Among the simple groups of order $\frac{1}{2}p^n(p^{2n}-1)$ here dealt with the particular case $p^n = 3^2$ gives the alternating group of degree 6. Herr Hölder in his memoir already quoted determines the group of isomorphisms of the alternating group for all degrees, and he finds that, as compared with all others, the alternating degree of degree 6 behaves in an exceptional manner and requires special treatment. There is no reason to regard the alternating groups of different degrees as characterized by common group properties, in the same sense that the groups of the modular equations for prime transformations are; though, no doubt, they have some common properties which make it convenient to deal with them as a class. When considered as one of a class with which it is connected by common group properties, it is here seen that the alternating group of degree 6 does not behave in an exceptional manner.

The Division of the Lemniscate. By G. B. Mathews.

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Since Abel's discovery* of the analogy which exists between the problem of the equisection of the lemniscate and the division of the circumference of a circle into equal parts, the lemniscate problem has attracted a good deal of attention: in particular, reference should be made to Schwering's important papers in *Crelle's Journal*, Vols. CVII., CX. But these researches are mainly analytical, and there is a certain interest in trying to develop the geometrical theory proper, so as to give the actual results for the section of the real period in a form suitable for geometrical construction. To do this in some of the simpler cases is the object of the present paper: the elements for the construction are found by analysis, and this is, of course, a kind of imperfection; but it is possible that the possession of explicit real formulæ which can be at once translated into geometry may lead to a purely geometrical method, at least when the section by rule and compass is possible.

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* Gauss was undoubtedly familiar with the theory: see *Disq. Arith.*, Art. 335, and *Werke*, III., p. 404 and following, with the editorial note, p. 496.

In order to follow out this idea consistently, we avoid the theory of complex multiplication, and take the modulus to be $\frac{1}{\sqrt{2}}$, so that the elliptic functions to be employed are those defined by

$$\mathrm{sn}^{-1}x = \int_0^x \frac{dx}{\sqrt{(1-x^2)(1-\frac{1}{2}x^2)}},$$

$$K = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-\frac{1}{2}x^2)}},$$

and the problem is to find the elliptic functions of the arguments $\frac{2rK}{n}$, where r, n are real integers.

If we take the lemniscate

$$r^2 = a^2 \cos 2\theta,$$

the arc measured from the vertex is

$$s = a\int_0^\theta \frac{d\theta}{\sqrt{\cos 2\theta}} = \frac{a}{\sqrt{2}}\int_0^\phi \frac{d\phi}{\sqrt{1-\frac{1}{2}\sin^2\phi}},$$

where $$\sin\phi = \sqrt{2}\sin\theta.$$

If we put $$\frac{s\sqrt{2}}{a} = u, \quad \phi = \mathrm{am}\, u,$$

then $$\sin\theta = \frac{1}{\sqrt{2}}\,\mathrm{sn}\, u, \quad \cos\theta = \mathrm{dn}\, u,$$

and the whole length of the lemniscate is

$$\frac{4Ka}{\sqrt{2}} = L,$$

say. Hence, if $s = \frac{L}{n}$, where n is a real integer, $u = \frac{4K}{n}$; it is convenient, however, to consider $2K$, not $4K$, as a period, so that the problem of n-section is to be understood as primarily the question of finding the elliptic functions of $\frac{2K}{n}$.

Following Halphen (*Fonc. Ell.*, I., p. 10; II., p. 386), let

$$U = x^2+y^2-R^2,$$

$$V = (x-\delta)^2+y^2-r^2,$$

so that $U=0$, $V=0$ represent two circles. Then the discriminant of $kU-V$ is

$$-R^2k^3+(2R^2+r^2-\delta^2)\,k^2-(R^2+2r^2-\delta^2)\,k+r^2,$$

and hence, in Halphen's notation,

$$k_1=\frac{2R^2+r^2-\delta^2}{R^2},$$

$$k_2=\frac{R^2+2r^2-\delta^2}{R^2},$$

$$k_3=\frac{r^2}{R^2}.$$

The condition that the elliptic functions associated with the pair of circles may be to modulus $\frac{1}{\sqrt{2}}$ is found to be

$$r^2=R^2-6R\delta+\delta^2;$$

and hence, putting $$\frac{\delta}{R}=c,$$

we have $$k_1=3\,(1-2c),$$

$$k_2=3-12c+c^2,$$

$$k_3=1-6c+c^2,$$

and Halphen's invariants X, Y, calculated by the corrected formula (*l.c.*, II., p. 649),

$$X=\frac{(4k_1k_3-k_2^2)^3}{2^8k_3^4},\quad Y=\frac{4k_1k_2k_3-k_2^3-8k_3^2}{2^3k_3^2},$$

are in the present case

$$X=\frac{(3-24c+6c^2-c^4)^3}{2^8\,(1-6c+c^2)^4},$$

$$Y=\frac{1-12c+5c^2-5c^4+12c^5-c^6}{2^3\,(1-6c+c^2)^2}.$$

It is convenient to change the parameter to d, where

$$d=\frac{1-c}{1+c}=\frac{R-\delta}{R+\delta};$$

the new values are

$$X = -\frac{(1+2d-4d^3-2d^4)^3}{2^4(1+d)^4(1-2d^2)^4},$$

$$Y = -\frac{d(1-2d^4)}{2(1+d)^2(1-2d^2)^2}.$$

If now we solve the equations

$$X = 0, \quad Y = 0, \quad X - Y = 0, \ \ldots$$

(see Halphen, II., p. 377), the corresponding values of d are those of

$$\operatorname{dn} \frac{2(r+2si)K}{n},$$

for $n = 3, 4, 5$, &c.; the positive real integers r, s assume, independently of each other, the values $0, 1, 2 \ldots n-1$ with the following limitations:—

If n is odd, the combination $(0, 0)$ is excluded;

If n is even, the combinations $(0, 0)$, $(0, \frac{1}{2}n)$, $(\frac{1}{2}n, 0)$ are to be excluded.

