On the Stability of a Frictionless Liquid. Theory of Critical Planes.

By

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1. In a recent paper*) Lord Rayleigh has attempted to investigate the conditions of stability of a frictionless liquid when there are *critical planes.* The statement of the problem is as follows: —

Liquid is flowing in steady motion between the two planes $y=0$ and $y=a$, the velocity being parallel to the axis of x and equal to V, where V is some function of y. A *small* disturbance is communicated to the liquid, it is required of investigate the conditions of stability, the motion being supposed to be in two dimensions.

In steady motion the component velocities are $(V, 0)$; after disturbance they an $(V+u, v)$, where u and v are small quantities *in the beginning of the disturbed motion*, and Lord Rayleigh proves that v is determined by the equation**)

$$(1)\qquad \left(\frac{n}{k}+V\right)\left(\frac{d^2 v}{dy^2}-k^2 v\right)=\frac{d^2 V}{dy^2}v.$$

This equation is obtained in the following manner. Whatever the character of the disturbance may be, the velocity v can be expressed by means of Fourier's theorem in a series of (or definite integrals involving) sines and cosines of x. It is therefore sufficient, so far as the coordinate x is concerned, to consider the typical term ε^{ikx}. Also if the disturbed motion is stable, the time factor must be a periodic function, and therefore expressible in the form ε^{int}. It will thus be found that if these substitutions be made, and the pressure and velocity u be eliminated from the equations of motion and continuity, the result will be expressed by (1).

In steady motion the molecular rotation is equal to $-\frac{1}{2}dV\,|\,dy$; accordingly if this quantity is constant, the right hand side of (1)

*) Proc. Lond. Math. Soc. vol. XXVII, p. 5.

**) Ibid. vol. XI, p. 57.

vanishes and the equation splits up into two factors, the second of which when equated to zero furnishes the differential equation for v. It may however happen that certain planes exist, which are called *critical* planes, at which the first factor vanishes; and at such a plane it cannot be asserted without further investigation that the second factor vanishes. This circumstance has led Lord Kelvin*) to throw doubts upon the universal application of results which are based upon the solution of the differential equation which is formed by equating the second factor to zero: and Lord Rayleigh has therefore attempted to meet this objection by examining what takes place when a critical plane exists, but the results on pages 10 and 11 are erroneous owing to the fact that they make the molecular rotation infinite at such planes. Some observations are added at the end of the paper which are apparently intended to meet this objection, but they miss the points at issue.

2. The fallacy of the results to which a wrong method of approximation has led him can be shown as follows: —

In questions relating to stability, the velocities produced by the disturbance are by hypothesis small quantities *in the beginning of the disturbed motion;* but a statement of this kind has no meaning unless the standard of measurement is defined. In cases like the present, the meaning of the phrase is, *a velocity whose numerical value is a small quantity in comparison with the numerical value of the velocity in steady motion.* Moreover by Newton's second Law of Motion, the forces required to produce these small velocities must be proportional to them, and the numerical values of these forces must therefore be small quantities compared with the numerical values of the forces required to generate the steady motion in a liquid at rest. Under these circumstances it is an obvious impossibility for a small disturbance to suddenly change the value of the molecular rotation from a finite to an infinite one; and any solution which leads to this result must necessarily be erroneous.

3. The quantity $d^2v \mid dy^2 - k^2v$ is proportional to the difference between the molecular rotation just before and just after disturbance, and must therefore be a small quantity in the beginning of the disturbed motion. Accordingly if the molecular rotation is *not* constant in steady motion, in which case $d^2V|dy^2$ will not be zero, it follows that at a critical plane we must have $d^2V|dy^2 = 0$ or $v = 0$. But the first condition is one which cannot be satisfied except for special values of V; and if the form of the function which determines the velocity in steady motion is *not* of this special form, it follows that $v = 0$ at a critical plane.

*) Phil. Maj. vol. XXIV, p. 275.

4. We are now prepared to work out the theory of critical planes.

For simplicity we shall suppose that the steady motion is such that there are no vortex sheets, in which case

$$(2) \qquad V = \varphi(y)$$

where φ is a given function which is finite and continuous throughout the space occupied by the liquid. If possible, let $y = c$ be a critical plane, the condition for which is that

$$(3) \qquad -n/k = \varphi(c).$$

This equation determines the value of the time constant n in terms of the wave constant k, provided a real value of c exists which lies between 0 and a. If no value of c exists which lies between these limits, equation (3) represents an impossible state of motion and a critical plane cannot exist.

Substituting the value of n/k from (2) in (1) and integrating, we shall obtain

$$(4) \qquad v = Af_1(y) + Bf_2(y)$$

where A and B are the constants of integration, and f_1, f_2 are two independent functions whose form depends upon that of φ. The boundary conditions require that $v = 0$, when $y = 0$ and $y = a$.

