeled.

The whole difficulty in the present-day presentation of secondary mathematics is a pedagogical one; and therefore the geometry teacher must be a sort of pedagogical Moses to lead those who may have been compelled to make bricks without straw, into a land of educational promise. The high school pupil is required to prove that "through a given point but one perpendicular can be drawn to a given line," a statement whose truth is just as evident to him as any one of the so-called axioms. The inconsistency is more apparent if we recall that just a few days later the pupil is tacitly requested to take for granted the fact that an angle has only one bisector. As far as consistency is concerned, one may equally as well require nine divided by three to be performed by the method of long division. However, I am not contending for proofs or their omission but for uniform consistency. Let us call things by their right names; and when an assumption is made let us be honest about it and call it such. Let the pupil understand that so-called self-evidence is not the Euclidian test for an axiom, but that it is an agreement as to fundamental, basic facts upon which to rear the logical sequence of theorems.

It is this subtle inconsistency upon the part of geometry teachers that gives rise to the oft-heard expressions, "Why, any fool can see that," and "I don't see any sense in such stuff;" statements which force the usual sermon on the value of mental training, end-points in culture, and the wonderful effect the study of geometry has had upon the lives of such famous historical characters as Napoleon, Washington, and Lincoln.

You will not wrongly understand me to be in favor of splitting hairs with the pupil on this subject, yet for the sake of the dignity of geometry as a secondary school subject on the one hand, and the pupils' welfare on the other, I am certain the time has come to eliminate these unnecessary evils.

Further evidence of the need of reform is found in the recent prophecy that "geometry must go out by the door Greek has already passed through but left ajar." The defects just pointed out are more or less superficial and are being gradually righted by makers of text-books.

The striking dissimilarity in the wording of fundamental definitions is wholly out of accord with a subject over two thousand years old. In my opinion there is but one other anomalous condition in the field of secondary mathematics comparable to it in mischievousness, and that is the promiscuous aggregation of algebraic symbols. Fortunately, the Central Association of Science and Mathematics Teachers is backing a movement to secure uniformity of usage in algebraic symbols at the present time, and it seems to me that some association might well take up the cognate problem of uniform definitions in geometry.

I am aware that at first thought a slight difference in the wording of definitions seems harmless; but a closer inspection reveals a decidedly different sequence of propositions. As an illustration, consider some of the various ways of defining parallels. If we hold with Dr. Johnson that parallels are lines that have the same direction and maintain always the same distance, we may readily say that the various pairs of corresponding and alternate angles formed by a transversal of such lines are equal by inspection; since the term direction is the essential element of the definition, and if a given line has a certain direction it can make but one set of angular magnitudes with a second direction, whether this second direction be represented by one or by more than one line.

If we agree that parallels are lines which cannot meet in finite space, we must supply rigid proofs for a number of theorems that show the equality of the above-mentioned pairs of angles. Notwithstanding this fact, the second definition is preferred on good authority, owing to the difficulty of assigning a concrete value to the term direction. It is used ambiguously, so those who are opposed to it say; but again an inconsistency creeps in, for while "direction" must be eliminated, owing to a double use of the term (since a line may have two directions if referred to any one of its internal points, and but one if referred to an extremity as in the case of the ray or half line), yet we use "adjacent" in two ways, and the term "vertex" may mean any one of the three angular points of a triangle.

It is one thing to use terms because reputed authorities have arbitrarily decided that we should do so, and quite another to use them because they actually produce desirable results when used in the class room. Throwing aside all semblance of quibbling, the pupil knows what we mean when we say "a line may be generated by a moving point;" and he also knows what we mean when we say "direction is that which controls the generation." While it is impossible to say what electricity really is, yet we all know some very concrete results of its use, and the case is not otherwise with the term "direction." In the mind of the pupil it is inseparably linked with the notion of straight lines, and we need not reject it any more than we should reject electricity. It is amusing, to say the least, to see an author using "left to left" and "right to right" in a mad endeavor to avoid the term "direction," and in a footnote on the same page have him speak of "similarly directed lines."

