clear. It will be noted that the places at which differential operations occur are those which mark the passing from one group of cases into the next in the case of a set of equations obtained by equating our Pfaffian expressions severally to zero. Further, it will be found that the more general derived functions that I have introduced play a similar part, in reference to these latter covariants, to that which the derived functions of a single expression play in reference to its covariants. It will also be noticed that the transformation theory applies. One of the main difficulties in working out general proofs of propositions in this subject is the extraordinary complication of the notation. The chief desideratum is the invention of a notation to express the general type of derived functions as convenient as that introduced by Cayley to denote the derived functions of a single expression.

Note on a Case of Divisibility of a Function of Two Variables by another Function.

By ARTHUR BERRY, Fellow of King's College, Cambridge. Received and read February 9th, 1899.

§ 1.

If $f = 0$, $\phi = 0$ are the equations, expressed in Cartesian coordinates, of two given algebraic curves, which have both simple and multiple points of intersection, and if $\psi = 0$ is the equation of a third curve passing through all these points, and satisfying certain further conditions at the multiple points of intersection, then we have the identity

$$\psi \equiv Af + B\phi$$

where A, B are polynomials in the coordinates x, y. The above mentioned conditions were first stated, and the theorem rigorously proved, by Noether.* A simpler proof of Noether's theorem was soon afterwards published by Halphen.† Dr. F. S. Macaulay has

* "Ueber einen Satz aus der Theorie der algebraischen Functionen," *Mathematische Annalen*, Vol. VI. (1873).

† "Sur une proposition d'Algèbre," *Bulletin de la Société Mathématique de France*, Vol. V. (1877); the article is reproduced in Benoist's French translation of Clebsch's *Vorlesungen über Geometrie*.

recently called my attention to a step in Halphen's argument which appears to require justification. The object of this note is to establish and to generalize the proposition which Halphen tacitly assumes.

§ 2.

If F, f, Φ, ϕ are four polynomials in two variables x, y, connected by the identity

$$Ff = \Phi\phi, \tag{1}$$

then, provided that f and ϕ have no polynomial as a common factor, we know that F must be divisible by ϕ and Φ by f, that is, that F/ϕ and Φ/f are themselves polynomials. But, if F and Φ are no longer polynomials, but infinite series proceeding by ascending positive integral powers of x, y (*i.e.*, ordinary power series), is the corresponding result true? In the first place, the algebraic notion of divisibility requires modification, when we are no longer dealing with polynomials. According to Weierstrass's definition,* an ordinary power series $p_1(x, y)$ is divisible by another $p_2(x, y)$, both of which converge in the neighbourhood of the origin,† if their quotient p_1/p_2 is expansible as a third power series $p_3(x, y)$, also convergent in the neighbourhood of the origin; this includes the ordinary algebraic definition. Now, if p_2 does not vanish at the origin, but is of the form $a_0 + a_1x + b_1y + \ldots$ where $a_0 \neq 0$, then it is known that $1/p_2$ is expressible in the form p_3, and therefore so also is p_1/p_2, whatever p_1 be. If, therefore,

$$\phi(0, 0) \neq 0,$$

we have

$$F/\phi = p_3(x, y),$$

and therefore also

$$\Phi/f = F/\phi = p_3(x, y);$$

but in Halphen's argument, f and ϕ both vanish at the origin, $f = 0$ and $\phi = 0$ being two curves which pass through the origin, and have multiple points of any order there; and therefore this

* "Einige auf die Theorie der analytischen Functionen mehrerer Veränderlichen sich beziehende Sätze," *Werke*, Vol. II., pp. 135–158. First printed in the *Abhandlungen aus der Functionenlehre* (1886). Weierstrass's results have been used by Stickelberger and by Baker in papers on Noether's theorem, *Mathematische Annalen*, Vols. XXX. (1887), XLII. (1893).

† By "neighbourhood of the origin," I mean the aggregate of values of x, y for which $|x|$, $|y|$ are less than certain finite positive quantities δ, δ', however small. I also use "point" as a convenient abbreviation for a pair of values of x, y.

procedure fails. Halphen, however, does not appear to notice this, and infers (without any attempt at proof) that Φ/f is a convergent power series.