Case I. We shall first suppose that $d^2V/dy^2 = 0$ at the critical plane; and also that neither of the functions f becomes infinite between $y = 0$ and $y = a$. In this case the boundary conditions will enable us to eliminate the two constants of integration, and we shall thus obtain an equation of the form

$$F(a, k, c) = 0$$

which is the equation for determining c. The condition for the existence of a critical plane is, that the above equation should have at least one real root lying between 0 and a.

It may however happen that one of the functions, say f_2, becomes infinite between the limits, in which case $B = 0$, and the boundary conditions then require that $f_1(0) = 0$ and $f_1(a) = 0$; is other words that 0 and a should be roots of the equation $f_1(y) = 0$. These conditions cannot be satisfied except for special forms of the function f_1, and when they cannot, the existence of a critical plane is impossible.

Case II. In this case d^2V/dy^2 does not vanish at a critical plane, and consequently v must satisfy the three conditions of vanishing when $y = 0$, $y = c$ and $y = a$; but as the value of v cannot certain more than two arbitrary constants, these three conditions cannot in general be satisfied, in which case a critical plane cannot exist.

5. If the steady motion is such that vortex sheets exist, the

equations must be separately applied to each region bounded by two consecutive vortex sheets; and the relations between the various constants of integration which occur in the solutions must be determined by means of the boundary conditions which exist at a vortex sheet.

6. When the molecular rotation is constant in steady motion, the problem is one which is best treated by special methods. In this case $d^2 V/dy^2$ is zero throughout the whole space occupied by the liquid, and consequently the second factor of (1) must vanish except at a critical plane.

In steady motion

(5) $$V = A - 2\omega y$$

where A is a constant, and ω is the constant molecular rotation. If $\omega + \zeta$ be the the molecular rotation after disturbance, ζ satisfies the equation

(6) $$\frac{d\zeta}{dt} + (V+u)\frac{d\zeta}{dx} + v\frac{d\zeta}{dy} = 0.$$

Since u and v are small quantities in the beginning of the disturbed motion, equation (6) reduces to

$$\frac{d\zeta}{dt} + V\frac{d\zeta}{dx} = 0$$

so that putting

$$\zeta = F(y)\,\varepsilon^{ikx+int}$$

where F is an undetermined function of y, we obtain

(7) $$(n/k + V)\,F(y) = 0.$$

This equation shows that $F(y)$ must be zero except at a critical plane.

Let ψ be the difference between Earnshaw's current function before and after disturbance; then

(8) $$\nabla^2\psi = -2\zeta = -2F(y)\,\varepsilon^{ikx+int}.$$

Equations (7) and (8) shew that ψ is the potential of a surface distribution of matter upon the critical plane, whose density is proportional to $F(c)$; consequently unless $F(c)$ be zero, the critical plane must be a vortex sheet. But it is known from the general principles of Hydrodynamics that a vortex sheet cannot be generated by a disturbing force; hence $F(c)$ must be zero unless the critical plane be a vortex sheet in steady motion.

7. To investigate the conditions for the existence of a critical plane, we must therefore suppose that a vortex sheet exists in steady motion. Let $y = c$ be the equation of the vortex sheet, and on the positive side of this plane let

$$V = V_1 = A - 2\omega(y-c)$$

and on the negative side, let

$$V = V_2 = B - 2\omega(y - c).$$

The condition that the vortex sheet should be a critical plane is that A or B should be equal to $-n|k$; and we shall adopt the former alternative. Writing $\psi\varepsilon^{ikx+int}$ for ψ in (8) we obtain

(9) $$\frac{d^2\psi}{dy^2} - k^2\psi = -2F(y)$$

and since $F(y) = 0$ except at the critical plane, are must have on the positive side of the plane

(10) $$\psi_1 = C \sinh k(y - a)$$

and on the negative side

(11) $$\psi_2 = D \sinh ky$$

the constants being determined so that v or $-d\psi|dx$ vanishes when $y = 0$ and $y = a$.

At the critical plane where $y = c$, we must have

$$-\frac{d\psi_1}{dy} + \frac{d\psi_2}{dy} = 2F(c)$$

which gives

(12) $$F(c) = \frac{1}{2}k\left\{C \cosh(a - c) + D \cosh kc\right\}$$

which determines the surface density $F(c)$.

After disturbance, the particles which originally lay on the critical plane will lie on a surface whose equation may be taken to be

(13) $$y - c - E\varepsilon^{ikx+int} = f(x, y, t) = 0.$$

This surface fulfills the conditions of a bounding surface, and therefore

(14) $$\frac{df}{dt} + V\frac{df}{dx} + v\frac{df}{dy} = 0.$$

Applying (14) to the positive side we get

$$E(n + kV_1) + Ck \sinh k(c - a) = 0;$$

and since by hypothesis

$$n + kV_1 = 0,$$

it follows that $C = 0$.

Applying (14) to the negative side we get

$$E(n + kV_2) + Dk \sinh kc = 0$$

which determines the relation between E and D.