The position of those who defend the lack of uniformly worded definitions in geometry becomes untenable when we consider the geometric increment to the definition. Here for the first time in the pupil's experience it becomes more than a defining process; not only does it separate the object in mind from all other concepts in the universe, but it also takes the added nature of substantial authority in proving theorems, and as such takes its place beside the axiom or assumption as an absolute fact, back of which we agree not to look. Since the term is to be regarded as a fixed point in the geometric compass, and its authority is to be unimpeachable, there must be no "confusion of tongues" in its expression.

The world-wide movement, now well under way, to improve the quality of geometry teaching is causing the most conscientious study and thorough examination of the foundation of mathematics on the pedagogical side.

It has been said quite recently that "the most significant single feature requiring attention is the teacher. What we need is not better mathematicians but better teachers." There can be no doubt about the fact that secondary school teachers need an awakening in the direction of a more thorough and intimate knowledge of the personal needs of the pupil. To know geometry is good, but to know the child mind is imperative.

To learn as definitely as may be the exact content of the pupil's mind, and to use this knowledge as a foundation upon which to rear our educational superstructure, has long been a recognized principle of teaching. But wide as has been its heralding, and insistently as have educators proclaimed it as the fundamental of fundamentals, we even to-day find it the least used and most abused axiom in the realm of pedagogy. How often are complex notions slurred over by vague definitions, and half-true statements allowed to disgrace the pages of an otherwise teachable text. And how often do we teachers permit these slurred-over notions to remain slurred over, allowing them to be used day after day by the pupil, who with implicit faith in the teacher and a reverence for the text-book akin to awe, blunders blindly on.

Is it any wonder that the first impressions of geometry upon the somewhat satiated mind of the secondary school boy or girl is that the whole thing is bosh and a sheer waste of time? And very often, when we consider the manner in which geometry is served up these days the judgment is correct. Moreover, it is prima-facie evidence that once aroused by fair means or foul, the boy or girl passing such a concise, accurate, and fundamental judgment is capable of starting with an hypothesis and arriving with amazing dexterity at the correct conclusion.

The teacher must choose the book that meets his own personal views of orthodox scientific truth, and reject all others. With this new and very complex problem to be solved it may readily be acknowledged that the teacher is the most significant single feature requiring attention.

What with personal views, mathematical monographs, and the reasonable demands of his pupils to be squarely dealt with, the teacher certainly has not the least of his professional duties to perform when he chooses a text; and having considered all phases of the question he decides and sighs with relief. But the new text is not to be thought of as a panacea; for there yet remain pedagogical errors to be corrected, and statements, not germane to the thought, abound in spite of the pains taken to select the best book on the market. Now it becomes apparent that the last word in text-book making has not been spoken for this generation, nor will it be until we teachers have a more thorough understanding of the habits, content, and working of the pupils' mind and have passed the good news along to the bookmakers so insistently that they dare not deny our demands.

If this forces a less commercial and more pedagogical book, addressed to sections rather than to the whole country, then so be it. Anything for the good of the pupil should be the motto of the twentieth century teacher.

There is a widely heralded modern proverb which declares that the entire school system should revolve about and center in the pupil; but the very fact of its present widespread popularity is to be understood as meaning that no such Utopia exists at the present time. Moreover, the paramount duty of every teacher to-day ought to be to make this maxim true. It is time to wake up and meet the pupil more than half way, and having found the pupil's vantage point, lead him out of darkness into light.

Now if I have been dealing in generalities more than is desirable, let me lower my sights and ask you, if you are a teacher, to take the trouble to find out how many or perhaps how few of your geometry pupils have a definite, workable notion of the definition. If you are discouraged by the experiment, try it on your fellow teachers. While I do not know either your pupils or your teachers personally, my guess is that, judged by a working standard, neither pupil nor teacher will satisfy you.

