finite number of quadrilaterals inscribed in the curve (see p. 186, Darboux, "Sur une classe remarquable de courbes et de surfaces algébriques").

If the trinodal quartic

$$y^2z^2+z^2x^2+x^2y^2+2xyz\,(ax+by+cz)=0$$

is capable of being written in the form (13), it is easily seen that we must have $c=1+ab$, or either of two similar relations.

9. The investigation in § 6 will include the case of a quartic with a triple point. For the equation (4) will represent such a quartic, if $a+b+c+d=0$, the triple point being $x=y=z=u$. The equation (8) then gives $(\vartheta+1)^2=0$, and is irrelevant, showing that there are no conics which touch the sides of an infinite number of triangles inscribed in a quartic with a triple point.

On the Theory of Screws in Elliptic Space. (*Supplementary Note.*)

By Arthur Buchheim, M.A.

[*Read November 13th*, 1884.]

At the January Meeting of the Society, I read a paper "On the Theory of Screws in Elliptic Space," which has since appeared in the *Proceedings*. My object was "to show that the *Ausdehnungslehre* supplies all the necessary materials for a calculus of screws in Elliptic Space." When I wrote that paper, I did not see how the same methods could (except in one obvious and unsatisfactory way) be extended to other kinds of space. In a paper on biquaternions, which is to appear in the *American Journal of Mathematics*, I have developed Clifford's calculus, in such a way as to make the methods and formulæ apply simultaneously to the three kinds of uniform space. While writing that paper, I saw how Grassmann's methods may be extended so as to give the metric formulæ for all kinds of space. This extension I explain in the present note.

The fundamental idea of the extension in question is that the symbol K no longer denotes the *Ergänzung* (complement) of a figure, but its polar with respect to the absolute. Besides this extension of the *Ausdehnungslehre*, I give a few kinematical investigations.

1.

In this note I use Professor Cayley's matrix notation, especially as applied to quadrics. Judging from the look of some recent work on Theta Functions, this notation does not seem to be as well known as it should be; I accordingly begin by explaining it.

Consider any symmetrical (or, as I prefer to call it, self-conjugate) matrix, and the substitution defined by it. To fix the ideas, suppose we have four variables; we take

$$(\xi_1, \xi_2, \xi_3, \xi_4) = \left(\begin{array}{cccc} a & h & g & l \\ h & b & f & m \\ g & f & c & n \\ l & m & n & d \end{array}\right) \between (x_1, x_2, x_3, x_4). \quad (*)$$

Now, using ξ to denote the set $(\xi_1, \xi_2, \xi_3, \xi_4)$, and, in the same way, using x to denote the set (x_1, x_2, x_3, x_4), and using A to denote the matrix of the substitution, I write this equation

$$\xi = Ax,$$

viz., this is a symbolic equation, to be understood as standing for the developed equation (*) above.

Now, let y be a new set: that is, we write y for (y_1, y_2, y_3, y_4), and

define $$xy = x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4.$$

Then we get $$x\xi = (abcdfghlmn \between x_1, x_2, x_3, x_4)^2.$$

But we had $$\xi = Ax.$$

Therefore, writing Ax^2 for $(Ax)\,x$, we can say that, if A denotes the matrix

$$\left(\begin{array}{cccc} a & h & g & l \\ h & b & f & m \\ g & f & c & n \\ l & m & n & d \end{array}\right)$$

then we can use Ax^2 to denote the quadric

$$(abcdfghlmn \between x_1, x_2, x_3, x_4)^2.$$

I call A the matrix of the quadric, and shall refer to the quadric as the quadric A.

It is hardly necessary to point out that, if x is a point, ξ is its polar plane with respect to the quadric A, and that the "tangential" equation is $A^{-1}\xi^2 = 0$, or that what precedes applies to sets containing any number of letters.

2.

