

Note on the thin astigmatic lens

This content has been downloaded from IOPscience. Please scroll down to see the full text.

1921 Trans. Opt. Soc. 23 56

(<http://iopscience.iop.org/1475-4878/23/1/305>)

View [the table of contents for this issue](#), or go to the [journal homepage](#) for more

Download details:

IP Address: 131.91.169.193

This content was downloaded on 01/09/2015 at 00:02

Please note that [terms and conditions apply](#).

NOTE ON THE THIN ASTIGMATIC LENS

BY THEODORE CHAUNDY, M.A.

MS. received, 1st June, 1921. Read, 10th November, 1921.

ABSTRACT. For the normal refraction of an aspherical wave-front at an aspherical surface three quantities are introduced which obey the Gaussian formula of symmetrical normal refraction.

THERE is a result in the first approximation theory of a thin astigmatic lens which does not seem to be so widely known as, I think, it should be. Essentially it may be stated as follows: An aspherical wave-front is refracted normally at an aspherical surface. If $1/r$, $1/s$ are the principal curvatures of the incident wave-front, their planes making angles α and $90^\circ + \alpha$ with a chosen reference-plane; if

$$1/R, 1/S, \theta, 90^\circ + \theta$$

are similar quantities for the refracting surface, and

$$1/r', 1/s', \alpha', 90^\circ + \alpha'$$

similar quantities for the emergent wave-front; then

$$u = 1/r + 1/s, u = (1/r - 1/s) \cos 2\alpha, u = (1/r - 1/s) \sin 2\alpha$$

all satisfy the Gaussian formula

$$\mu' u' - \mu u = (\mu' - \mu) U.$$

To establish this I need the analytical expression of the fundamental laws of refraction, viz.,

$$\left. \begin{aligned} \mu' l' &= \mu l + \kappa L \\ \mu' m' &= \mu m + \kappa M \\ \mu' n' &= \mu n + \kappa N \end{aligned} \right\} \dots (1).$$

Here (l, m, n) and (l', m', n') are the direction cosines of the incident and emergent rays, (L, M, N) those of the normal to the refracting surface and κ a quantity whose evaluation does not here concern us.

Now for normal refraction I may take the origin of three dimensional Cartesian coordinates as point of incidence and the x -axis as common normal. We may thus take the equation to the incident wave-front, the refracting surface and the emergent wave-front to be

$$2x = ay^2 + 2hyz + bz^2 \dots (2),$$

$$2x = Ay^2 + 2Hyz + Bz^2 \dots (3),$$

$$2x = a'y^2 + 2h'yz + b'z^2 \dots (4).$$

Consider a point (x, y, z) of the incident wave-front near the origin, i.e. y, z are small quantities of the first order and x —from equation (2)—small of the second order. The normal at (x, y, z) to (2) has direction cosines proportional to

$$1, -(ay + hz), -(hy + bz).$$

Since the sum of the squares of these quantities differs from unity by a small quantity of the second order, we may regard them as *actual* direction cosines. Any point on this normal has coordinates

$$x + \lambda, y - \lambda (ay + hz), z - \lambda (hy + bz),$$

where λ is some parameter. This cuts (3) where

$$2x + 2\lambda = Ay^2 + 2Hyx + Bz^2 + \text{quantities of higher order.}$$

Hence λ is of the second order, and to a first approximation the point of incidence on (3) is

$$(x + \lambda, y, z).$$

The normal of refraction has therefore direction cosines

$$1, -(Ay + Hx), -(Hy + Bz),$$

and the emergent ray has direction cosines

$$1, -(a'y + h'z), -(h'y + b'z).$$

Hence by equation (1)

$$\begin{aligned} \mu' &= \mu + \kappa, \\ \mu' (a'y + h'z) &= \mu (ay + hz) + \kappa (Ay + Hx), \\ \mu' (h'y + b'z) &= \mu (hy + bz) + \kappa (Hy + Bz). \end{aligned}$$

These equations are true for all small values of y, z , if (4) truly represents the emergent wave-front. Hence we may replace them by

$$\left. \begin{aligned} \mu'a' &= \mu a + (\mu' - \mu) A \\ \mu'h' &= \mu h + (\mu' - \mu) H \\ \mu'b' &= \mu b + (\mu' - \mu) B \end{aligned} \right\} \dots (5).$$

Now, since the principal curvatures of (2) are $1/r, 1/s$ lying in planes making with yOx , say, angles $\alpha, 90^\circ + \alpha$, its equation may be rewritten

$$2x = (y \cos \alpha + z \sin \alpha)^2/r + (y \sin \alpha - z \cos \alpha)^2/s,$$

so that

$$\begin{aligned} a &= \cos^2 \alpha/r + \sin^2 \alpha/s, \\ h &= \sin \alpha \cos \alpha (1/r - 1/s), \\ b &= \sin^2 \alpha/r - \cos^2 \alpha/s. \end{aligned}$$

Hence

$$\begin{aligned} a + b &= 1/r + 1/s, \\ a - b &= (1/r - 1/s) \cos 2\alpha, \\ 2h &= (1/r - 1/s) \sin 2\alpha, \end{aligned}$$

and similarly for the other surfaces.

Substitution in equation (5) gives the promised result,

$$\mu' (1/r' + 1/s') = \mu (1/r + 1/s) + (\mu' - \mu) (1/R + 1/S) \dots (6),$$

$$\mu' (1/r' - 1/s') \cos 2\alpha' = \mu (1/r - 1/s) \cos 2\alpha + (\mu' - \mu) (1/R - 1/S) \cos 2\theta \dots (7),$$

$$\mu' (1/r' - 1/s') \sin 2\alpha' = \mu (1/r - 1/s) \sin 2\alpha + (\mu' - \mu) (1/R - 1/S) \sin 2\theta \dots (8).$$

We may replace equations (7) and (8) by the simple complex equation (using i for $\sqrt{-1}$)

$$\mu' (1/r' - 1/s') e^{2i\alpha'} = \mu (1/r - 1/s) e^{2i\alpha} + (\mu' - \mu) (1/R - 1/S) e^{2i\theta} \dots (9).$$

Equation (6) which shews that the mean curvature obeys the Gaussian formula is generally known, I believe. But I have no references to the corresponding property of what we may call the "components of curvature-difference," as represented by equations (7), (8). These equations represent, in point of fact, a vector-addition. That is to say, if we draw in a plane a length AB proportional to $\mu (1/r - 1/s)$ and in a direction 2α , and then a length BC proportional to $(\mu' - \mu) (1/R - 1/S)$ and in a direction 2θ , then AC is proportional to $\mu' (1/r' - 1/s')$, and is in a direction $2\alpha'$.

Finally, if we have a succession of normal refractions, occurring effectively at the same point—e.g. as occasioned by passage through a *thin* astigmatic lens or by a series of such lenses in close contact—then equations of the types of (6), (7), (8) are applicable to each refraction, and in full analogy with Gauss's results we see that the "mean powers" and the "components of power-difference" of the constituent surfaces are additive to give the "mean power" and the "components of power-difference" of the complete system.