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Source: *The Mathematical Gazette*, Vol. 11, No. 161 (Dec., 1922), pp. 188-193
Published by: Mathematical Association
Stable URL: http://www.jstor.org/stable/3603025
Accessed: 05-11-2015 14:18 UTC

THE COMPLETE ANGLE AND GEOMETRICAL GENERALITY.*

By D. K. Picken, M.A.

1. The very great importance to Elementary Euclidean Geometry of the figure intermediate between the Straight Line and the Triangle appears to have been missed: viz. of the plane figure formed by two intersecting straight lines, each unbounded both ways. For this type of figure we use the name "Complete Angle."

If the two lines be denoted by l_1 and l_2, we may denote Complete Angles by (l_1, l_2) and (l_2, l_1); and we shall find it important to distinguish between these two "opposite" figures. Again, denoting the unbounded straight line through the points A and B by AB—and *not* distinguishing between AB and BA—we have such notation as (AB, CD) for Complete Angles. But we shall find that quite the best notation is the unqualified AOB—equivalent to (OA, OB)—for the Complete Angle formed by two straight lines, of which A, B are points respectively and O the intersection. Thus if $A_1, A_2, \ldots$ denote different positions of A on its line, etc., A_1OB_1, A_2OB_2, etc., denote one and the same Complete Angle. This we shall see to be an important fact.

The place of the Complete Angle in the geometrical system gives it priority of right to the unqualified three-symbol "expression." The ordinary Angle-figure, in which both lines are bounded one way by their intersection O, may be called for distinction a "Simple Angle" and denoted by $\angle^{\text{le}} AOB$; the corresponding elementary (positive or negative) angle-quantity † by $\angle AOB$. The triangle may be denoted by $\triangle^{\text{le}} AOB$; its area by $\triangle AOB$. Some such scheme of notation is necessary to sound scientific presentation of the facts of Elementary Geometry.

2. It will appear that nearly all the Angle propositions of Geometry can be most simply and most generally stated in terms of congruence of Complete Angles—expressed by

$$(l_1, l_2) \equiv (l'_1, l_2'), \quad AOB \equiv A'O'B',\ddagger \text{ etc.}$$

More particularly, the Angle propositions for which the familiar ambiguity "equal or supplementary" has been characteristic take a more accurate form, which resolves that ambiguity and exhibits the fact that it has operated as a barrier to the development of important geometrical theory. In this connection we note that the congruence $AOB \equiv A'O'B'$ implies that $\angle AOB$ and $\angle A'O'B'$ (positive or negative angles) are either (1) equal or (2) *differ* by straight angle (S);

$$i.e. \ \angle AOB \sim \angle A'O'B' = 0 \text{ or } S;$$

or, again, admitting all the possible trigonometrical angles,

$$\angle AOB = \angle A'O'B' + n \,.\, S; \ n \text{ integral.}$$

It is the fact that the second alternative has always been expressed as a sum (viz., ignoring sign of the angles, $\angle AOB + \angle A'O'B' = S$; *i.e.* "supplementary" angles), rather than as a difference, that has interfered with a proper development of the geometrical theory of Angle.

[*Note.*—In the development sketched out below, no diagrams are used. They are, in fact, quite unnecessary, if the fundamental ideas are clearly understood. Readers can supply what they consider helpful. *The methods are essentially quite general*; and, therefore, *the simplest figure which exhibits*

* This paper is an abridged form of a much longer paper on "The Euclidean Geometry of Angle," for which space could not be found in the *Gazette*.

† The important distinction between Angle-figure and angle-quantity can be usefully indicated by means of the capital and the small initial letters.

‡ This implies $BOA \equiv B'O'A'$; but $AOB \not\equiv B'O'A'$. See § 1.

all the essential facts may always be used. We shall see throughout that *the maximum of mathematical simplicity is consistent with maximum generality.*]

3. (i) The Angle-relations of the congruence of two triangles ABC, $A'B'C'$ are either

$$(1)\ BAC \equiv B'A'C', \quad CBA \equiv C'B'A', \quad ACB \equiv A'C'B',$$

or

$$(2)\ BAC \equiv C'A'B', \quad CBA \equiv A'B'C', \quad ACB \equiv B'C'A'.$$

These two types (in Plane Geometry) may conveniently be distinguished by the terms (1) "congruent" and (2) "contra-congruent." (So, again, "similar" and "contra-similar.")