As before, I denote the points of reference by e_1, e_2, e_3, e_4, and the complement (*Ergänzung*) by K. I call to mind that, if the four coordinates of a plane are ξ_1, ξ_2, ξ_3, ξ_4, the plane itself is

$$\xi = \xi_1 Ke_1 + \xi_2 Ke_2 + \xi_3 Ke_3 + \xi_4 Ke_4.$$

Now consider the polar plane of e_1 with respect to the quadric A of (1); its coordinates are $(ahgl)$: that is, the plane is

$$aKe_1 + hKe_2 + gKe_3 + lKe_4;$$

and we get similar expressions for the polar planes of the three other vertices of the tetrahedron of reference.

We shall take the quadric A as absolute, and we denote the polar planes of e_1, &c. by ωe_1, &c. We have, by what precedes,

$$(\omega e_1, \omega e_2, \omega e_3, \omega e_4) = \left(\begin{matrix} a & h & g & l \\ h & b & f & m \\ g & f & c & n \\ l & m & n & d \end{matrix} \right\rangle\!\!\!\!\!\!\!\chi\; Ke_1, Ke_2, Ke_3, Ke_4);$$

say, this is

$$(\omega e_1, \omega e_2, \omega e_3, \omega e_4) = (A \between Ke_1, Ke_2, Ke_3, Ke_4),$$

and then, if we define that

$$\omega \Sigma x_i e_i = \Sigma x_i (\omega e_i),$$

ω is obviously an operator which changes any point into its polar.

In exactly the same way, if $\omega e_2 e_3 e_4$ denotes the pole of $e_2 e_3 e_4$, and $\omega e_2 e_3$ denotes the polar of $e_2 e_3$, we get

$$(\omega e_2 e_3 e_4,\ \omega e_3 e_1 e_4,\ \omega e_1 e_2 e_4,\ \omega e_3 e_2 e_1) = (A' \between e_1, e_2, e_3, e_4),$$

$$(\omega e_2 e_3, \omega e_3 e_1, \omega e_1 e_2, \omega e_1 e_4, \omega e_2 e_4, \omega e_3 e_4) = (A'' \between e_1 e_4, e_2 e_4, e_3 e_4, e_2 e_3, e_3 e_1, e_1 e_2),$$

where A', A'' are self-conjugate matrices of the fourth and sixth sides respectively, and are, in fact, the matrices of the plane and line equations of the absolute.

If A is the matrix unity, A', A'' are also the units of their own orders, and each equation of the absolute is of the form $\Sigma x^2 = 0$, and then we have simply $\omega = K$; and for any other form of the absolute ω has the same meaning as K has for the special form.

Now, though the absolute was not explicitly used there, it is tolerably obvious that all the formulæ of the paper on the Theory of Screws in Elliptic Space have reference to this special form of the equation of the absolute. To see this, we have only to consider the expression for the distance between two points; this is given by

$$\cos(xy) = \frac{xKy}{\sqrt{xKx}\sqrt{yKy}}$$

$$= \frac{x_1y_1+x_2y_2+x_3y_3+x_4y_4}{(x_1^2+x_2^2+x_3^2+x_4^2)^{\frac{1}{2}}(y_1^2+y_2^2+y_3^2+y_4^2)^{\frac{1}{2}}}.$$

But this is what the ordinary expression for the distance becomes if we take $x_1^2+x_2^2+x_3^2+x_4^2 = 0$ as the equation of the absolute.

It is now obvious that the formulæ of the paper become applicable to any form of the equation of the absolute, provided that for K, denoting the complement, we substitute ω, denoting the polar with respect to the absolute.

I use ω partly because it is necessary to have two symbols for the two things, and partly to make the formulæ look like the biquaternion formulæ of the paper above referred to. In the following section, I take a special form of the equation of the absolute, and show how we get formulæ applicable to the three kinds of space.

3.