Further, it is to be remembered that the combinations (r, s), $(n-r, n-s)$ lead to the same value of d, and this only occurs once in the equation we have to consider. Moreover, when n is even, (r, s) coincides with $(n-r, n-s)$ for $r = s = \frac{n}{2}$.

Therefore the degree of the equation in d is

$$\tfrac{1}{2}(n^2-1) \text{ if } n \text{ is odd,}$$

$$\tfrac{1}{2}n^2-3 \quad \text{if } n \text{ is even.}$$

If n is composite, the d-equation contains irrelevant factors arising from the divisors of n, but by taking the cases in their natural order these can be eliminated as they arise. Let the final equation, cleared of these factors, be

$$F(d) = 0.$$

Then all the equations $F(d) = 0$ are irreducible in the ordinary domain of rationality, that is, in the domain of ordinary rational numbers; but they are all Abelian, and therefore are algebraically solvable. The questions that have to be answered relate to the groups of the equations, and the arithmetical nature of the irrational quantities which the roots necessarily involve.

First, as to the groups. In the complex theory all the roots are arranged in an Abelian cycle, and this is further sub-divided into minor groups. At present we are concerned with the real roots, and it is clear that these form an Abelian cycle of their own, when $n = p$, an odd prime. If g is a primitive root of p, the quantities we want to find are

$$\mathrm{dn}\,\frac{2K}{p},\quad \mathrm{dn}\,\frac{4K}{p},\quad \dots\quad \mathrm{dn}\,\frac{2\,(p-1)\,K}{2p},$$

and these are identical, save as to order, with

$$\mathrm{dn}\,\frac{2K}{p},\quad \mathrm{dn}\,\frac{2gK}{p},\quad \mathrm{dn}\,\frac{2g^2K}{p},\quad \dots\quad \mathrm{dn}\,\frac{2g^{\frac{1}{2}(p-3)}K}{p},$$

and, if we now write
$$p' = \tfrac{1}{2}\,(p-1),$$
$$d_s = \mathrm{dn}\,\frac{2g^sK}{p},$$
we have
$$d_1 = \vartheta\,(d_0),$$
$$d_2 = \vartheta\,(d_1),$$
$$\dots\quad \dots\quad \dots$$
$$d_{p'-1} = \vartheta\,(d_{p'-2}),$$
$$d_0 = d_{p'} = \vartheta\,(d_{p'-1}),$$

where ϑ denotes a rational function, which can be obtained by the multiplication theorem. Of course, in practice, the smallest value of g is the most convenient.

All this, and much besides, is illustrated by the worked out examples which follow.

I. $n = 3$.

The equation is $X = 0$, which may be written in the form

$$2\,(d^2+d-\tfrac{1}{2})^2 = \tfrac{3}{2}\,;$$

whence
$$(2d+1)^2 = 3\pm 2\sqrt{3},$$
$$d = \tfrac{1}{2}\left\{-1\pm\sqrt{3\pm 2\sqrt{3}}\right\}.$$

The real roots are

$$d_0 = \mathrm{dn}\,\frac{2K}{3} = \tfrac{1}{2}\left\{-1+\sqrt{3+2\sqrt{3}}\right\}$$
$$= \tfrac{1}{2}\left\{-1+\sqrt[4]{3}\,\frac{\sqrt{3}+1}{\sqrt{2}}\right\} = \cdot 77123 = \cos 39^\circ 32' 8'',$$

and $$d_0' = \operatorname{dn}\frac{4iK}{3} = \tfrac{1}{2}\{-1-\sqrt{3+2\sqrt{3}}\}$$

$$= \tfrac{1}{2}\left\{-1-\sqrt[4]{3}\,\frac{\sqrt{3}+1}{\sqrt{2}}\right\}.$$

The other roots are given by

$$d = \tfrac{1}{2}\left\{-1 \pm i\sqrt[4]{3}\,\frac{\sqrt{3}-1}{\sqrt{2}}\right\},$$

and correspond to the arguments

$$\frac{(2+4i)\,K}{3},\quad \frac{(4+4i)\,K}{3}.$$

II. $n = 4$.

The equation is $Y = 0$, or

$$d\,(1-2d^4) = 0;$$

whence $$d = 0,\quad \pm\frac{1}{\sqrt[4]{2}},\quad \pm\frac{i}{\sqrt[4]{2}},$$

corresponding to the arguments

$$(1+i)\,K,\quad \frac{K}{2},\quad \frac{K}{2}+2iK,\quad \frac{(3+2i)\,K}{2},\quad \frac{(1+2i)\,K}{2},$$

respectively.

If $$\cos\phi = \frac{1}{\sqrt[4]{2}},$$

$$\phi = 32^\circ\, 45'\, 54'',$$

and ϕ may of course, as in the last case, be constructed geometrically.

III. $n = 5$.

This is the first case of special interest, and requires discussion in detail.

