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## SIGN IN ELEMENTARY ANALYTICAL GEOMETRY.

BY F. G. BROWN, B.A., B.Sc.

THE question of sign constitutes a real difficulty to the intelligent boy at the outset of his study of Coordinate Geometry. At the beginning of his Trigonometry he is told  $OP$  must be considered always positive, but later on he will find some authorities giving a point in the third quadrant as  $(-r, \theta)$ , while others prefer  $(r, \theta + \pi)$ . The perpendicular distance of  $(h, k)$  from  $ax + by + c = 0$  is given by  $\pm (ah + bk + c)/(a^2 + b^2)^{\frac{1}{2}}$ , and sign seems to matter, but usually the pupil is told that he only wants to know how far off  $(h, k)$  is, and he is advised to stick to the absolute value. But a little later on he wants the equations of the bisectors of the angles between two given lines, and then he is blamed for not remembering that signs matter a good deal.

It seems advisable to have at the outset a clear distinction between terminated and non-terminated lines, the former having either one or both end-points. Thus  $OX$ , the initial line of coordinates, is terminated at  $O$ , and we agree that a movement to the right along  $OX$  is positive. Then, any line  $OP$ , terminated at both ends, and rotating about  $O$  counter-clockwise, keeps the positive sign it started with. Hence, if we draw the full rectangular axes of ordinary Cartesian geometry,  $OP$  is still positive when it lies along  $OX$ , or  $OY'$ . Then, obeying the usual conventions as to the signs of coordinates, we have, putting  $OP = r = |x + iy|$ ,

Quadrant.	1st	2nd	3rd	4th
Signs of $x, y$	$+$ $+$	$-$ $+$	$-$ $-$	$+$ $-$
By definition, $\sin \theta = y/r$ $\cos \theta = x/r$	$+$ $+$	$-$ $+$	$-$ $-$	$+$ $-$
$\tan \theta = y/x$	$+$	$-$	$+$	$-$

I do not use the cumbersome forms  $OM/OP$ , etc., nor do I use the phrases "side opposite," "side adjacent."

The results tabulated above are obtained solely by means of *terminated* lines. When we enter upon a discussion of the equation of a straight line we find that  $ax + by + c = 0$  represents a non-terminated line.

Two cases arise :

(1) Lines parallel to the  $Y$ -axis :  $x = k$ .

Since the upwards direction is positive along  $Y'OY$ , we have no hesitation in applying the same rule to  $x = k$ .

(2) Lines not-parallel to the  $Y$ -axis :  $y = mx + b$ .

This is where the first difficulty is encountered, and I find the best way of meeting it is to extend the sign-conventions to oblique axes, the line in question being the new axis of  $Y$ . Hence "upwards" is still positive. Another way of putting it is to say that the line is to be considered as having reached its position by a positive rotation about its point of intersection with the  $X$ -axis starting from the position  $X'OX$ . Thus the angle of inclination with the  $X$ -axis lies between  $O$  and  $\pi$ ; strictly

$$0 \leq \theta_1 < \frac{\pi}{2}; \quad \frac{\pi}{2} < \theta_2 < \pi;$$

$\theta_1$  being acute,  $\theta_2$  obtuse. Hence the sine of the angle of inclination is always positive, but the tangent is positive for  $\theta_1$ , negative for  $\theta_2$ . Having put  $x = k$  into a separate class, we need no discussion as to what happens when

$\theta = \frac{\pi}{2}$ . (It should be noted that when we say "sin  $\theta$ ," we imply that we have made the "xyr" construction, using a terminated length of the non-terminated line as our  $r$ .)

The information "the gradient is to be  $m$ " allows me to cover the entire plane with parallel strokes.

The information "the line is to go through  $x_1, y_1$ " allows me to cover the entire plane with strokes through  $x_1, y_1$ . (Note that if I draw lines *terminated* at  $x_1, y_1$  I can say I draw them in *all* directions, but if I draw non-terminated lines through  $x_1, y_1$ —and this is the class we wish to consider—I must restrict the angle to  $0 \leq \theta < \pi$ ; otherwise we should cover the plane twice.)

