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On the Satellite of a Line relatively to a Cubic.

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Introduction.

1. In a "Memoir," of some length, "On the Diameters of a Plane Cubic" (*Phil. Trans.*, 1888 A., pp. 166–170), the equation of the Satellite of a line was incidentally arrived at in a form (48)(38)(37) [(25) of the present paper], involving the square of the primitive line as an implicit factor. It did not come within the scope of that memoir to discuss the reduction of this form with the generality which its importance authorised, nor were the subsidiary results, on which that general reduction is based, at that time published. I therefore limited myself to verifying the reduction given by reference to special forms of the cubic.

The results referred to having since appeared, as quoted hereinafter, I show in the present paper how the general reduction may be effected; the method of which appears to have some novelty and interest *per se;* and to save the necessity of reference to the "Memoir," as well as to obtain the equation in a slightly simplified form at once —viz., one in which only the first power of the line is implicitly involved—I revert to what is, in fact, the original investigation by which I arrived at it. Four forms of the Satellite Line are deduced (24), (25), (26), (27), in the last two of which the product of the "Quippian" of u and the line L enters.

2. Let

$$u \equiv a_1x^3+b_2y^3+c_3z^3+3a_2x^2y+3a_3x^2z+3b_1y^2x+3b_3y^2z+3c_1z^2x$$
$$+3c_2z^2y+6exyz=0 \text{ ....(1)},$$

be the equation of the plane cubic, and

$$3u_1 = \partial u/\partial x, \quad 3u_2 = \partial u/\partial y, \quad 3u_3 = \partial u/\partial z \text{ ....(2)},$$

$$\left.\begin{array}{l} 6a = \partial^2 u/\partial x^2, \quad 6b = \partial^2 u/\partial y^2, \quad 6c = \partial^2 u/\partial z^2 \\ 6f = \partial^2 u/\partial y\,\partial z, \quad 6g = \partial^2 u/\partial z\,\partial x, \quad 6h = \partial^2 u/\partial x\,\partial y \end{array}\right\} \text{ ....(3)}.$$

Further, let $L \equiv lx+my+nz$ ....(4)

be any linear form; then

$$s \equiv (bc-f^2)\,l^2+\ldots+2\,(gh-af)\,mn+\ldots \text{ .... (5)}$$

will be the envelope of polar lines of points on $L = 0$, or, briefly, the "Poloid" of L. If s be written in the normal form

$$\mathrm{a}x^2+\mathrm{b}y^2+\mathrm{c}z^2+2\mathrm{f}yz+2\mathrm{g}zx+2\mathrm{h}xy \text{ ....(6)},$$

then, as I have pointed out in the "Memoir,"

$$\mathrm{a}=0, \quad \mathrm{b}=0, \quad \mathrm{c}=0$$

are the reciprocals of u_1, u_2, u_3, and

$$2\mathrm{f}=0, \quad 2\mathrm{g}=0, \quad 2\mathrm{h}=0$$

the contravariant conics of u_2u_3, u_3u_1, u_1u_2, respectively; these coefficients being connected by the relations

$$\left.\begin{array}{l} \dfrac{\partial \mathrm{a}}{\partial l}+\dfrac{\partial \mathrm{h}}{\partial m}+\dfrac{\partial \mathrm{g}}{\partial n} \equiv 0, \\ \dfrac{\partial \mathrm{h}}{\partial l}+\dfrac{\partial \mathrm{b}}{\partial m}+\dfrac{\partial \mathrm{f}}{\partial n} \equiv 0, \\ \dfrac{\partial \mathrm{g}}{\partial l}+\dfrac{\partial \mathrm{f}}{\partial m}+\dfrac{\partial \mathrm{c}}{\partial n} \equiv 0, \end{array}\right\} \text{ ....(7)},$$

and by those among their second differential coefficients which these involve, as

$$\frac{\partial^2 \mathrm{a}}{\partial l^2}+\frac{\partial^2 \mathrm{h}}{\partial l\,\partial m}+\frac{\partial^2 \mathrm{g}}{\partial n\,\partial l} \equiv 0 \text{ ....(8)},$$

or, as they (7, 8) may be more briefly written,

$$\left.\begin{aligned} 2a_1+2h_2+2g_3 &\equiv 0, \\ 2h_1+2b_2+2f_3 &\equiv 0, \\ 2g_1+2f_2+2c_3 &\equiv 0, \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots(9),$$