The equation is $$Y - X = 0,$$

or $$(1+2d-4d^3-2d^4)^3 - 8d\,(1+d)^2\,(1-2d^2)^2\,(1-2d^4) = 0,$$

which is of the twelfth degree. The two real roots we have to discover are

$$d_1 = \operatorname{dn}\frac{2K}{5},\quad d_2 = \operatorname{dn}\frac{4K}{5}.$$

Now, since $$\mathrm{dn}\,(2K-u) = \mathrm{dn}\,u,$$

and $$\frac{8K}{5} = 2K - \frac{2K}{5},$$

it follows that d_1, d_2 form an Abelian cycle of two terms; and, by applying the duplication formula, we find that

$$d_2 = -\frac{1-2d_1^2+2d_1^4}{1-4d_1^2+2d_1^4} = \vartheta\,(d_1), \text{ say},$$

and, similarly, $d_1 = \vartheta\,(d_2)$.

Now, let x_1, x_2 be the roots of a quadratic equation

$$x^2+ax+b = 0,$$

and suppose $$x_1 = \vartheta\,(x_2), \quad x_2 = \vartheta\,(x_1),$$

ϑ being as above defined. Then, since

$$x_1 = \frac{b}{x_2} \quad \text{and} \quad x_2 = \frac{b}{x_1},$$

it follows that x_1, x_2 both satisfy

$$\frac{b}{x} = \vartheta\,(x) = -\frac{1-2x^2+2x^4}{1-4x^2+2x^4}.$$

Hence the polynomial

$$2x^5+2bx^4-2x^3-4bx^2+x+b$$

must be exactly divisible by x^2+ax+b. This leads to the conditions that

$$(a^2+1)(a^2+a-1)^2 = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(i.)},$$

and that b is a common root of

$$2\,(a+1)\,b^2-2\,(a^2-1)\,b-(a-1) = 0 \;\ldots\ldots\ldots\ldots\text{(ii.)},$$

$$2b^2-2\,(a^2+2a-2)\,b+(2a^3-2a+1) = 0 \;\ldots\ldots\text{(iii.)}.$$

The solutions are as follows—

$$a = i, \qquad b = -\frac{1-i}{2},$$

$$a = -i, \qquad b = -\frac{1+i}{2},$$

$$a = \frac{-1+\sqrt{5}}{2}, \quad 2b^2+(3-\sqrt{5})\,b+(\sqrt{5}-2) = 0,$$

$$a = \frac{-1-\sqrt{5}}{2}, \quad 2b^2+(3+\sqrt{5})\,b-(\sqrt{5}+2) = 0,$$

and both roots of each quadratic are available, because for these two values of a the equations (ii.) and (iii.) are identical.

The solution of $2b^2+(3-\sqrt{5})\,b+(\sqrt{5}-2)=0$

is
$$b=\frac{-3+\sqrt{5}\pm\sqrt{30-14\sqrt{5}}}{4}$$
$$=\frac{-3+\sqrt{5}\pm i\,\sqrt[4]{5}\,(3-\sqrt{5})}{4},$$

and that of the other quadratic is
$$b=\frac{-3-\sqrt{5}\pm\sqrt[4]{5}\,(3+\sqrt{5})}{4},$$

so that altogether there are six quadratics in x, namely,
$$2x^2+2ix-(1-i)=0 \quad\ldots\ldots\ldots\ldots(1),$$
$$2x^2-2ix-(1+i)=0 \quad\ldots\ldots\ldots\ldots(2),$$
$$4x^2+2(\sqrt{5}+1)\,x-3+\sqrt{5}\pm i\,(3-\sqrt{5})\,\sqrt[4]{5}=0 \quad\ldots\ldots(3,\ 4),$$
$$4x^2-2(\sqrt{5}+1)\,x-3-\sqrt{5}\pm\ (3+\sqrt{5})\,\sqrt[4]{5}=0 \quad\ldots\ldots(5,\ 6).$$

These, then, are the six quadratic factors of the original equation in x.

The only quadratics which can have real roots are the last pair. Choosing
$$4x^2-2(\sqrt{5}+1)\,x-3-\sqrt{5}+(3+\sqrt{5})\,\sqrt[4]{5}=0,$$
we have
$$(4x-\sqrt{5}-1)^2=6\,(3+\sqrt{5})-4\,(3+\sqrt{5})\,\sqrt[4]{5}$$
$$=(\sqrt{5}+1)^2\,\{3-2\sqrt[4]{5}\};$$
and therefore
$$x=\frac{\sqrt{5}+1}{4}\,\{1\pm\sqrt{3-2\sqrt[4]{5}}\}.$$

Both the roots are real positive proper fractions, and therefore correspond to the arguments $\frac{2K}{5}$, $\frac{4K}{5}$ respectively.

It may be observed that
$$\frac{\sqrt{5}+1}{4}=\cos\frac{\pi}{5},$$
and that $\sqrt{3-2\sqrt[4]{5}}$ is an algebraical unit.

It is easy to see how the values of x may be constructed geometrically.