Combining the two demands I cannot escape drawing just the one line required, and the opportunity is taken of impressing the law that the equation of the straight line contains *two* constants. If the pupil is smart enough to be told how this applies to  $x=k$ , then explain; otherwise you may wriggle out of the difficulty by reminding him that we have divided lines logically into two classes, and we are now dealing with lines not parallel to the  $Y$ -axis.

To "fix" a line, therefore, you must give me both (1) a fixed gradient  $m$  and (2) a fixed point  $(x_1, y_1)$ . Any other point on the line may be called  $(x, y)$ , and it is impossible to deduce the relation which must exist between  $(x, y)$  and  $(x_1, y_1)$  and  $m$  except by equating two different ways of stating the gradient. Once this fact is realised we shall refuse to wade through the misplaced ingenuities of the "proofs," of the ordinary text-books, for the "different forms"; all such proofs are camouflages. (Much of the difficulty felt by the beginner in this subject is due to the psychological effect of a writer's low opinion of a reader's intelligence.)

We have

$$\text{gradient} = \tan \theta = m.$$

Also

$$\text{gradient} = (y - y_1)/(x - x_1);$$

$$\therefore \frac{y - y_1}{x - x_1} = m. \dots\dots\dots(1)$$

I take this to be standard, although later on, partly with an eye to Solid Geometry, I write

$$\frac{y - y_1}{x - x_1} = \frac{\sin \theta}{\cos \theta}, \text{ and then } \frac{x - x_1}{\cos \theta} = \frac{y - y_1}{\cos \left( \frac{\pi}{2} - \theta \right)} = r.$$

The next four "forms" are merely variations of (1) due to the special position of  $x_1, y_1$ .

$$\text{Thus, } x_1, y_1 \text{ being the origin, } \frac{y}{x} = m, \dots\dots\dots(2)$$

$$x_1, y_1 \text{ being } 0, b, \frac{y - b}{x} = m, \dots\dots\dots(3)$$

$$x_1, y_1 \text{ being } a, 0, \frac{y}{x - a} = m, \dots\dots\dots(4)$$

$x_1, y_1$  being  $p \cos a, p \sin a$ ,

$$\frac{y - p \sin a}{x - p \cos a} = m = \tan \theta = \tan \left( \frac{\pi}{2} + a \right) = -\cot a. \dots\dots\dots(5)$$

Simplify to taste. My own preference is for delay of elegance of form until the idea is firmly embedded, just as in elementary dynamics a boy should be forced to write  $s = (v^2 - u^2)/2a$  in order that he may be constantly reminded of the factors *average velocity*  $(v + u)/2$  and *time*  $(v - u)/a$ . Elegant formulae appeal to the memory, but mathematical progress is quicker in the end if all elementary formulae are framed to remind the pupil, so far as may be, of the principles which were used in establishing them.

The five forms given above contain explicitly the *gradient* of the line. If the fixing of the line is made to depend on *two given points*, it is quite easy to see that  $m$  is obtainable from these two given points, and then either point is available for use in the standard form (1). Thus

$$\frac{y - y_1}{x - x_1} = \frac{y - y_2}{x - x_2} = \frac{y_2 - y_1}{x_2 - x_1} \quad (\text{i.e. } m), \dots\dots\dots(6)$$

any two of which will, in fact, give the required relation, *i.e.* the equation of the straight line. In passing it may be remarked that if we are so blind as to omit to give the above in the Algebra lesson as a practical example of the beloved " $k$ " method, then at least we should make our pupils check the above statement by straight-out multiplication. I have had many students deny offhand that the first two equated will give the correct result.

The Intercept-Form is merely a special case of (6), but I prefer to say, "Consider  $x/a + y/b = 1$ ; when  $x$  is 0,  $y$  is  $b$ ; when  $y$  is 0,  $x$  is  $a$ ; therefore, etc."

Of course we must include all the forms under the general  $ax + by + c = 0$ , and we must on no account omit to discuss the availability of the equation for every point on the line, and the existence upon the line of every point whose coordinates satisfy the equation.

Now, as regards direction, the advantage of the standard form (1), and of deriving all others from it, lies in the fact that on the left-hand side we have a *ratio*, and it is immaterial whether we write

$$\frac{y - b}{x - 0} = m \quad \text{or} \quad \frac{b - y}{0 - x} = m.$$

To the beginner this is no small matter, and he should be practised in deriving the standard equation with the fixed point in all quadrants and for all slopes (except  $\theta = \pi/2$ ).