I shall now, to my own and my readers' relief, discard suffixes, and use the ordinary notations $(xyzw)$ for the coordinates of a point, and $(lmnp)$, $(abcfgh)$ for planes and screws respectively. I also use α, β, γ, δ for e_1, e_2, e_3, e_4.

Let the point equation of the absolute be

$$e^2(x^2+y^2+z^2)+w^2 = 0.$$

Then the plane equation will be

$$l^2+m^2+n^2+e^2p^2 = 0,$$

and the line equation will be

$$f^2+g^2+h^2+e^2(a^2+b^2+c^2) = 0.$$

It is obvious that $e^2 = \pm 1$ gives elliptic and hyperbolic space. It is not quite so obvious that $e^2 = 0$ gives parabolic space, if $w = 0$ is the plane infinity; but it is not hard to see if we remember,—

(1) That the absolute is a curve, viz., the "circle at infinity" taken twice over, so that its point equation is (plane infinity)$^2 = 0$;

(2) That therefore a line touches the absolute if it cuts the "circle at infinity"; and that

(3) f, g, h are proportional to the direction cosines of $(abcfgh)$.

We have

$$(\omega\alpha,\ \omega\beta,\ \omega\gamma,\ \omega\delta) = \left(\begin{array}{cccc} e^2, & 0, & 0, & 0 \\ 0, & e^2, & 0, & 0 \\ 0, & 0, & e^2, & 0 \\ 0, & 0, & 0, & 1 \end{array}\right\between \beta\gamma\delta,\ \gamma\alpha\delta,\ \alpha\beta\delta,\ \gamma\beta\alpha),$$

$$(\omega\beta\gamma\delta,\ \omega\gamma\alpha\delta,\ \omega\alpha\beta\delta,\ \omega\gamma\beta\alpha) = \left(\begin{array}{cccc} 1, & 0, & 0, & 0 \\ 0, & 1, & 0, & 0 \\ 0, & 0, & 1, & 0 \\ 0, & 0, & 0, & e^2 \end{array}\right\between \alpha,\ \beta,\ \gamma,\ \delta),$$

$$(\omega\beta\gamma,\ \omega\gamma\alpha,\ \omega\alpha\beta,\ \omega\alpha\delta,\ \omega\beta\delta,\ \omega\gamma\delta) = \left(\begin{array}{cccccc} 0, & 0, & 0, & e^2, & 0, & 0 \\ 0, & 0, & 0, & 0, & e^2, & 0 \\ 0, & 0, & 0, & 0, & 0, & e^2 \\ 1, & 0, & 0, & 0, & 0, & 0 \\ 0, & 1, & 0, & 0, & 0, & 0 \\ 0, & 0, & 1, & 0, & 0, & 0 \end{array}\right\between \beta\gamma,\ \gamma\alpha,\ \alpha\beta,\ \alpha\delta,\ \beta\delta,\ \gamma\delta).$$

It is necessary to change the definition of $[ab]$ (p. 91) and to write

$$e^{-1}\sin e\,[ab] = \frac{T(ab)}{Ta\,Tb} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (\theta),$$

where, of course, $\qquad Ta = +\sqrt{a\,.\,\omega a}.$

To show how the formulæ apply to parabolic space, I take the expression for the distance between two points and the expression for the axis of a screw.

We take $a = (xyzw)$, $b = (x'y'z'w')$; and we must remember that, in parabolic space, $w = 0$ is the plane infinity, so that $w =$ const. for all points not at infinity, and we can take $w = 1$.