The numerical values are

$$x_1 = \cdot 8870484 = \cos 27^\circ 29' 43'',$$

$$x_2 = \cdot 7309856 = \cos 43^\circ 1' 52''.$$

For the sake of comparison we will consider the quinquisection problem in the light of the complex theory. Here the divisor is, in the first instance, not 5 but $1+2i$, and the solution depends upon any one of the equations

$$\operatorname{sn}(1+2i)\,u = 0,$$

$$\operatorname{dn}(1+2i)\,u = \pm 1,$$

$$\operatorname{cn}(1+2i)\,u = \pm 1.$$

Now, if we make use of the formulæ

$$\operatorname{sn} iu = \frac{i \operatorname{sn} u}{\operatorname{cn} u}, \quad \operatorname{cn} iu = \frac{1}{\operatorname{cn} u}, \quad \operatorname{dn} iu = \frac{\operatorname{dn} u}{\operatorname{cn} u},$$

the addition theorem gives

$$\operatorname{sn}(1+2i)\,u = \frac{2(\operatorname{sn} u \operatorname{dn} 2u + i \operatorname{cn} u \operatorname{dn} u \operatorname{sn} 2u \operatorname{cn} 2u)}{2 \operatorname{cn}^2 2u + \operatorname{sn}^2 u \operatorname{sn}^2 2u}$$

$$= \frac{P(\operatorname{sn} u)}{Q(\operatorname{sn} u)},$$

where $P(x) = -x\{(1-2i)x^8 - 2(1-7i)x^6 + (1-32i)x^4 + 4(1+7i)x^2 - 4(1+2i)\}$,

$$Q(x) = 5x^8 - 20x^6 + 28x^4 - 16x^2 + 4.$$

We know from the general theory that $P(x)$ and $Q(x)$ must have a common factor of the fourth degree; this is found to be

$$(1-2i)x^4 - 2(1-2i)x^2 - 2i;$$

therefore

$$\operatorname{sn}(1+2i)\,u = -\frac{\operatorname{sn} u\{\operatorname{sn}^4 u - 2(2-i)\operatorname{sn}^2 u - 2i(1+2i)\}}{(1+2i)\operatorname{sn}^4 u - 2(1+2i)\operatorname{sn}^2 u + 2i}$$

$$= -\frac{\operatorname{sn} u\{\operatorname{sn}^4 u + 2i(1+2i)\operatorname{sn}^2 u - 2i(1+2i)\}}{(1+2i)\operatorname{sn}^4 u - 2(1+2i)\operatorname{sn}^2 u + 2i}.$$

Equating this to zero, and supposing that sn u does not vanish,

$$\begin{aligned}\operatorname{sn}^2 u &= -i(1+2i) \pm \sqrt{-(1+2i)^2+2i(1+2i)}\\ &= -i(1+2i) \pm i\sqrt{1+2i},\\ \operatorname{sn} u &= \pm\sqrt{-i(1+2i)\pm i\sqrt{1+2i}}.\end{aligned}$$

Since i is a primitive root of $1+2i$, the four values of sn u are those for which

$$\begin{aligned}u &\equiv \pm\frac{2K}{1+2i},\quad \pm\frac{2iK}{1+2i},\\ &\equiv \pm\frac{2(1-2i)K}{5},\quad \pm\frac{2(2+i)K}{5}.\end{aligned}$$

The four values of sn u may be plotted off by rule and compass in the plane of the complex variable; and this is, in a sense, the proper geometrical solution of the quinquisection problem.

To find cn u and dn u, we have

$$\begin{aligned}\operatorname{cn}^2 u &= 1-\operatorname{sn}^2 u\\ &= -1+i\mp i\sqrt{1+2i},\\ \operatorname{dn}^2 u &= 1-\tfrac{1}{2}\operatorname{sn}^2 u\\ &= \frac{i}{2}\{1\pm\sqrt{1+2i}\} = \tfrac{1}{4}\{i\pm\sqrt{1+2i}\}^2.\end{aligned}$$

If we like, we may reduce sn u, cn u, dn u to the form $\alpha+\beta i$; the results are

$$\operatorname{sn} u = \pm\left[\sqrt{\tfrac{1}{2}\left\{\sqrt[4]{5}\left(\frac{\sqrt{5}+1}{2}+\sqrt{\frac{\sqrt{5}-1}{2}}\right)\pm\left(1+\sqrt{\frac{\sqrt{5}+1}{2}}\right)\right\}}\right.$$
$$\left.+\sqrt{\tfrac{1}{2}\left\{\sqrt[4]{5}\left(\quad\right)\mp\left(1+\sqrt{\frac{\sqrt{5}+1}{2}}\right)\right\}}\,i\right],$$

$$\operatorname{cn} u = \pm\tfrac{1}{2}\left[\left\{1\pm\sqrt{\frac{\sqrt{5}+1}{2}}\mp\sqrt{\frac{\sqrt{5}-1}{2}}\right\}\right.$$
$$\left.+\left\{1\pm\sqrt{\frac{\sqrt{5}+1}{2}}\pm\sqrt{\frac{\sqrt{5}-1}{2}}\right\}i\right],$$

$$\begin{aligned}\operatorname{dn} u &= \pm\tfrac{1}{2}\left\{i\pm\left(\sqrt{\frac{\sqrt{5}+1}{2}}+i\sqrt{\frac{\sqrt{5}-1}{2}}\right)\right\}\\ &= \pm\tfrac{1}{2}\left\{\sqrt{\frac{\sqrt{5}+1}{2}}+\left(\sqrt{\frac{\sqrt{5}-1}{2}}\pm 1\right)i\right\}.\end{aligned}$$

The values of $\operatorname{cn} u$, $\operatorname{dn} u$ may be neatly expressed in the trigonometrical forms

$$\operatorname{cn} u = \pm \tfrac{1}{2} \left[\left(1 \pm \frac{4}{\sqrt[4]{5}} \sin \frac{\pi}{10} \cos \frac{3\pi}{10}\right) + \left(1 \pm \frac{4}{\sqrt[4]{5}} \cos \frac{\pi}{10} \sin \frac{3\pi}{10}\right) i \right],$$

$$\operatorname{dn} u = \pm \tfrac{1}{2} \left[\frac{2}{\sqrt[4]{5}} \sin \frac{2\pi}{5} + \left(\frac{2}{\sqrt[4]{5}} \sin \frac{\pi}{5} \pm 1 \right) i \right];$$

there does not appear to be any correspondingly simple expression for $\operatorname{sn} u$.