We already know that the  $Y$ -axis divides the whole plane, so that measurements *from the axis* and parallel to the  $X$ -axis are positive if to the right, negative if to the left. Drawing a line obliquely across the  $X$ -axis we follow exactly the same rule. Hence the distance, parallel to the  $X$ -axis, from  $Q(x_2, y_1)$  on the line  $ax + by + c = 0$  to  $P(x_1, y_1)$  is  $x_1 - x_2$  and is always positive (negative) when  $P$  is to the right (left) of the given line.

A limiting case arises when the given line is the  $X$ -axis itself, or is parallel to it, but in this case there can be no right nor left-hand side to the line. We can merely say that for any two points on such a line a measurement to the right (left) is pos. (neg.). Just as in studying the equation of the straight line we considered it expedient to put all lines whose slope is infinite into a separate class, so now we have a separate class whose slope is zero, and any member of this class divides the plane into *upper* and *lower*, instead of left and right.

Linc.	In the equation, substitute	Resulting sign of L.H.S. of equation.	Therefore,
$y = 0$	$x_1 = \text{anything.}$ $y_1 = \text{pos. number.}$	+	Upper side +
$y = 0$	$x_1 = \text{anything.}$ $y_1 = \text{neg. number.}$	-	Lower side -
$y - 2 = 0$	$x_1 = \text{anything.}$ $y_1 = 4.$	+	Upper side +
$y + 3 = 0$	$x_1 = \text{anything.}$ $y_1 = -4.$	-	Lower side -

Having thus disposed of all lines whose slope is zero, we deal with the general equation  $ax + by + c = 0$ , excluding only the class thus disposed of.

Let  $x_2, y_1$  be a point on the line.

Then  $ax_2 + by_1 + c = 0$ .

Let  $x_1, y_1$  be a given point not on the line.

$$\begin{aligned} \text{Then } ax_1 + by_1 + c &= ax_1 + by_1 + c - 0 \\ &= ax_1 + by_1 + c - (ax_2 + by_1 + c) \\ &= a(x_1 - x_2). \end{aligned}$$

Now, if  $x_1, y_1$  is to the right of the given line,  $x_1 - x_2$  is positive, and if we take care to have  $a$  positive when writing  $ax + by + c = 0$ , we shall then ensure that the substitution of  $x_1, y_1$  in the left side of this equation will lead to a positive result, whereas  $x_1 - x_2$  is negative if  $x_1, y_1$  is to the left, and with the same precaution as to the sign of  $a$  we shall get the desired negative result. (Another device is to divide through by  $a$  so as to leave the coefficient of  $x$  positive unity, as is usually done in the first step of the solution of the general quadratic.) In this way there can be no ambiguity whatever as to which side of a given line is positive.

*The perpendicular from  $ax + by + c = 0$  to the point  $x_1, y_1$ .*

Take, as before,  $x_2, y_1$  a point on the line.

$$\text{Then } x_2 = -\frac{by_1 + c}{a}$$

and

$$x_1 - x_2 = \frac{ax_1 + by_1 + c}{a}.$$

Now  $p/(x_1 - x_2) = \sin \theta$ , where  $\theta$  is the angle of inclination of the line with the  $X$ -axis, and, in our system,  $\theta$  is always positive and less than  $\pi$ , hence  $\sin \theta$  is always positive.

But  $\tan \theta = -a/b$ ; hence  $\sin \theta = \text{the absolute value of } a/\sqrt{(a^2 + b^2)}$ .

$$\text{Hence } p = (x_1 - x_2) \sin \theta = \frac{ax_1 + by_1 + c}{a} \cdot \frac{a}{\sqrt{(a^2 + b^2)}}.$$

When  $(ax_1 + by_1 + c)/a$  is  $+$ ,  $p$  is  $+$ , because  $\sin \theta (= a/\sqrt{(a^2 + b^2)})$  is  $+$ .