Just ask the bright boy right in the midst of a demonstration how he knows that statement he has just made is true. If not too badly nonplussed he may answer correctly and say, "By definition." But ask how he knows that to be true. He will possibly say, "The book said so." And now, fellow teachers, is your chance; ply him with questions regarding the book, and suggest that he would probably believe 2 and 2 make 5 if he but saw it in a text-book. Then will he wake up, and not only will this be so, but about twenty other young Americans will all be awake at the same time.

After hearing all each one has to say, and after you have learned that "definition means to define," and this splendid definition of a definition has been illustrated by the remarkable statement that a horse is an animal with four legs, begin with any definition in the book, say, the one of a triangle, and have everyone in the class but yourself repeat it verbatim. Then dissect it. Follow this with the book definition of a quadrilateral; have the class repeat it and dissect it. Compare the two definitions; contrast them; let the class see wherein they are alike, and wherein they are different. Now, the very fact that the boy is awake, and that everybody else is awake, will cause some member of the class to rise to the situation and state that a definition is a statement of a set of conditions to which a convenient name has been given.

At this period, having properly praised the discoverer and illustrated his reply by a number of the most familiar definitions in the book, it will be time to seek the origin of the definition. This will necessarily require more than one recitation period, and will force a consideration of axioms and postulates. If you are well supplied with supplementary texts, you will be able to put a sufficient number of copies into the hands of the class for comparison. Just why some books give a list of assumptions, while others call them axioms and postulates, will prove of great interest to them; and right here let me state that the pupils are in great danger of learning in a manner not easy to forget, that some things are just as self-evident as the so-called axiom and the teacher's chance of a a lifetime to beget a true notion of the foundations of geometry has arrived.

Just to keep up the interest and to emphasize the foregoing, have the class hunt up the definition of circle, and let everybody but the teacher read what he finds. It doesn't really matter what he finds; the vital thing is that everyone is awake. One boy recognizes a familiar friend in the statement that "a circle is a portion of a plane bounded by a closed curve every point of which is equally distant from a point within called the center." Another finds a stranger and proceeds to introduce to the class the statement that "a circle is a closed curve every point of which is equally distant from a central point in the plane." A third leaves the class wondering what next, having read that "a circle is composed of both a portion of a plane and the closed curve that bounds it."

Here, fellow teachers, are three distinctly different definitions of a circle, and all within the pages of books labeled geometry. And the startling thing is not that they are worded differently, but that they are fundamentally different. Get an expression of opinion from the class on the merits of these definitions. If you have not tried it, you may be surprised at the readiness with which the pupils grasp the essential nature of the definition. Just a little steering will cause them to conclude that the definition is an agreement, made by persons interested in the subject, concerning a certain set of conditions which shall be called by a certain name, and the statement of which must be so carefully worded as to shut out all other ideas in the universe.

Let them understand also that they themselves have both the ability and the right to make agreements, provided they stand willing to take all consequences of the agreement. This of course presupposes the teacher to be guiding them intelligently while they form this judgment.

In the succeeding demonstrations see to it that the pupils state the nature and content of the definition. For variety call their attention to the old definition of a trapezoid, extant in the seventies, namely, a quadrilateral having either one or two pairs of parallel sides; thus effectually distinguishing it from the parallelogram which must have two pairs of parallel sides. Some such treatment will tend to destroy the prevalent book worship, and to place the definition where it belongs, on an equal footing with the axiom as authority. But greater than either of these results will be the fact that the pupil has been aroused to a notion of his own ability to think accurately and definitely to some purpose. Again, let me say that, in my judgment, the time for definite action has arrived. Let a committee, consisting of men of unquestioned scholarship and approved class room experience, be appointed to draft a code of sane, sensible, sound, and scientifically-stated definitions for use in teaching geometry, and let every effort be made to incorporate them into every text-book on the subject.

Time was when it was thought impossible to have uniform examinations for Ohio teachers, but, thanks to progress, that time has passed, and the day is coming when the geometry definition will come into its own.