The most elegant method of all, from the point of view of the complex theory, is to introduce the complex fifth roots of unity. If we put

$$\omega = e^{2\pi i/5},$$

$$\xi = \omega + i\omega^2 - i\omega^3 - \omega^4,$$

it is found that

$$\xi^2 = -(1+2i)\sqrt{5},$$

and we may put

$$\frac{i\xi}{\sqrt[4]{5}} = \sqrt{1+2i};$$

hence

$$\operatorname{dn} u = \pm \frac{i}{2} \left\{ 1 \pm \frac{\xi}{\sqrt[4]{5}} \right\},$$

$$\operatorname{cn} u = \pm \frac{1+i}{2} \left\{ 1 \mp \frac{i\xi}{\sqrt[4]{5}} \right\},$$

$$\operatorname{sn}^2 u = 2 - i \pm \frac{\xi}{\sqrt[4]{5}}.$$

The quartic of which $\operatorname{dn} u$ is a root is

$$2d^4 - 2id^2 + i = 0,$$

which breaks up into the two quadratics

$$2d^2 - 2id - (1+i) = 0,$$

$$2d^2 + 2id - (1+i) = 0.$$

Of these the first is the same as the second of the six quadratics obtained from the geometrical theory; the other is obtained from the first of the six by changing the sign of the middle term. This may be accounted for by the fact that, strictly speaking, we ought to write

$$\frac{R-\delta}{R+\delta} = \pm d,$$

thus obtaining twelve quadratics, in accordance with the analytical theory.

As to the arguments to be associated with the different values of d, it will be found that, if we write for convenience

$$\epsilon = \sqrt{\frac{\sqrt{5}+1}{2}}, \quad \epsilon' = \sqrt{\frac{\sqrt{5}-1}{2}},$$

$$d_1 = \tfrac{1}{2}\{\epsilon+(\epsilon'+1)i\} = \frac{i}{2}\left\{1-\frac{\xi}{\sqrt[4]{5}}\right\} = \mathrm{dn}\,\frac{(2-2i)K}{1+2i},$$

$$d_2 = -\tfrac{1}{2}\{\epsilon+(\epsilon'-1)i\} = \frac{i}{2}\left\{1+\frac{\xi}{\sqrt[4]{5}}\right\} = \mathrm{dn}\,\frac{(4-4i)K}{1+2i},$$

$$d_3 = -\tfrac{1}{2}\{\epsilon+(\epsilon'+1)i\} = -\frac{i}{2}\left\{1-\frac{\xi}{\sqrt[4]{5}}\right\} = \mathrm{dn}\,\frac{2K}{1+2i},$$

$$d_4 = \tfrac{1}{2}\{\epsilon+(\epsilon'-1)i\} = -\frac{i}{2}\left\{1+\frac{\xi}{\sqrt[4]{5}}\right\} = \mathrm{dn}\,\frac{2iK}{1+2i}.$$

We are now in a position to deduce a solution of the real quinquisection. Putting

$$u = \frac{2K}{1+2i}, \qquad v = \frac{2K}{1-2i},$$

$$u+v = \frac{4K}{5}, \qquad u-v = -\frac{8iK}{5}.$$

Now, by the addition theorem,

$$\mathrm{dn}\,(u+v)+\mathrm{dn}\,(u-v) = \frac{2\,\mathrm{dn}\,u\,\mathrm{dn}\,v}{-1+2\,(\mathrm{dn}^2 u+\mathrm{dn}^2 v)-2\,\mathrm{dn}^2 u\,\mathrm{dn}^2 v},$$

$$1+\mathrm{dn}\,(u+v)\,\mathrm{dn}\,(u-v) = \frac{\mathrm{dn}^2 u+\mathrm{dn}^2 v}{-1+2\,(\mathrm{dn}^2 u+\mathrm{dn}^2 v)-2\,\mathrm{dn}^2 u\,\mathrm{dn}^2 v}.$$

In the present case

$$\mathrm{dn}\,u = -\tfrac{1}{2}\{\epsilon+(\epsilon'+1)i\},$$

$$\mathrm{dn}\,v = -\tfrac{1}{2}\{\epsilon-(\epsilon'+1)i\};$$

and therefore

$$\mathrm{dn}\,u\,\mathrm{dn}\,v = \tfrac{1}{4}\{\epsilon^2+\epsilon'^2+2\epsilon'+1\} = \tfrac{1}{4}\{\sqrt{5}+1+2\epsilon'\},$$

$$\mathrm{dn}^2 u\,\mathrm{dn}^2 v = \tfrac{1}{4}(\sqrt{5}+1)(1+\epsilon'),$$

$$\mathrm{dn}^2 u+\mathrm{dn}^2 v = -\epsilon'.$$

Putting in these values, and performing the necessary reductions, we find

$$\mathrm{dn}\,(u+v)+\mathrm{dn}\,(u-v) = \frac{\sqrt{5}+1}{2}-\frac{3+\sqrt{5}}{2}\,\epsilon',$$

$$\mathrm{dn}\,(u+v)\,\mathrm{dn}\,(u-v) = \frac{\sqrt{5}+1}{4}-\frac{\sqrt{5}+2}{2}\,\epsilon'.$$