We have

$$a = x\alpha + y\beta + z\gamma + w\delta,$$

$$\omega a = e^2 x\beta\gamma\delta + e^2 y\gamma\alpha\delta + e^2 z\alpha\beta\delta + w\gamma\beta\alpha.$$

Therefore $\qquad T^2 a = a\omega a = e^2(x^2+y^2+z^2) + w^2.$

Moreover, if $(abcfgh)$ are the coordinates of the line ab, we get easily

$$T^2(ab) = f^2 + g^2 + h^2 + e^2(a^2+b^2+c^2).$$

Therefore, putting in the values of the coordinates, we get

$$e^{-2}\sin^2 e\,[ab] = \frac{\begin{bmatrix} (xw'-x'w)^2+(yw'-y'w)^2+(zw'-z'w)^2 \\ +e^2\{(yz'-y'z)^2+(zx'-z'x)^2+(xy'-x'y)^2\}\end{bmatrix}}{\{e^2(x^2+y^2+z^2)+w^2\}\{e^2(x'^2+y'^2+z'^2)+w'^2\}}.$$

Now, if we take $e=0$ and $w=w'=1$, we get

$$[ab]^2 = (x-x')^2+(y-y')^2+(z-z')^2,$$

which is right.

Now take the expression for the axis of a screw: using (θ) above instead of the equation on p. 91, the expression for the axis on p. 94

becomes $$b = a\cos\frac{e\phi}{2} - e^{-1}\,\omega a\sin\frac{e\phi}{2},$$

where $$e^{-1}\sin e\phi = \frac{a^2}{T^2a}.$$

Now we need only consider the case $e=0$, and then we have

$$b = a - \omega a\,.\,\frac{\phi}{2},$$

$$\phi = \frac{a^2}{T^2a},$$

or $$b = a - \frac{a^2}{2T^2a}\,.\,\omega a = a - \lambda\,.\,\omega a \text{ say.}$$

Now let the coordinates of a be $(abcfgh)$: then the coordinates of ωa are $(e^2f,\ e^2g,\ e^2h,\ a, b, c)$, and we get, since $e=0$,

$$b = (a,\ b,\ c,\ f-\lambda a,\ g-\lambda b,\ h-\lambda c).$$

But $$\lambda = \frac{a^2}{2T^2a} = \frac{2\,(af+bg+ch)}{2\,(f^2+g^2+h^2)}.$$

Now, in a paper "On the application of Quaternions to the Theory of the Linear Complex and the Linear Congruence" (*Mess. of Math.*, Vol. XII., p. 129), I have (allowing for a mistake in sign) given

$$\beta,\quad \alpha - \frac{S\alpha\beta}{\beta^2}\,.\,\beta$$

as the vector coordinates of the axis of the screw (α, β), where $\alpha = fi+gj+hk$, $\beta = ai+bj+ck$: and these values obviously agree with the value of b just given. There are one or two points in which the formulæ of the paper on the Theory of Screws require modification.

The two conditions for parallelism are

$$a \pm e^{-1}\omega a = \lambda\,(b \pm e^{-1}\omega b),$$

and the definition of a vector is

$$a \pm e^{-1}\omega a = 0.$$

Writing
$$\xi = \frac{1+e^{-1}\omega}{2},$$

$$\eta = \frac{1-e^{-1}\omega}{2},$$

I find it convenient to replace Clifford's *right* and *left* by ξ, η used as prefixes: that is to say, a, b are ξ-parallel if $\xi a = \lambda \xi b$, and a is a ξ-vector if $\xi a = 0$: there are, of course, corresponding definitions of η-parallels, and of the η-vector.

It is easy to prove and important to notice, that, if a, b are any two screws, we have

$$\omega a \,.\, \omega b = e^{2}\,(ab).$$

For the next section the following definition and theorem are required:—If x is any point, and if a is any screw, xa is called the plane corresponding to x with respect to a: if a is a line, xa is the plane joining x, a: in the same way, if x is a plane, we have a point xa which, in the case in which a is a line, becomes the point of intersection of the line and plane.

If $x \equiv (xyzw)$ and $a \equiv (abcfgh)$ and $xa \equiv (lmnp)$, we have

$$(lmnp) = \left(\begin{matrix} 0, & h, & -g, & a \\ -h, & 0, & f, & b \\ g, & -f, & 0, & c \\ -a, & -b & -c, & 0 \end{matrix}\right)\!\!\big(xyzw\big).$$

If $(xyzw)$ is on the line $(abcfgh)$, $(lmnp) \equiv 0$.