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In brief, the sign of  $p$  (the perpendicular from the line to the point) is entirely dependent on the sign of  $(ax_1 + by_1 + c)/a$ , i.e. of  $x_1 - x_2$ .

$$\text{Thus, } p \text{ from } 2x + 3y + 6 = 0 \text{ to } 1, 1 = \frac{2 + 3 + 6}{2} \cdot \frac{2}{\sqrt{13}} \quad (\text{pos.}),$$

$$,, \text{ from } -2x + 3y + 6 = 0 \text{ to } 1, 1 = -\frac{2 + 3 + 6}{2} \cdot \frac{2}{\sqrt{13}} \quad (\text{neg.})$$

$$(\text{if preferred, write } -\frac{2}{\sqrt{13}} \text{ in this last case),}$$

$$,, \text{ from } 2x - 3y + 6 = 0 \text{ to } 1, 1 = \frac{2 - 3 + 6}{2} \cdot \frac{2}{\sqrt{13}} \quad (\text{pos.}),$$

$$,, \text{ from } 2x + 3y - 6 = 0 \text{ to } 1, 1 = \frac{2 + 3 - 6}{2} \cdot \frac{2}{\sqrt{13}} \quad (\text{neg.}),$$

$$,, \text{ from } -2x - 3y - 6 = 0 \text{ to } 1, 1 = -\frac{2 - 3 - 6}{2} \cdot \frac{2}{\sqrt{13}} \quad (\text{pos.}).$$

On plotting the lines it will be found that  $p$  in every case conforms to the right-or-left-hand-side-of-the-line convention.

Also in every case it will be found that  $p$  from the line to the origin  $(0, 0)$  will have the same sign as any other  $p$  from the line to a point on the same side of the line as the origin.

This last result may cause some difficulty, since  $p$  is a terminated line, and we appear to have agreed that a vector rotating about the origin shall remain positive in all positions. But there is no real contradiction. Our rule was laid down to enable us to define the trigonometrical ratios, and, in so far as these ratios have to do with the determination of  $p$ , we are concerned only with the ratios of  $\theta$ —the slope of the line  $ax + by + c = 0$  and not the slope of  $p$ . The difficulty occurs again when we use Hesse's Normal Form (where "Normal" is so often misinterpreted as "perpendicular" instead of "standard" or "regular")

$$x \cos \alpha + y \sin \alpha = p.$$

Here we are concerned with the angle ( $\alpha$ ) which  $p$  traverses, starting from its initial position along  $OX$ , and hence we must consider  $p$  essentially positive. Furthermore the angle  $\alpha$  is not confined within the limits we imposed for  $\theta$ , the angle of slope of a non-terminated line, but is able to take all values. The form should be introduced by showing that we can "fix" a line if we agree it shall be at right angles to, and at the outer end of, a given spoke.

The only excuse for the early introduction of Hesse's Form into most standard texts seems to be its utilisation for the finding of the perpendicular distance from (to) a line to (from) a given point. My own preference is for delay, until higher powers of analysis enable the pupil to use the form in connection with the circle. It seems bad pedagogy to use Hesse for the fundamental result, then to formulate a memory-rule whereby this result can be applied to  $ax + by + c = 0$ , and then to use this latter rule for the determination of the equations of the bisectors of "the angles between two lines."

*The Equation of the Bisector of the Angle between two given lines.*

Let

$$A = ax + by + c = 0$$

and

$$B = dx + ey + f = 0$$

be the given lines, and suppose that the gradient of  $B$  is greater than that of  $A$ . Then the angle between the lines is the positive angle through which  $B$  has turned from coincidence with  $A$ , and it will always be found that if  $x_1, y_1$  be a point on the bisector of this angle, then the perpendiculars from  $A$  and  $B$  respectively to the point  $x_1, y_1$  will have opposite signs. These perpendiculars are equal in absolute value. Hence we have

$$\frac{ax_1 + by_1 + c}{a} \cdot \frac{a}{\sqrt{(a^2 + b^2)}} = - \frac{dx_1 + ey_1 + f}{d} \cdot \frac{d}{\sqrt{(d^2 + e^2)}},$$

and since this is true for any point  $x_1, y_1$  on the bisector we have the equation of the bisector by dropping the subscripts.

From the Trigonometrical point of view the angle between two non-terminated lines is multivalued; from the Euclidean point of view there are four angles; for the purposes of Analytical Geometry there is only one angle.