$$1.2.a_{11}+1.2.h_{12}+1.2.g_{31} = 0 \ldots\ldots\ldots\ldots(10).$$

The condition that s should be touched by the line

$$\xi x+\eta y+\zeta z = 0,$$

is written

$$\sigma \equiv A\xi^2+B\eta^2+C\zeta^2+2F\eta\zeta+2G\zeta\xi+2H\xi\eta = 0 \quad \ldots\ldots(11),$$

viz., $\qquad A = \mathrm{bc}-\mathrm{f}^2 \ldots,\quad F = \mathrm{gh}-\mathrm{af} \ldots \quad \ldots\ldots\ldots\ldots(12),$

and the condition that $\quad lx+my+nz = 0$

should touch $\qquad u = 0,$

$$5v \equiv A'l^2+\ldots+2F'mn+\ldots = 0 \ldots\ldots\ldots\ldots\ldots(13),$$

where $$6A' = \frac{\partial^2 v}{\partial l^2}\ldots \quad 6F' = \frac{\partial^2 v}{\partial m\,\partial n}\ldots \quad \ldots\ldots\ldots\ldots(14).$$

3. It is known that $v+4\sigma$ (in which ξ, η, ζ are replaced by l, m, n),

or, $$(A'+20A)\,l^2+\ldots+2\,(F'+20F)\,mn+\ldots \equiv 0 \ldots\ldots\ldots(15);$$

but $A' \ldots F' \ldots$ are expressed in terms of $A \ldots F \ldots l, m, n$ through the equations

$$A' = -4\left(A+l\frac{\partial A}{\partial l}+m\frac{\partial H}{\partial l}+n\frac{\partial G}{\partial l}\right) \ldots\ldots\ldots\ldots\ldots(16),$$

$$\left.\begin{aligned} F' &= -4\left(F+l\frac{\partial H}{\partial n}+m\frac{\partial B}{\partial n}+n\frac{\partial F}{\partial n}\right) \\ &= -4\left(F+l\frac{\partial G}{\partial m}+m\frac{\partial F}{\partial m}+n\frac{\partial C}{\partial m}\right) \end{aligned}\right\} \ldots\ldots\ldots\ldots(17),$$

with analogous formulæ for B', C', G', H', as I have shown [*Lond. Math. Soc. Proc.*, Vol. XIX., p. 502 (95), (97)]; or in terms of l, m, n and the coefficients of s,

$$\left.\begin{aligned} A' = -4\{3(\mathrm{bc}-\mathrm{f}^2)+(\mathrm{bc}_{11}+\mathrm{cb}_{11}-2\mathrm{ff}_{11})\,l^2+\ldots \\ +2\,(\mathrm{gh}_{11}+\mathrm{hg}_{11}-\mathrm{af}_{11}-\mathrm{fa}_{11})\,mn+\ldots\} \\ F' = -4\{3(\mathrm{gh}-\mathrm{af})+(\mathrm{bc}_{23}+\mathrm{cb}_{23}-2\mathrm{ff}_{23})\,l^2+\ldots \\ +2\,(\mathrm{gh}_{23}+\mathrm{hg}_{23}-\mathrm{af}_{23}-\mathrm{fa}_{23})\,mn+\ldots\} \end{aligned}\right\}\ldots(18),$$

where, as in (10),

$$\mathrm{c}_{11} = \partial^2\mathrm{c}/2\partial l^2 \ldots \mathrm{c}_{23} = \partial^2\mathrm{c}/2\partial m\partial n \ldots .$$

Equation of the Satellite Line.

4. If, now, $x'y'z'$ is a point in which

$$L \equiv lx+my+nz = 0$$

meets the cubic $u = 0$, the tangent at this point is (3)

$$ax'^2+by'^2+cz'^2+2fy'z'+2gz'x'+2hx'y' = 0\,;$$

and, since this also touches the Poloid of L (6, 11) if (xyz) is any other point on the tangent to u at $x'y'z'$,

$$A\,(z'y-y'z)^2+\ldots+2F\,(x'z-z'x)(y'x-x'y)+\ldots = 0.$$

Eliminating $x'y'z'$ among the above equations, and

$$lx'+my'+nz' = 0,$$

$$\{(bc'+cb'-2ff')\,l^2+\ldots\}^2-4\,\{(bc-f^2)\,l^2+\ldots\}\,\{(b'c'-f'^2)\,l^2+\ldots\} = 0 \qquad \ldots\ldots\ldots\ldots(19),$$