The plane $(lmnp)$ always passes through $(xyzw)$.

4.

A motion* is defined as a linear transformation not altering the absolute: therefore, if $(xyzw)$ moves to $(x'y'z'w')$, we have

$$(x'y'z'w') = (1+X)(xyzw),$$

where X is a matrix.

I now suppose the motion to be infinitesimal, and for $1+X$ I write

* *Cf.* Lindemann, *Math. Ann.*, Band VII.

$\lambda+\epsilon$, where λ is a scalar matrix, differing infinitesimally from unity, and ϵ is an infinitesimal matrix to be determined. Now let A be the matrix of the absolute: then, since $\lambda+\epsilon$ is to be an automorphic of A, we have, using ϵ' to denote the conjugate of ϵ,

$$(\lambda+\epsilon')\,A\,(\lambda+\epsilon) = \rho A,$$

ρ being a scalar, and $\rho-1$ infinitesimal; multiplying out, and neglecting infinitesimals of the second order, we get

$$\lambda^2 A+\epsilon' A+A\epsilon = \rho A.$$

Therefore all the conditions of the problem are satisfied, provided we take

$$\lambda^2 = \rho,$$

$$A\epsilon+\epsilon' A = 0.$$

Now this last condition asserts that $A\epsilon$ is a skew matrix, say

$$A\epsilon = \eta = \begin{vmatrix} 0 & h & -g & a \\ -h & 0 & f & b \\ g & -f & 0 & c \\ -a & -b & -c & 0 \end{vmatrix}.$$

Then, if $\eta\,(xyzw)$ represents a plane, $\epsilon\,(xyzw) = A^{-1}\eta\,(xyzw)$ will represent the pole of that plane. Now we have

$$(x'y'z'w') = \lambda\,(xyzw)+\epsilon\,(xyzw),$$

that is, the new position of the point $(xyzw)$ is on the line joining the point to the pole of the plane $\eta\,(xyzw)$; but it is obvious, from (3), that this plane is the plane corresponding to $(xyzw)$ with respect to the screw $(abcfgh)$; moreover the line joining any point to the pole of a plane is (by definition) at right angles to the plane. Combining all this, we get the theorem:—Every infinitesimal motion of a rigid body is defined by a certain screw, in such wise that every point of the body moves along the normal to the plane corresponding to the point with respect to the screw, and that every plane of the body turns about the normal to the point corresponding to the plane with respect to the polar screw.

On this I remark:—(1) The second part is inserted in virtue of the principle of duality; (2) The normal to a point, in a plane, is the intersection of the plane with the polar plane of the point; (3) If a is any screw, ωa is the polar screw.

If a is a line, it is obvious that a point moves at right angles to the plane joining it to the line, and that the motion is of the nature of a rotation about the line: for we have

$$(x'y'z'w') = (\lambda + A^{-1}\eta \between xyzw);$$

but, if $(abcfgh)$ is a line, we have for all points on it $\eta\,(xyzw) \equiv 0$, and therefore $A^{-1}\,\eta\,(xyzw) \equiv 0$, and therefore

$$(x'y'z'w') = (\lambda \between xyzw).$$

Therefore all points on the line are unaltered, that is to say, the motion is a rotation about the line. But it appears, in precisely the same way, that all planes through the polar of the line are unaltered so that the motion is at the same time a translation along the polar. It is a well known and easily proved theorem, that any rotation about a line is at the same time a translation along its polar. For, since the absolute is unaltered, if a point P moves to P', the plane ωP will move to $\omega P'$; and therefore, if all points $\lambda P_1 + \mu P_2$ are unaltered, all planes $\lambda\omega P_1 + \mu\omega P_2$ are unaltered: but the points are the points of a straight line, and the planes are the planes through its polar; moreover, a motion which does not affect the points of a line is a rotation about the line, and a motion which does not affect the planes through a line is a translation along the line. The theorem is therefore proved.