It may be noticed incidentally that the question of the divisibility of functions of two or more variables is much more troublesome than in the case of one variable, since in the former case, if two functions are zero of the same order at any point, their quotient is not in general a determinate finite quantity, but is wholly indeterminate. The behaviour at the origin of such a simple function as $(x+y)/(x-y)$ illustrates this.

Halphen's assumption may, however, be justified by methods given by Weierstrass in the paper already quoted.*

$F(x, y)$ being a power series as defined (assumed not divisible by x), $F(0, y)$ is an ordinary power series in y only; let the lowest power of y which occurs be y^m, where m is an integer $\geqslant 0$, so that m is the number of vanishing roots of the equation $F(0, y) = 0$, or, in geometrical language, is the number of points in which the curve $F = 0$ meets the axis of y at the origin; for shortness, let us call m the y-order of F at the origin. Then, adapting Weierstrass's general result to the case of two variables, we know that, corresponding to any positive number δ' less than a certain finite number, we can choose a finite positive number δ such that, for any value of x for which $|x| < \delta$, the equation $F(x, y) = 0$ has m roots y, and no more, for which $|y| < \delta'$. These roots are put *en évidence* by expressing F in the form

$$F(x,y) \equiv F_1(x,y)\, e^{G_1(x,y)},$$

where G_1 is an ordinary power series convergent in the region considered, and F_1 is a polynomial in y of order m, of the form

$$y^m + p_1 y^{m-1} + \ldots + p_m,$$

where the coefficients are ordinary power series in x; and, when $|x| < \delta$, F_1 vanishes for m values of y, which satisfy the condition $|y| < \delta'$. The result holds also if F is a polynomial.

We now choose δ, δ', so as to satisfy the conditions of Weierstrass's theorem for the two functions F, φ of equation (1), and so that further, within the region defined by $|x| < \delta$, $|y| < \delta'$, f, ϕ only vanish simultaneously at the origin. This would, of course, be impossible if

* See also Harkness and Morley, *Treatise on the Theory of Functions*, § 88; and for a different method of treatment, Picard, *Traité d'Analyse*, Vol. II., pp. 241–246.

f, ϕ had a common factor. It is implied that F, Φ converge throughout the region thus defined. We can also, by a linear transformation, arrange so that x is not a factor of any of the four functions.

We now have
$$\left.\begin{aligned} F &\equiv F_1(x, y)\, e^{G_1(x, y)} \\ \phi &\equiv \phi_1(x, y)\, e^{G_2(x, y)} \end{aligned}\right\}, \tag{2}$$

where F_1, ϕ_1 are polynomials in y of orders m, n (these being the y-orders of F, ϕ at the origin), the coefficients being as before power series in x, and the exponentials are neither zero nor infinity, in the region considered.

Let us further resolve ϕ_1 into factors $(y-y_1)(y-y_2)\dots(y-y_n)$, where each root y_i is a function of x, which vanishes when $x=0$, and satisfies the condition $|y_i| < \delta'$, as long as $|x| < \delta$; any number of the functions y_i may be equal to one another.

The fundamental equation (1) now becomes
$$F_1(x, y) f(x, y) = \phi_1(x, y)\, \Phi\, e^{G(x, y)}, \tag{3}$$
where G is written for $G_2 - G_1$.

Now let x have any value in the region other than zero; then there are n values of y in the region for which the right-hand side of (3) vanishes; by hypothesis none of these pairs of values of x and y make f vanish; therefore there are at least n values of y in the region for which F_1 vanishes; therefore the order of F_1 is at least equal to n; therefore the y-order of F_1 is at least equal to n, that is $F_1(0, y)$ vanishes at least n times at the origin.

In equation (3), replace y by the function y_1 (which vanishes when $x=0$); then the right-hand side vanishes for all values of x in the region, and, since f can only vanish when $x=0$, F_1 vanishes when $0 < |x| < \delta$; but we have just proved that it vanishes when $x=0$. Therefore $F_1(x, y_1)$ vanishes for all values of x in the region.