(ii) For the Isosceles Triangle, such that $AB = AC$, the Angle-relation is

$$ABC \equiv BCA \text{ or } CBA \equiv ACB, \quad i.e. \ (BC, BA) \equiv (CA, CB).$$

(To see the precise bearing of those congruences, the sides of the triangle must be extended beyond the vertices.)

(iii) (1) The Bisectors of the Complete Angle AOB are the *two* lines OC such that

$$AOC \equiv COB \text{ (and } BOC \equiv COA).$$

(2) And from the limiting case when A, O, B are collinear we have the fact that

$$OC \perp OA \text{ implies } AOC \equiv COA,$$

or

$$l_1 \perp l_2 \quad ,, \quad (l_1, l_2) \equiv (l_2, l_1).$$

(3) Conversely, if $AOB \equiv BOA$,

then either (1) O, A, B are collinear points

or (2) $OA \perp OB$.

(iv) The Right Angle relation of (iii) (2) may, further, clearly be extended thus:

If $$l_1 \perp l_2 \text{ and } l_3 \perp l_4,$$

then $$(l_1, l_2) \equiv (l_3, l_4).$$

4. The definition (§ 1) of the Complete Angle notation has the following important corollary:

If A, B, C, D *be* (coplanar) *points such that*

$$ABC \equiv ABD,$$

the three points B, C, D *are collinear.*

[*Note.*—The two Angle expressions differ only in the symbols C, D at one extreme; and the *points* C, D are collinear with that which belongs to the common middle symbol B.]

This is one of a group of very important Complete Angle propositions. See §§ 9, 14 below.

5. The geometrical facts underlying Trigonometrical addition of angles give the following important propositions:

(i) If $$AOB \equiv A'O'B' \text{ and } BOC \equiv B'O'C',$$

then $$AOC \equiv A'O'C'.$$

And so, again, if AOB, BOC, ... $KOL \equiv A'O'B'$, $B'O'C'$, ... $K'O'L'$,

then $$AOL \equiv A'O'L'.$$

(ii) A precisely analogous proposition, with contra-congruence in place of congruence:

If $AOB \equiv B'O'A'$ and $BOC \equiv C'O'B'$, then $AOC \equiv C'O'A'$, etc.

(iii) By a combination of (i) and (ii), if $AOB \equiv COD$, and we use $BOC \equiv BOC$ as auxiliary, so that AOB, BOC are contra-congruent to DOC, COB, then

$$AOC \text{ is contra-congruent to } DOB,$$

and therefore $$AOC \equiv BOD.$$

These theorems have a striking analogy to the fundamental Ratio-theorems for which the Euclid references are "*ex aequali*" and "*componendo.*" But see further § 7 below.

6. (i) The fundamental Angle-proposition of Parallels takes the following simple general form:

Parallel lines form, with any transversal, congruent corresponding Complete Angles; *i.e.* if $l_1 \| l_2$, and l any transversal, $(l, l_1) \equiv (l, l_2)$; and conversely.

And the following form is an important corollary:

(ii) *If* A, B, C, D *be* (coplanar) *points such that*

$$ABC \equiv DCB \quad (\text{and } BCD \equiv CBA),$$

then $AB \| CD$.

[The form $(AB, BC) \equiv (CD, BC)$ is the key.]

Note that also $ABD \equiv CDB$; $BAC \equiv DCA$; $BAD \equiv CDA$.

(iii) If coplanar lines l_1, l_2, l_3, l_4 are such that $l_1 \| l_3$ and $l_2 \| l_4$, then

$$(l_1, l_2) \equiv (l_1, l_4) \equiv (l_3, l_2) \equiv (l_3, l_4).$$

This is the Angle property of the Parallelogram—also expressible thus:

$$ABC \equiv DCB \equiv CDA \equiv BAD.$$

7. The propositions of § 5 may now be given important generalisation.

(i) If $\quad (l_1, l_2) \equiv (l_1', l_2') \quad$ and $\quad (l_2, l_3) \equiv (l_2', l_3')$,

then $\quad (l_1, l_3) \equiv (l_1', l_3')$.