It is worth while to consider space of more than three dimensions. It is obvious that all the investigations of this section apply, and we get the theorems:—

"Every infinitesimal motion of a rigid body in a space of n dimensions is defined by a certain form a of order $n-2$ in the units of reference, in such wise that any point x moves along the normal to the $(n-1)$-flat xa."

"Every rotation about an r-flat is at the same time a rotation about the polar $(n-r-1)$-flat."

5.

I now take the equation of the absolute in the canonical form

$$e^2(x^2+y^2+z^2)+w^2 = 0,$$

and I take $(abcfgh)$ as the coordinates of the screw defining the motion.

Then, if $(xyzw)$ moves to $(x'y'z'w')$, we have

$$(x'y'z'w') = \left(\begin{array}{cccc} \lambda & h & -g & a \\ -h & \lambda & f & b \\ g & -f & \lambda & c \\ -e^2a & -e^2b & -e^2c & \lambda \end{array} \between xyzw\right);$$

and then, if (A, B, C, F, G, H) becomes (A', B', C', F', G', H'), we have

$$(A'B'C'F'G'H') = \left(\begin{array}{cccccc} \lambda^2 & h & -g & 0 & c & -b \\ -h & \lambda^2 & f & -c & 0 & a \\ g & -f & \lambda^2 & b & -a & 0 \\ 0 & e^2c & -e^2b & \lambda^2 & h & -g \\ -e^2c & 0 & e^2a & -h & \lambda^2 & f \\ e^2b & -e^2a & 0 & g & -f & \lambda^2 \end{array}\right| \between ABCFGH)$$

we can verify at once that the two screws $(abcfgh)$, $(e^2f, e^2g, e^2h, a, b, c)$ are transformed into themselves.

Moreover, it is a known property, which can be easily verified in the present instance, that if a transformation of line-coordinates is derived, as this is, from a transformation of point-coordinates, then $AF+BG+CH$, and *therefore* $AF'+A'F+BG'+B'G+CH'+C'H$ are invariants; therefore, if we call the screw $(abcfgh)$, A, and say that two screws x, y are reciprocal (in involution) if xy vanishes, we get as the first result that every screw reciprocal to A or to ωA is transformed into a screw reciprocal to A or to wA.

Moreover, since A is absolutely unaltered by the motion, the axes of A are also unaffected by it, and therefore any point on an axis (plane through an axis) is transformed into a point on the same axis (plane through the same axis): therefore every infinitesimal motion is a rotation about an axis of the screw defining the motion, and therefore also a translation along an axis of the screw.

We get also

$$(A' \pm e^{-1}F', B' \pm e^{-1}G', C' \pm e^{-1}H')$$

$$= \left(\begin{array}{ccc} \lambda^2 & \pm e(c \pm e^{-1}h) & \mp e(b \pm e^{-1}g \\ \mp e(c \pm e^{-1}h) & \lambda^2 & \pm e(a \pm e^{-1}f) \\ \pm e(b \pm e^{-1}g) & \mp e(a \pm e^{-1}f) & \lambda^2 \end{array}\right| \between A \pm e^{-1}F, B \pm e^{-1}G, C \pm e^{-1}H).$$

Therefore

(1) If $A \pm e^{-1}F = B \pm e^{-1}G = C \pm e^{-1}H = 0$, $A' \pm e^{-1}F'$, $B' \pm e^{-1}G'$, $C' \pm e^{-1}H'$, all vanish: therefore ξ-vectors are transformed into ξ-vectors; η-vectors into η-vectors.