Therefore $\mathrm{dn}\,\frac{4K}{5}$, $\mathrm{dn}\,\frac{8iK}{5}$ are the roots of

$$d^2-\left(\frac{\sqrt{5}+1}{2}-\frac{3+\sqrt{5}}{2}\epsilon'\right)d+\left(\frac{\sqrt{5}+1}{4}-\frac{\sqrt{5}+2}{2}\epsilon'\right)=0.$$

Guided by previous results, we assume

$$d=\frac{\sqrt{5}+1}{4}\,t;$$

the equation then becomes

$$t^2-2\,(1-\epsilon)\,t+(\sqrt{5}-1-2\epsilon)=0,$$

and hence

$$t = 1-\epsilon\pm\sqrt[4]{5}\,.\,\epsilon'$$

$$=1-\sqrt{\frac{\sqrt{5}+1}{2}}\pm\sqrt{\frac{5-\sqrt{5}}{2}}.$$

Since $\mathrm{dn}\,\frac{4K}{5}$ is positive,

$$\mathrm{dn}\,\frac{4K}{5}=\left(1-\sqrt{\frac{\sqrt{5}+1}{2}}+\sqrt{\frac{5-\sqrt{5}}{2}}\right)\frac{\sqrt{5}+1}{4},$$

$$\mathrm{dn}\,\frac{8iK}{5}=\left(1-\sqrt{\frac{\sqrt{5}+1}{2}}-\sqrt{\frac{5-\sqrt{5}}{2}}\right)\frac{\sqrt{5}+1}{4}=-\mathrm{dn}\,\frac{2iK}{5}.$$

The value previously obtained for $\mathrm{dn}\,\frac{4K}{5}$ was

$$\mathrm{dn}\,\frac{4K}{5}=\frac{\sqrt{5}+1}{4}\left(1-\sqrt{3-2\sqrt[4]{5}}\right);$$

it is easy to show that the two values coincide; thus, writing for the moment

$$\theta=\sqrt[4]{5},$$

we ought to have

$$\theta\epsilon'-\epsilon=-\sqrt{3-2\theta}.$$

The signs agree, and, on squaring,

$$\theta^2\epsilon'^2-2\theta+\epsilon^2=3-2\theta,$$

or

$$\theta^2\,\frac{\sqrt{5}-1}{2}=3-\frac{\sqrt{5}+1}{2}=\frac{5-\sqrt{5}}{2},$$

that is,

$$\theta^2=\sqrt{5},$$

and the relation is verified.

In the same way,

$$\operatorname{dn}\frac{8iK}{5}=\frac{\sqrt{5}+1}{4}\,(1-\sqrt{3+2\sqrt[4]{5}})=-\operatorname{dn}\frac{2iK}{5}.$$

It is now possible to construct a table of the values of $\operatorname{sn} u$, $\operatorname{cn} u$, $\operatorname{dn} u$ for all the arguments

$$\frac{2\,(m+ni)\,K}{5};$$

but, on account of the trouble of reducing the expressions to their simplest forms, no attempt has been made to form a complete table. A few of the formulæ, however, are given. For convenience, we write

$$\epsilon=\sqrt{\frac{\sqrt{5}+1}{2}},\quad \epsilon'=\sqrt{\frac{\sqrt{5}-1}{2}},$$

$$\eta=\sqrt{3+2\sqrt[4]{5}},\quad \eta'=\sqrt{3-2\sqrt[4]{5}},$$

where observe that

$$\eta\eta'=\sqrt{5}-2,\quad \eta+\eta'=2\epsilon,\quad \eta-\eta'=2\epsilon'\sqrt[4]{5}.$$

so that we have

$$\left.\begin{aligned}\eta&=\epsilon+\epsilon'\theta\\ \eta'&=\epsilon-\epsilon'\theta\end{aligned}\right\},\qquad \theta=\sqrt[4]{5}.$$

$$\operatorname{dn}\,\frac{2K}{5}=\tfrac{1}{2}\epsilon^2\,(1+\eta'),$$

$$\operatorname{dn}\,\frac{4K}{5}=\tfrac{1}{2}\epsilon^2\,(1-\eta'),$$

$$\operatorname{dn}\frac{2iK}{5}=\tfrac{1}{2}\epsilon^2\,(\eta-1),$$

$$\operatorname{dn}\frac{4iK}{5}=\tfrac{1}{2}\epsilon^2\,(\eta+1);$$

$$\mathrm{cn}\,\frac{2K}{5} = \frac{1+\eta'}{\eta-1} = \frac{\epsilon^2+\epsilon-1}{\theta+1},$$

$$\mathrm{cn}\,\frac{4K}{5} = \frac{1-\eta'}{\eta+1} = -\frac{\epsilon^2-\epsilon-1}{\theta+1},$$

$$\mathrm{cn}\,\frac{2iK}{5} = \frac{\eta-1}{1+\eta'} = -\frac{\epsilon^2-\epsilon-1}{\theta-1},$$

$$\mathrm{cn}\,\frac{4iK}{5} = \frac{\eta+1}{1-\eta'} = \frac{\epsilon^2+\epsilon-1}{\theta-1};$$