And there is an obvious extension to two sets of n lines.

(ii) So for contra-congruence relations.

(iii) And from these,

$$\text{if} \quad (l_1, l_2) \equiv (l_3, l_4), \quad \text{then } (l_1, l_3) \equiv (l_2, l_4).$$

These generalised analogues of the Ratio theorems are powerful geometrical instruments.

[*Note.*—We may regard the "alternated" form of the Parallels relation $(l, l_1) \equiv (l, l_2)$, viz. $(l, l) \equiv (l_1, l_2)$, as *defining* a use of the Complete Angle expression (l_1, l_2) in the limiting case of parallelism.]

(iv) By § 3 (iv), if $l_1 \perp l_2$ and $l_3 \perp l_4$,

$$(l_1, l_3) \equiv (l_2, l_4) \quad \text{and} \quad (l_1, l_4) \equiv (l_2, l_3).$$

8. (i) The Angle property of the Circle takes the following simple general form:

The necessary and sufficient condition that four (coplanar) *points* A, B, C, D *be concyclic is*

$$ABC \equiv ADC$$

$$(\text{or} \quad BCD \equiv BAD \quad \text{or} \quad ABD \equiv ACD, \text{ etc.}).$$

It is easy to see that this is the proper expression of the "concyclic" relation. But a really satisfactory proof (which must be omitted from this summary) is not so obvious.

Cor. In particular, using § 3 (iv), if $AB \perp BC$ and $AD \perp DC$, A. B, C, D are concyclic.

(ii) If AD be the tangent at A to the circle ABC,

$$DAC\,(\equiv AAC) \equiv ABC, \text{ by § 4 and § 8 (i)};$$

and, conversely, $\quad ABC \equiv DAC$

is the condition that AD *be tangential to the circle* ABC.

(iii) An obvious important corollary is that

$$ABC \equiv CDA$$

is the condition for congruent (*but not coincident*) *circles* ABC, ACD with common chord AC.

[*Note.*—In § 8 (i) $\angle ABC$ may be equal, but may not be (trigonometrically) "supplementary," to $\angle ADC$. In § 8 (iii) $\angle ABC$ may be supplementary, but may not be equal, to $\angle ADC$.]

9. On the basis of the results of §§ 3, 4, 6 and 8, *every* congruence $PQR \equiv UVW$, in which the Angle expressions are 3-permutations of the four symbols A, B, C, D, can be given a simple geometrical interpretation.

It is, of course, sufficient to consider the congruence of ABC with every other Complete Angle of the type in question. To take only one example, different from those of the sections in question:

If $ABC \equiv CAD$, then $CEA \equiv CAD$, if the circles ABC, ACE be congruent [§ 8 (iii)]; and, by § 8 (ii), AD touches circle ACE. Hence AD touches the circle through A, C congruent (but not coincident) with circle ABC.

10. Elementary Orthocentre theory can not (so far as the writer has been able to learn) be given *general* treatment on standard methods. The following proofs are typical applications of the new principles of generality.

(i) If AL, BM, CN be the "perpendiculars" of the triangle ABC, and B, C, M, N, therefore, concyclic, etc., let O be the intersection of lines BM, CN, and K that of lines AO, BC; then A, O, M, N are concyclic; $\therefore$ by §§ 4 and 8,

$$KAM \equiv OAM \equiv ONM \equiv CNM \equiv CBM \equiv KBM;\text{*}$$

$$\therefore A, B, K, M \text{ are concyclic, and } AKB \equiv AMB;$$

$$\therefore AK \perp KB, \text{ and } K \text{ coincides with } L.$$

(ii) $BOC \equiv MON$—the same Complete Angle (§ 1)
$\equiv MAN \equiv CAB.$

Hence the circles ABC, OBC are congruent; and so, also, OCA, OAB congruent with them.

