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416. Formulae for the Area and Half-Angles of a Triangle ABC Obtained by Means of the Equivalent Isosceles Triangle ADE, viz. the Isosceles Triangle Which Has the Same Area and the Same Angle A as the Given Triangle

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Writing the formula  $a^2 = b^2 + c^2 - 2bc \cos A$

in the form 
$$a^2 = (b+c)^2 \left( 1 - \frac{4bc \cos^2 \frac{A}{2}}{(b+c)^2} \right),$$

so that  $a = (b+c) \sin \phi$ , if  $\cos \phi = \frac{\sqrt{bc} \cdot \cos \frac{1}{2}A}{\frac{1}{2}(b+c)}$ ,

then, since  $AF = \sqrt{bc} \cos \frac{1}{2}A$ , we see that if  $FD$  is produced to  $H$ , so that  $FH = \frac{1}{2}a$ , then  $AH = \frac{1}{2}(b+c)$ , and the angle  $HAF = \phi$ .

The line  $AH$  is of special interest. For let the bisector of the angle  $A$  meet  $BC$  in  $L$  and the circumcircle of  $ABC$  in  $O$ , then the circle with centre  $O$ , radius  $OC$ , will pass through  $B$  and  $C'$ . Draw  $LT'$  perpendicular to  $AL$ , to cut this circle in  $T$ , and join  $AT$ .

Then 
$$\begin{aligned} bc &= AL^2 + BL \cdot LC \quad (\text{Euc. VI. B.}) \\ &= AL^2 + LT^2 = AT^2; \end{aligned}$$

$\therefore AT$  touches the circle  $BCO'$  at  $T$ , and its length  $= \sqrt{bc} = AD = AE$ .

(This construction is incidentally a good one for constructing the triangle  $ADE$ .)

Also, since  $2\triangle ABL + 2\triangle ALC = 2\triangle ABC$ ,  
we have  $AL(b+c) \sin \frac{1}{2}A = bc \sin A$ ;

$$\therefore AL = \frac{2bc \cos \frac{1}{2}A}{b+c}, * \quad \therefore \frac{AL}{AT} = \cos \phi.$$

Hence, if the tangent  $AT$  be produced, it will cut  $DF$  produced in  $H$ , such that  $AH = AF \sec \phi = \frac{1}{2}(b+c)$ , and consequently  $FH = \frac{1}{2}a$ , as before.

I have not seen a geometrical construction of this auxiliary angle  $\phi$  before, and its simplicity, being merely the angle between the tangent from  $A$  to the circle  $BCO'$  and the bisector  $AL$ , is very enchanting.

It is easy to give a construction for the other auxiliary angle  $\phi'$  given by the formula

$$a^2 = (b-c)^2 + 4bc \sin^2 \frac{1}{2}A,$$

whereby  $a = (b-c) \sec \phi'$ , if  $\tan \phi' = \frac{\sqrt{bc} \sin \frac{1}{2}A}{\frac{1}{2}(b-c)} = \frac{DF}{\frac{1}{2}(b-c)}$ ,

by taking a point  $K$  on the bisector, such that  $FK = \frac{1}{2}(b-c)$ , and consequently  $DK = \frac{1}{2}a$ ; then  $DKF = \phi'$ .

This, however, is rather artificial, and therefore of no special interest, whereas  $\phi$  came naturally during the process of finding the lengths of  $AD$ ,  $AE$ ; moreover, the circle concerned is an important one, as it cuts  $AL$  in the in-centre and an ex-centre of  $ABC$ .  
A. LODGE.

#### 417. [C. 2. a.] Geometrical Integration of $\sec \theta d\theta$ .

Let  $Q$  be a point in the arc  $AQB$  of a quadrant of a circle  $AOB$  of radius  $a$ ,  $\angle AOQ = \theta$ ,  $Q'$  a pt. near to  $Q$ .

Draw  $QN$  perpr. to  $OA$ , and let  $BQ, BQ', QQ'$  meet  $OA$  produced in  $P, P', T$ .  
Then  $PQ \cdot PB = PO^2 - OQ^2 = PB^2 - 2OQ^2$ ;

$$\therefore BP \cdot BQ = 2OQ^2 = BP' \cdot BQ';$$

$$\therefore P, P', Q', Q \text{ are concyclic.}$$

$$\therefore \text{(i) } \triangle BQQ', BPP' \text{ are similar, and (ii) } \angle TQP = \angle TP'Q'.$$

From (i),  $\frac{PP'}{QQ'} = \frac{BP}{BQ}$ ,

$$\frac{1}{a} \cdot \frac{d(OP)}{d\theta} = \frac{BP}{BQ} = \frac{OP}{ON};$$

\* It might just be noticed that  $AS, AD$  (or  $AT$ ), and  $AH$  are the harmonic, geometric, and arithmetic means between  $b, c$ .