(2) If
$$\frac{A \pm e^{-1}F}{A_1 \pm e^{-1}F_1} = \frac{B \pm e^{-1}G}{B_1 \pm e^{-1}G_1} = \frac{C \pm e^{-1}H}{C_1 \pm e^{-1}H_1},$$
we have also
$$\frac{A' \pm e^{-1}F'}{A'_1 \pm e^{-1}F'_1} = \frac{B' \pm e^{-1}G'}{B'_1 \pm e^{-1}G'_1} = \frac{C' \pm e^{-1}H'}{C'_1 \pm e^{-1}H'_1};$$

that is, parallel lines or screws become parallel lines or screws.

(3) If $a \pm e^{-1}f = b \pm e^{-1}g = c \pm e^{-1}h = 0$,

we have $$\frac{A' \pm e^{-1}F'}{A \pm e^{-1}F} = \frac{B' \pm e^{-1}G'}{B \pm e^{-1}G} = \frac{C' \pm e^{-1}H'}{C \pm e^{-1}H} = \lambda^2.$$

Therefore, if the screw defining the motion is a vector, every screw becomes a parallel screw.

I now find the distance through which a point moves. I use quaternions to shorten the calculation.

We have, for the distance between the points,

$$e^{-2}\sin^2 e\,[PP'] = \frac{\left[\begin{array}{c}(xw'-x'w)^2+(yw'-y'w)^2+(zw'-z'w)^2 \\ +e^2\{(yz'-y'z)^2+(zx'-z'x)^2+(xy'-x'y)^2\}\end{array}\right]}{\{e^2(x^2+y^2+z^2)+w^2\}\{e^2(x'^2+y'^2+z'^2)+w'^2\}}.$$

If $\rho = xi+yj+zk$, $\rho' = x'i+y'j+z'k$, this is obviously

$$\frac{e^2T^2V\rho\rho'+T^2(w\rho'-w'\rho)}{(e^2T^2\rho+w^2)(e^2T^2\rho'+w'^2)}.$$

Now, in the present case, we have

$$(x'y'z'w') = (\lambda x+hy-gz+aw,\ \lambda y+fz-hx+bw,\ \lambda z+gx-fy+cw,$$
$$\lambda z-e^2ax-e^2by-e^2cz).$$

Now take $a = fi+gj+hk$, $a' = ai+bj+ck$: then we have

$$\rho' = \lambda\rho+V\rho a+wa',$$
$$w' = \lambda w+e^2S\rho a'.$$

We have therefore, if we omit terms which obviously cancel, to calculate

$$X = e^2T^2V\rho\,(V\rho a+wa')+T^2(e^2\rho S\rho a'-wV\rho a-w^2a').$$

Now the first term is

$$e^2T^2\rho T^2(V\rho a+wa')-e^2S^2\rho\,(V\rho a+wa')$$
$$= e^2T^2\rho\,(T^2V\rho a-2wS\rho aa'+w^2T^2a')-e^2w^2S^2\rho a'.$$

The second term is

$$e^4T^2\rho S^2\rho a'+2e^2w^2S^2\rho a'+w^2T^2V\rho a-2w^3S\rho aa'+w^4T^2a'.$$

This gives

$$X = (e^2T^2\rho+w^2)(T^2V\rho a+w^2T^2a'-2wS\rho aa'+e^2S^2\rho a')$$
$$= (e^2T^2\rho+w^2)\,\{T^2(V\rho a+wa')+e^2S^2\rho a'\}.$$

Moreover, neglecting terms of the second order,

$$e^2T^2\rho'+w'^2=\lambda^2(e^2T^2\rho+w^2).$$

Therefore $$e^{-2}\sin^2 ePP'=\frac{T^2(V\rho\alpha+w\alpha')+e^2S^2\rho\alpha'}{\lambda^2(e^2T^2\rho+w^2)} \quad\text{..............}(\delta).$$

Now introduce the notation of the rest of this paper, calling the screw of the motion A, and we get

$$e^{-1}\sin ePP'=\frac{T(PA)}{\lambda TP}=\frac{TA}{\lambda}.\sin[PA].$$

If A is a vector, we have $\alpha=\pm e\alpha'$, and we get from (δ)

$$\begin{aligned}e^{-2}\sin^2 ePP' &= \frac{e^2T^2V\rho\alpha'+w^2T^2\alpha'+e^2S^2\rho\alpha'}{\lambda^2(e^2T^2\rho+w^2)}\\ &=\frac{(e^2T^2\rho+w^2)T^2\alpha'}{\lambda^2(e^2T^2\rho+w^2)}\\ &=\frac{T^2\alpha'}{\lambda^2}.\end{aligned}$$

Therefore the translations of all points are equal.