Hence, for all such values of x,
$$\frac{F_1(x, y)}{y-y_1} = \frac{F_1(x, y) - F_1(x, y_1)}{y - y_1},$$

and, F_1 being a polynomial in y, this becomes, by actual division, a polynomial
$$y^{m-1} + p_1' y^{m-2} + \dots + p_{m-1}',$$

where the coefficients are rational integral functions of y_1 and of the coefficients of F_1. Repeating the process with each factor of ϕ_1 (which need not be distinct) we find that F_1/ϕ_1 is a polynomial in y

of order $m-n\ (\geqslant 0)$, the coefficients of which are rational integral functions of $y_1, y_2, \ldots, y_n$ and of the coefficient of F_1; but they are symmetric functions of $y_1, \ldots, y_n$; therefore they are rational integral functions of the coefficients of F_1, ϕ_1, which coefficients are ordinary power series in x; therefore so are also the coefficients in our new polynomial.

We have thus proved that F_1/ϕ_1 is an ordinary power series in x, y, convergent in our region; therefore also

$$F/\phi \equiv e^{-G_1(x,y)} F_1/\phi_1$$

is such a power series.

This proves the result required and supplies the gap in Halphen's argument.

§ 3.

The result thus obtained admits of some easy generalizations.

In the first place, if f, ϕ, instead of being polynomials, are ordinary power series in x, y convergent in the region considered, the proof is unaltered provided that the condition that f, ϕ should have no common algebraical factor is suitably modified. This condition was only used to ensure that f and ϕ should have no common zero except the origin in the region considered. If therefore f and ϕ are such that every point (α, β) for which they both vanish is separated from the origin by a finite interval, *i.e.*, satisfies an inequality $|\alpha| + |\beta| > \delta''$, where δ'' is some finite positive number, the proof holds without alteration.

Moreover, by treating any point (x_0, y_0) in the interior of the region of convergence of the four series f, ϕ, F, Φ as origin, provided that we can draw round x_0, y_0 a finite region, at no point of which (with the possible exception of x_0, y_0 itself) f and ϕ vanish simultaneously, we see that F/ϕ or Φ/f is expressible as an ordinary convergent power series $p(x-x_0, y-y_0)$. We may further treat the functions as analytical functions defined by the original power series and their "continuations," and the result will still hold for the interior of any region common to the four regions of continuity thus obtained. The general form of the theorem obtained may now be enunciated as follows:—

If f, ϕ, F, Φ be four analytical functions of two variables x, y, defined each by an ordinary power series and its continuations, and if throughout any region R common to the regions of continuity of the four functions, then: (1) the identity $Ff = \Phi\phi$ subsists, and (2) every point common to $f=0$, $\varphi=0$ is separated by a finite interval from every other such point, then the quotient $F/\phi \equiv \Phi/f$ is an analytical function, the region of continuity of which comprises all points lying in the interior (as distinguished from the boundary) of the region R, and which is therefore expansible in an ordinary power series $p\,(x-x_0, y-y_0)$, in the neighbourhood of every point x_0, y_0 in the interior of R.

[Since this paper was presented to the Society some references have been added and some small alterations have been made with the view to removing some obscurities and dealing with possible cases of exception. I am indebted to the referee for suggesting these improvements].

A Note on Minimal Surfaces. *By* T. J. I'A. Bromwich, St. John's College, Cambridge. Received February 6th, 1899. Read February 9th, 1899.

We shall first investigate the condition for a minimal surface in tangential coordinates. That is, we attempt to find the condition that the envelope of the plane $lx+my+nz=p$ may be a minimal surface, where, of course, p is homogeneous of degree unity in l, m, n. The point of contact of the plane with its envelope is given by

$$x=\frac{\partial p}{\partial l},\quad y=\frac{\partial p}{\partial m},\quad z=\frac{\partial p}{\partial n},$$

and the corresponding normal is

$$\frac{\xi-x}{l}=\frac{\eta-y}{m}=\frac{\zeta-z}{n}=\frac{-\rho}{\sqrt{l^2+m^2+n^2}}=-\lambda, \text{ say},$$

where ξ, η, ζ are the current coordinates of a point on the normal at distance ρ from x, y, z. The value of ρ will give a principal radius of curvature of the envelope if ξ, η, ζ lies on the consecutive normal, or if

$$\left(\frac{\partial^2 p}{\partial l^2}-\lambda\right)dl+\frac{\partial^2 p}{\partial l\,\partial m}dm+\frac{\partial^2 p}{\partial l\,\partial n}dn-l\,d\lambda=0,$$

with two similar conditions.