The simple familiar proposition that if the line AL intersect the circle ABC again at Q, L is the mid-pt. of OQ, is ordinarily given proof which is not general (*i.e.* not capable of statement without specific reference to a diagram). The Complete Angle proof is as follows: As in (i),

$$OBC \equiv OBL \equiv ONL \equiv CNL \equiv CAL \equiv CAQ \equiv CBQ;$$

$\therefore$ (§ 3) BC is a bisector of the Complete Angle OBQ and bisects OQ (which $\perp$ it).

11. Simson's Line theory provides the most characteristic application of these ideas (which were in fact arrived at in a discussion of the difficulty of obtaining a *general* proof of the elementary theorem of Simson's Line).

(i) If P be concyclic with A, B, C and PX, PY, PZ the perpendiculars to the lines BC, CA, AB, then, from the facts that B, C, X are collinear, etc., and A, P, Y, Z concyclic, etc., we have (§§ 4 and 8),

$$PXY \equiv PCY \equiv PCA \equiv PBA \equiv PBZ \equiv PXZ;$$

$$\therefore X, Y, Z \text{ are collinear.}$$

(ii) If the line PX intersect the circle ABC again at R,

$$ARX \equiv ARP \equiv ABP \equiv ZBP \equiv ZXP \equiv ZXR;$$

$$\therefore (\S\ 6)\ AR \parallel ZX.$$

(iii) The proof that Simson's Line bisects OP can be similarly generalised.

(iv) The extension to "isogonals" may be made in the following terms:

If X', Y', Z', on the lines BC, CA, AB respectively, be such that

$$(BC, PX') \equiv (CA, PY') \equiv (AB, PZ'),$$

then X', Y', Z' are collinear points.

For, by hypothesis, $AY'P \equiv AZ'P$; $\therefore A$, P, Y', Z' are concyclic, etc.; and the argument of (i) and (ii) is therefore again applicable.

[It will be observed that the term "isogonal" is given a proper precision, in terms of the Complete Angle.]

* These chains of absolutely general congruences—exhibited in §§ 10, 11—are characteristic of the power of the method.

(v) The generalisation for a point P *not* on the circle may be made in the following stages:

(1) If the lines AP, BP, CP intersect the circle again at U, V, W, the triangles UVW, XYZ are *similar* (in the precise sense of § 3).

(2) The scale-ratio of this similarity $= D.\Pi : PA.PB.PC$, if D be the diameter of circle ABC, and Π the "power-area" of the point P with respect to that circle; and therefore $\delta.\Pi = PA.PB.PC$, if δ be the diameter of circle XYZ.

(3) $\triangle XYZ : \triangle ABC = \Pi : D^2$;

$\therefore\ \triangle XYZ \propto \Pi$ when P varies.

[*Note.*—It is interesting to consider these results for a variety of special positions of P.]

12. An important element of generality is introduced into Ratio theory. Only the briefest reference is here possible.

(i) (1) If a transversal intersect lines OX, OY at A, B and the bisector of the Simple Angle XOY at K,

$$AK : KB = OA : OB,$$

in which *signs are ascribed to* OA, OB by reference to the standard directions OX, OY of the given lines.

(2) $$\triangle OAB = \tfrac{1}{2}\,.\sin XOY\,.\,OA\,.\,OB,$$

in which *signs of angle, lengths and area* are taken into account.

(ii) (1) The general proof of the fundamental theorem of Projective Geometry at once follows—in the form

$$(AB : BC)/(AD : DC) = (\sin XOY/\sin YOZ)/(\sin XOW/\sin WOZ),$$

$ABCD$ being a variable transversal of the given pencil O, $XYZW$.

(2) The projective properties of the Circle are immediately deducible from § 8.

[*Note.*—In the generality of these Projective Geometry theorems—generality which is, of course, absolutely essential to their use—the main ideas of this paper have undoubtedly been implicit; but the elementary implications have, for some reason, not been tracked down. Where the false ambiguity "equal or supplementary" (quite false in this context, where the trigonometric principles are essential) has been employed, the consequent error (in sign of a sine) has been automatically corrected by double occurrence!]

(iii) One of the most interesting exercises in the ideas—probably not of intrinsic importance—is provided by the generalised Ptolemy's theorem, in which from four given points twelve different triangles are derived, with sides in a specific proportion. These are proved to consist of six pairs of *congruent* triangles; and the six non-congruent triangles are proved all *similar* (in the precise sense of § 3).