From these results it follows that, to the first order of small quantities, the liquid on the positive side of the vortex sheet is uninfluenced by the disturbance. The only effect of the latter is to displace the particles of liquid which originally formed the vortex sheet into the sinuous surface (14), whilst the surface itself moves forward with the same velocity V_1 which the particles on its positive side had in steady motion.

8. We have lastly to determine the value of c, which fixes the position of the critical plane.

The pressure equation is

$$\frac{du}{dt} + v\frac{dV}{dy} = -\frac{1}{\varrho}\frac{dp}{dx} - V\frac{du}{dx}$$

whence recollecting that x and t enter in the form of the factor $\varepsilon^{ikx+int}$, we easily obtain:

$$\left(\frac{n}{k} + V\right)\frac{dv}{dy} - v\frac{dV}{dy} = -\frac{ikp}{\varrho}, \tag{15}$$

whence the condition of continuity of pressure requires that the left hand side of (15) should be continuous. Since v_1 and $dv_1 | dy$ are zero, the condition becomes

$$\left(\frac{n}{k} + V_2\right)\frac{dv_2}{dy} + 2\omega v_2 = 0$$

or

$$\tanh kc = k(V_1 - V_2) \,|\, 2\omega. \tag{16}$$

Now c must be a positive quantity lying between a and zero; moreover since $2\pi | k$ is the wave-length of the disturbance, k must also be positive; hence the *first* condition that the vortex sheet may be critical plane is that

$$0 < k(V_1 - V_2) \,|\, 2\omega < 1. \tag{17}$$

If this condition is satisfied, let m be the value of kc furnished by (16); then if λ be the wave-length, we shall have $c = \lambda m | 2\pi$; and since c cannot be greater than a, we obtain the second condition that

$$\lambda < 2\pi a \,|\, m. \tag{18}$$

Unless both the conditions (17) and (18) are satisfied, a critical plane cannot exist.

9. With regard to the method of procedure, it may be objected that the hypothesis that x and t enter into the solution in the form of the factor $\varepsilon^{ikx+int}$ is a pure assumption, that it would be easy to invent examples in which this is not the case. The answer to this objection is, that by making this hypothesis in the first instance, a particular solution can be obtained which can always be generalized by Fourier's theorem or in certain cases by definite integrals of a simpler form. It is also necessary to recollect that before embarking in any mathematical investigations, we must first settle the practical

question as to how the disturbed motion is to be generated. To talk about an arbitrary disturbance is beside the mark, untill we have first invented or imagined some machinery by which this operation can be effected; for if we omit to do this, it may be found that some proposed disturbed motion is an impossibility, and our mathematics will show this by leading us into all sorts of difficulties and apparently inexplicable results.

10. In the class of problems discussed by Lord Rayleigh, a plane of discontinuity $y = c$ usually exists, at which the steady motion changes its character by reason of the molecular rotation or the tangential velocity V being discontinous. We shall therefore suppose that the disturbed motion is generated by means of an impulsive pressure applied to this plane, which produces a velocity $v = \varphi(x)$ at the plane in question.

The differential equation for v is satisfied by

$$v = F(k, y) \cos(kx - nt) \tag{19}$$

and the equation of continuity shows that

$$u = -k^{-1} F'(k, y) \sin(kx - nt)$$

which is an example of the general dynamical theorem, that an impulsive force of one type may produce a velocity of a different type.

The differential equation for F being of the second order will contain two constants. If therefore we confine our attention, for the present, to the liquid lying between the planes $y = c$ and $y = a$, where the latter is one of the bounding planes, the constants must be determined so that

$$F(k, a) = 0,$$

we may therefore write in the place of (19)

$$v = \mathsf{A} f(k, y) \cos(kx - nt)$$

where

$$f(k, a) = 0.$$

The usual process of solution gives a relation of the form

$$n = \psi(k);$$

and when the steady motion is stable, n must be real for all values of k.

The constant A may have any value which is independent of x, y and t; we shall therefore put

$$\mathsf{A} = \frac{2\varphi(\lambda)\cos\lambda k}{\pi f(k, c)},$$

which shows that

$$v = \frac{2 f(k, y)\, \varphi(\lambda) \cos\lambda k \cos\{kx - t\psi(k)\}}{\pi f(k, c)}$$

is a solution of (1), and therefore

$$v = \frac{2}{\pi}\int_0^\infty dk \int_{-\infty}^\infty \frac{f(k, y)\,\varphi(\lambda) \cos \lambda k \cos\{kx - t\psi(k)\}}{f(k, c)}\, d\lambda$$

is also a solution. The initial value of v at the plane $y = c$, is obtained by putting $y = c$ and $t = 0$, which gives

$$v = \frac{2}{\pi}\int_0^\infty dk \int_{-\infty}^\infty \varphi(\lambda) \cos \lambda k \cos kx\, d\lambda$$

$$= \varphi(x)$$

by Fourier's theorem, which is the proposed initial value of v at the plane in question.

The solution for the liquid lying between the planes $y = 0$ and $y = c$ can be obtained in a similar manner.

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