$$\mathrm{sn}\,\frac{2K}{5} = \frac{2\sqrt{\theta-\epsilon}}{\eta-1},$$

$$\mathrm{sn}\,\frac{4K}{5} = \frac{2\sqrt{\theta+\epsilon}}{\eta+1},$$

$$\mathrm{sn}\,\frac{2iK}{5} = \frac{2i\sqrt{\theta-\epsilon}}{1+\eta'},$$

$$\mathrm{sn}\,\frac{4iK}{5} = \frac{2i\sqrt{\theta-\epsilon}}{1-\eta'}.$$

It is interesting to verify these results by comparing them with those of Gauss and Schwering. The function which Gauss calls sin lemn v, and Schwering sin am v, is the inverse of the function

$$v = \int_0^x \frac{dx}{\sqrt{(1-x^4)}} = (\sin \mathrm{lemn})^{-1}\, x;$$

we shall write this

$$\mathrm{sl}\, v = x,$$

and put

$$\mathrm{cl}\, v = \sqrt{1-x^2}, \quad \mathrm{dl}\, v = \sqrt{1+x^2},$$

$$\int_0^1 \frac{dx}{\sqrt{1-x^4}} = \varpi = \frac{K}{\sqrt{2}}.$$

Then, if the arguments u, v relate to the same point on the lemniscate,

$$u = \sqrt{2}\,(\varpi - v),$$

$$\mathrm{sn}\, u = \mathrm{cl}\, v,$$

$$\mathrm{cn}\, u = \mathrm{sl}\, v,$$

$$\mathrm{dn}\, u = \frac{1}{\sqrt{2}}\,\mathrm{dl}\, v;$$

and when $$u = \frac{2mK}{n},$$

$$v = \frac{(n-2m)\varpi}{n}.$$

Schwering gives a numerical result which in this notation is

$$\operatorname{sl}\frac{\varpi}{5} = \cdot 2620824 = \operatorname{cn}\frac{4K}{5},$$

and hence we find

$$\operatorname{dn}\frac{4K}{5} = \sqrt{\tfrac{1}{2}\left(1+\operatorname{cn}^2\frac{4K}{5}\right)} = \cdot 7310017.$$

The value found above (p. 375) was

$$\cdot 7309856,$$

so that there is a discrepancy of about ·0000161 which must be accounted for by errors of calculation, as the agreement is too close for any mistake in the argument of the function.

To take an example of a different kind, Gauss gives a number of quinquisection formulæ (*Werke*, III., p. 421) of which the first is, in our notation, equivalent to

$$\left\{\operatorname{cn}^2\frac{K}{5} + \operatorname{cn}^2\frac{3K}{5}\right\}^2 = 14\sqrt{5}-30 = \sqrt{5}\,(3-\sqrt{5})^2;$$

whence $$\operatorname{cn}^2\frac{K}{5} + \operatorname{cn}^2\frac{3K}{5} = \theta\,(3-\sqrt{5}) = 2\theta\epsilon'^4.$$

To verify this, write

$$\operatorname{dn}\frac{2K}{5} = d_1, \quad \operatorname{dn}\frac{4K}{5} = d_2,$$

$$\operatorname{cn}\frac{2K}{5} = c_1, \quad \operatorname{cn}\frac{4K}{5} = c_2;$$

then, observing that $$\operatorname{cn}^2 u = \frac{\operatorname{dn}2u + \operatorname{cn}2u}{1+\operatorname{dn}2u},$$

we find $$\operatorname{cn}^2\frac{K}{5} + \operatorname{cn}^2\frac{3K}{5} = \frac{d_1+c_1}{1+d_1} + \frac{d_2-c_2}{1+d_2}$$

$$= \frac{d_1+d_2+2d_1d_2+c_1-c_2+c_1d_2-c_2d_1}{1+d_1+d_2+d_1d_2}.$$

From the short table given above

$$d_1+d_2=\epsilon^2,\quad d_1d_2=\tfrac{1}{4}\epsilon^4(1-\eta'^2)=\tfrac{1}{2}\epsilon^4(\theta-1),$$

$$c_1-c_2=\frac{2(\epsilon^2-1)}{\theta+1},\quad c_1d_2-c_2d_1=\frac{\epsilon^2(1-\eta'^2)}{\eta^2-1}=\frac{\epsilon^2(\theta-1)}{\theta+1}.$$

Therefore the numerator of the fraction is

$$\epsilon^2+\epsilon^4(\theta-1)+\frac{2(\epsilon^2-1)+\epsilon^2(\theta-1)}{\theta+1}$$

$$=\frac{1}{\theta+1}\left[\epsilon^2+\epsilon^4(\sqrt{5}-1)+2\epsilon^2-2-\epsilon^2+2\epsilon^2\theta\right]$$

$$=\frac{1}{\theta+1}(4\epsilon^2-2+2\epsilon^2\theta)=2\epsilon^2+\frac{2e'^2}{\theta+1}=2\epsilon^2+\theta-1$$

$$=\sqrt{5}+\theta=\theta(\theta+1).$$

In a similar way, it may be shown that the denominator is equal to $\frac{1}{2}\epsilon^4(1+\theta)$; hence

$$\text{cn}^2\frac{K}{5}+\text{cn}^2\frac{3K}{5}=\frac{2\theta}{\epsilon^4}=2\theta\epsilon'^4,$$

which completes the verification.