Moreover we have

$$\begin{aligned}&\rho w'-\ 'w-eV\rho\rho'\\ &=e^2\rho S\rho\alpha'-wV\rho\alpha-w^2\alpha'-eV\rho V\rho\alpha-ewV\rho\alpha'\\ &=e^2\rho S\rho\alpha'-wV\rho\alpha+w^2\alpha'+e\rho S\rho\alpha-e\alpha\rho^2-ewV\rho\alpha'\\ &=e\rho(S\rho\alpha+eS\rho\alpha')-w(V\rho\alpha+eV\rho\alpha')-(e\alpha\rho^2+\alpha'w^2).\end{aligned}$$

Therefore, if $\alpha+e\alpha'=0$, we get

$$\rho w'-\rho'w-eV\rho\rho'=-\alpha'(w^2+e^2T^2\rho^2).$$

That is, the left-hand side is constant to a factor *près*; therefore, if the screw of a motion is a vector, the translations of all points are parallel.

In precisely the same way we can prove that, if the screw of a motion is a vector, the axes of rotation of all planes are parallel, and that the amount of rotation is constant for all planes, and equal to the amount of translation of all points.

Now consider the motion of a line: let the coordinates of the line be $(ABCFGH)$, and let $Fi+Gj+Hk=\varpi$, $Ai+Bj+Ck=\rho$; then, if ϖ', ρ' are the corresponding vectors for the new position of the line, we have

$$\varpi'=e^2V\rho\alpha'+V\varpi\alpha+\lambda^2\varpi,$$

$$\rho'=V\rho\alpha+V\varpi\alpha'+\lambda^2\rho.$$

But $e^2 V\rho\alpha' + V\varpi\alpha$, $V\rho\alpha + V\varpi\alpha'$ are the vector coordinates of a screw which I have elsewhere (in the memoir on Biquaternions, already referred to) called the axis of the cylindroid determined by the screws (ϖ, ρ) and (α, α').

Therefore we can say that the new position of a line A is in the cylindroid containing A and the axis of the cylindroid $(A\alpha)$ if α is the screw defining the motion.

In conclusion, I prove Professor Ball's theorem that every ξ-vector is reciprocal to every η-vector.

Let a be a ξ-vector, b an η-vector: then we have

$$\omega a = -ea,$$

$$\omega b = eb.$$

Therefore $$(\omega a)(\omega b) = -e^2 ab.$$

But we have universally

$$(\omega a)(\omega b) = e^2 ab.$$

Therefore, unless e vanishes,

$$ab = -ab,$$

or $$ab = 0.$$

On the Motion of a Viscous Fluid contained in a Spherical Vessel. *By* Horace Lamb, M.A., F.R.S.

[*Read November* 13*th*, 1884.]

In several of the most important problems in Viscosity which have as yet been solved, the fluid is supposed limited, whether externally or internally, by a single spherical, or nearly spherical, boundary. For instance, we have the case of a ball pendulum oscillating in an unlimited mass of fluid (Stokes), the case of a hollow spherical shell filled with liquid, and oscillating by the torsion of a suspending wire (Helmholtz), and so on.* In a previous communication† to the Society, I have given formulæ by means of which all problems of this

* See Hicks, "Report on Hydrodynamics," *B. A. Rep.*, 1882.

† *Proceedings*, T. xiii., p. 51.