(iv) The right-angled triangle, divided by a perpendicular, is an elementary instance in which the two sub-triangles are *similar*, and are *contra-similar* to the original triangle.

13. The Complete Angle is obviously a valuable instrument for importing generality into the Angle properties of the Conic.

(i) The properties culminating in the theorem that

If the line P_1P_2 intersect directrix in D, and the tangents at P_1, P_2 intersect in T, SD and ST are the bisectors of Complete Angle P_1SP_2

can be given simple general proof.

(ii) Following a standard notation, the triangles SPG, PMS are *contra-similar*. And from this more precise form of a well-known property we get general proof of the theorem that tangent and normal at P are the bisectors of the Complete Angle SPS'.

(iii) (1) The generality of the proposition on the inclinations of two tangents to the focal radii of their intersection, T, has always been a crux. By the method of Complete Angle congruences we give general proof to it, in the precise form

$$(t_1, ST) \equiv (S'T, t_2) \quad \text{or} \quad P_1TS \equiv S'TP_2$$

—of which the "alternated" form exhibits the fact that either tangent may, of course, be associated with either focus.

(2) And the treatment of the corresponding facts for the Parabola is likewise general.

14. To sum up—the "Complete Angle" brings a new method to bear upon the Euclidean Geometry of Angle. The power of the method is epitomised in the congruence relations

$$(1)\ ABC \equiv ABD, \quad (2)\ ABC \equiv DCB, \quad (3)\ ABC \equiv ADC,$$

giving general expression to the geometrical relations

$$(1)\ B, C, D \text{ collinear}, \quad (2)\ AB \parallel CD, \quad (3)\ A, B, C, D \text{ concyclic};$$

and in the "*ex aequali*" and "*componendo*" relations of § 7 (the importance of which is concealed by the condensation of this paper).

The applications are practically coextensive with the relevance of the Angle concept in Geometry; and Complete Angle congruences almost entirely replace quantitative relations. D. K. Picken.

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144. "A nonius, or, as our author rather affectedly calls it, a vernier."—*Edin. Review*, 1806, ii. p. 97. (Review of Wollaston's *Phil. Trans.* paper, 1802.) [The "nonius," better known in later years as a "verñier," was invented by Pedro Nuñez (Nunes, Nunnius, Nonius), who was Professor of Mathematics at the University of Coimbra, and Cosmographo Major to King Emmanuel of Portugal. Nuñez described in 1542 an instrument of his invention for the division of arcs, but the instrument claimed as his own by Vernier in the volume given below and in his *La Construction, l'usage et les propriétés du quadrant de mathêmatique*, etc. (1634), is a very great improvement on the "nonius."]

The following is from the *N.E.D.*: Paul Vernier, French mathematician, 1580-1637, describes it in a tract on *Quadrant Nouveau de Mathêmatique*, 1631, erroneously called a Nonius. 1766. Instructions for Halley's Quadrant, 17. A scale of divisions graduated on the chamfered edge or sloped side of the index, which scale is called the Vernier. 1774. M. Mackenzie, Maritime Survey, 28. 1798. Phil. Trans. lxxxviii. 473. Pedro Nuñez, 1492-1577. De Crepusculis, 1542 A.D., contrivance for gradation of mathematical instruments. Vernier an improved form of this. 1750. Phil. Trans. xlvi.

145. Nicole, one of the authors of the famous *Logic of Port-Royal*, relates the following: "One day I told Madame de Longueville that I could prove that there are at least two people living in Paris with the same number of hairs on their heads. She asserted that I could never prove this without counting them first.

"My premisses are these. No head has more than 200,000 hairs, and the worst provided has one. Consider 200,000 heads, none having the same number of hairs. Then each must have a number of hairs equal to some number from 1 to 200,000 both included. Of course if any have the same number of hairs my bet is won. Now take one more person, who has not more than 200,000 hairs on his head. His number must be one of the numbers 1 to 200,000 included. As the inhabitants of Paris are nearer 800,000 than 200,000, there are many heads with an equal number of hairs."