It is found in a similar way that

$$\text{cn}^2\frac{K}{5}-\text{cn}^2\frac{3K}{5}=2\epsilon'^5;$$

whence $$\left(\text{cn}^2\frac{K}{5}-\text{cn}^2\frac{3K}{5}\right)^2=4\epsilon'^{10}=10\sqrt{5}-22,$$

which is Gauss's second formula.

By combining these results we obtain

$$\text{cn}^2\frac{K}{5}=\epsilon'^4(\theta+\epsilon'),\quad \text{cn}^2\frac{3K}{5}=\epsilon'^4(\theta-\epsilon'),$$

so that $$\text{cn}\,\frac{K}{5}=\frac{\sqrt{5}-1}{2}\sqrt{\sqrt[4]{5}+\sqrt{\frac{\sqrt{5}-1}{2}}}=\epsilon'^2\sqrt{\theta+\epsilon'},$$

$$\text{cn}\,\frac{3K}{5}=\frac{\sqrt{5}-1}{2}\sqrt{\sqrt[4]{5}-\sqrt{\frac{\sqrt{5}-1}{2}}}=\epsilon'^2\sqrt{\theta-\epsilon'}.$$

Thursday, May 14th, 1896.

Major P. A. MACMAHON, R.A., F.R.S., President, in the Chair.

The following gentlemen were admitted into the Society:—Mr. F. W. Dyson, Royal Observatory, Greenwich; Mr. G. H. J. Hurst, Eton College; and Mr. F. W. Russell, University College School.

Mr. Baker spoke upon "The Bitangents of a Plane Quartic Curve and the Straight Lines of a Cubic Surface."

A paper by Prof. E. W. Brown, "On the Application of the Principal Function to the Solution of Delaunay's Canonical System of Equations," was, in the absence of the author, taken as read.

Short impromptu communications were made by the President, Prof. Hill, Col. Cunningham, Mr. Hammond, and Mr. Tucker.

The following presents had been received since the April meeting:

"Proceedings of the Royal Society," Vol. LIX., No. 356.

"Journal of the Institute of Actuaries," Vol. XXXII., No. 181, Pt. 5, April, 1896.

"Bulletin of the American Mathematical Society," 2nd Series, Vol. II., No. 7; April, 1896.

"Encyclopädie der Mathematischen Wissenschaften," 2 Probeartikele, 8vo; Leipzig, 1896. iii.C 6*a*, "Flächer die Herordnung," von W. F. Meyer. ii.A 7*b*, "Potential-theorie (Theorie der Laplace-Poisson'schen Differential Gleichung)," von H. Burkhardt und W. F. Meyer.

"Bulletin de la Société Mathématique de France," Tome XXIV., Nos. 2, 3; Paris, 1896.

"Bulletin des Sciences Mathématiques," Tome XX., Avril, 1896; Paris.

"Berichte über die Verhandlungen der Königl. Sächs. Gesellschaft der Wissenschaften zu Leipzig," I; 1896.

"Beiblätter zu den Annalen der Physik und Chemie," Bd. XX., St. 4; Leipzig, 1896.

"Jahrbuch über die Fortschritte der Mathematik," Bd. XXV., Heft 1; Berlin, 1896.

"Proceedings of the Physical Society of London," Vol. XIV., Pt. 5, May, 1896.

"Memoirs and Proceedings of the Manchester Literary and Philosophical Society," Vol. X., No. 2, 1896.

"Atti della reale Accademia dei Lincei—Rendiconti," Vol. V., Fasc. 7, Sem. 1; Roma, 1896.

"Educational Times," May, 1896.

"Journal für die reine und angewandte Mathematik," Bd. cxvi., Heft 2; Berlin, 1896.

"Annali di Matematica, Milano," Tome xxiv., Fasc. 2; 1896.

"Indian Engineering," Vol. xix., Nos. 14, 15, 16, April 4th to April 18th, 1896.

On the Application of the Principal Function to the Solution of Delaunay's Canonical System of Equations. By Ernest W. Brown. Read May 14th, 1896. Received April 28th, 1896.

(i.) The system of canonical equations adopted by Delaunay is

$$\frac{dL}{dt} = \frac{\partial R}{\partial l}, \quad \frac{dG}{dt} = \frac{\partial R}{\partial g}, \quad \frac{dH}{dt} = \frac{\partial R}{\partial h};$$

$$\frac{dl}{dt} = -\frac{\partial R}{\partial L}, \quad \frac{dg}{dt} = -\frac{\partial R}{\partial G}, \quad \frac{dh}{dt} = -\frac{\partial R}{\partial H}.$$

In order to solve these equations, he puts

$$R = -B - A\cos\theta + R_1,$$

where $$\theta = il + i'g + i''h + i'''n't + q.$$

Here $-A\cos\theta$ is any periodic term of the disturbing function R, and $-B$ is the non-periodic portion of R; B, A are functions of L, G, H only, i, i', i'', i''' are positive or negative integers, n' is the solar mean motion, and q is a constant depending on the solar epoch and perigee. The equations are solved by neglecting R_1, and Delaunay then inquires what variable values the six arbitrary constants, introduced into the solution, are to have when R_1 is not neglected. These arbitraries are so chosen that the new equations are canonical. In order to satisfy this condition, three only of the six arbitraries may be chosen at will, the manner in which the other three arbitraries enter into the solution being then determinate. Delaunay adopts the constant terms in the expressions for l, g, h as three of the arbitraries, and the equations are afterwards transformed so that the new variables are the non-periodic, instead of the constant, parts of l, g, h.