One of the lines has rotated  $\theta_1$ , the other  $\theta_2$ , and the angle between them is  $\theta_1 - \theta_2$ . I always insist on an inspection of the given equations to find which is the greater angle; then this angle (say  $\theta_1$ ) is the exterior angle of a triangle in which one of the interior angles is  $\theta_2$ , or is vertically opposite to  $\theta_2$ , and the other is the angle between the lines.

The bisector of the supplementary angle between the lines is of course given by equating the same expressions as above but with like signs, and it is a useful problem to discover why this must always be.

Ex.

$$x + 3y - 6 = 0,$$

$$3x + y + 2 = 0.$$

The bisector is

$$\frac{x + 3y - 6}{1} \cdot \frac{1}{\sqrt{10}} = - \frac{3x + y + 2}{3} \cdot \frac{3}{\sqrt{10}}$$

or

$$x + y - 1 = 0. \dots\dots\dots(1)$$

The bisector of the supplementary angle is

$$x - y + 4 = 0, \dots\dots\dots(2)$$

and these are "the bisectors of the angles between, etc." Examiners often ask a candidate to "distinguish" between the bisectors, and there are wordy rules about the bisector of "the angle containing the origin." The above method contains a truer distinction.

It is not claimed that the conventions indicated in this brief paper are entirely satisfactory; Bôcher in his *Plane Analytic Geometry* indicates that neither his own convention "nor any other one which could be made" can be satisfactory. But I prefer to take  $\sqrt{a^2 + b^2}$  as being always unambiguous and to make the sign of a perpendicular from a line to a point depend solely upon the value of  $x_1 - x_2$  (i.e. abscissa of point minus the abscissa of the point on the line where  $y = y_1$  cuts it). The use of  $\sin \theta$  makes this possible. It will be noted too that the perpendicular from  $y = 2$  to  $(3, 0)$  turns out to be negative, as it should, and the perpendicular from  $y = 0$  to  $(1, 1)$  is positive, but the application of my rules to these cases introduces the form 0/0, and it is best to follow the ordinary rule of signs, familiar from graph-work, with the comforting assurance that these special cases do not form exceptions, nor do we get a sudden change of value on proceeding to the limit.

Immediately you try to frame satisfactory rules for the perpendicular from a point to a line you run up against the fundamental convention of Cartesians, which requires  $(x, y)$  to be arrived at by travelling from the axes and not towards them.

Sydney.

F. G. BROWN.

**88. The American Cocker**—"I have heard devils can be raised with Daboll's arithmetic. . . . That's my small experience as far as the Massachusetts calendar, and Bowditch's navigator, and Daboll's arithmetic go."—p. 515, Herman Melville's *Moby Dick* (World's Classics).

**89. Tautochronism**—"It was in the left-hand try-pot of the Pequod, with the soapstone diligently circling round me, that I was first indirectly struck by the remarkable fact, that in geometry all bodies gliding along the cycloid, my soapstone for example, will descend from any point in precisely the same time."—Herman Melville, *Moby Dick*, p. 502.

**90.** Sir Isaac Newton, it is said, spoke with much contempt (but surely without just grounds) of those two accomplished scholars and critics (Bentley and Hare) for squabbling (as he expressed it), about an old play-book (Terence). —[Whiston mentions this in his *Memoirs of Dr. Clarke*: Warton, Note to *Martinus Scriblerus*, c. ix.]

**91.** It was said of the first mathematicians that they opened a field in which their successors go on advancing and behold the horizon receding at every step. He who enters on this fair field must be ever pressing forward and consider nothing as done while anything remains undone.—Graves' *Hamilton*, i. 112.

**92. Captain Cook**.—Christie's sale last week included an important MS. of Captain James Cook, "Arithmetical Trigonometry" and "Arithmetical Dialling," 1763, the property of the late Mr. H. H. Emson, for which Messrs. Quaritch paid £200. The MS. extends to 97 pages folio, and contains elaborately and beautifully drawn diagrams, which bear out the statement in the *Dictionary of National Biography* that Captain Cook might "under other circumstances have proved himself as eminent as a surveyor as he actually did as an explorer." He was also something of a versifier, for he has proclaimed his ownership of this MS. in two verses on the first page. Some MSS. of Captain Cook were sold at Puttick and Simpson's on March 10, 1868, and this may have been among them.—*Times Lit. Supp.*, Aug. 4, 1921.