if $\quad b' = Cz^2+Ax^2-2Gzx \ldots f' = -Ayz-Fx^2+Gxy+Hzx \ldots\ldots(20).$

The result of these substitutions in

$$4\,\{(b'c'-f'^2)\,l^2+\ldots+2\,(g'h'-a'f')\,mn+\ldots\}$$

is $\quad 4L^2\,\{(BC-F^2)\,x^2+\ldots+2\,(GH-AF)\,yz+\ldots\},$

or $\quad 4L^2\,(\mathrm{abc}+2\mathrm{fgh}-\mathrm{af}^2-\mathrm{bg}^2-\mathrm{ch}^2)(\mathrm{a}x^2+\ldots+2\mathrm{f}yz+\ldots)\ldots\ldots(21).$

It was shown by Cayley (*Phil. Trans.*, 1857, p. 432), by means of the canonical form of u, and is proved generally in the "Memoir" (*Phil. Trans.*, 1888, A., p. 160), that

$$4\,(\mathrm{abc}+2\mathrm{fgh}-\mathrm{af}^2-\mathrm{bg}^2-\mathrm{ch}^2) = P^2,$$

P being the "Pippian" of u.

Thus (5), (6), (21), the product

$$4\{(bc-f^2)\,l^2+\dots\}\{(b'c'-f'^2)\,l^2+\dots\} = P^2L^2s \dots\dots\dots (22).$$

But, by the same substitutions (20),

$$(b'c+bc'-2ff')\,l^2+\dots+2\,(fg'+f'g-ch'-c'h)\,lm$$

becomes (2), (3), identically

$$(Al^2+\dots+2Fmn+\dots)\,u$$

$$+L^2(Aa+Bb+Cc+2Ff+2Gg+Hh)$$

$$-2L\{(Al+Hm+Gn)\,u_1+(Hl+Bm+Fn)\,u_2+(Gl+Fm+Cn)\,u_3\}^*;$$

so that (22), the eliminant (19), is the product of the factors (ϖ, v),

$$(Al^2+\dots)\,u+L^2(Aa+\dots)-L\left[2\{(Al+\dots)\,u_1+\dots\}\pm Ps\right]\dots(23).$$

5. It will be shown below (p. 254) that the terms

$$2\{(Al+\dots)\,u_1+(Hl+\dots)\,u_2+(Gl+\dots)\,u_3\}+Ps$$

$$\equiv\left\{2\,(Aa+\dots+2Ff+\dots)+\tfrac{1}{2}\Sigma\frac{\partial P}{\partial l}\frac{\partial s}{\partial x}\right\}L \dots\dots\dots\dots (\text{i.});$$

so that the first of the two factors (23) becomes, identically,

$$\varpi=(Al^2+\dots)\,u-\frac{L^2}{2}\left\{2\,(Aa+\dots+2Ff+\dots)+\frac{\partial P}{\partial l}\frac{\partial s}{\partial x}+\dots\right\};$$

viz., this is the equation of the three tangents at the points in which L meets u, the other factor (v) being (as will also be shown, p. 261) a proper (nodal) cubic concomitant of u and L.

Thus it appears that one form of the satellite of the line L is

$$2\,(Aa+Bb+Cc+2Ff+2Gg+2Hh)+\Sigma\frac{\partial P}{\partial l}\frac{\partial s}{\partial x}=0 \dots\dots\dots (24).$$

Other Forms of the Equation of the Satellite.

6. It will further be shown below (pp. 256–7) that (14)

$$-\Sigma\frac{\partial P}{\partial l}\frac{\partial s}{\partial x}\equiv L\left(\mathrm{a}\frac{\partial^2 P}{\partial l^2}+\dots+2\mathrm{f}\frac{\partial^2 P}{\partial m\,\partial n}+\dots\right)$$

$$+\Sigma\,(A'+24A)\,a+2\Sigma\,(F'+24F)\,f \dots\dots\dots\dots\dots\dots (\text{ii.}),$$

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* Observe that { } is the u-polar conic of the s-pole of L.

wherein the term, multiplied by L, the result of substituting symbols of differentiation with respect to l, m, n for x, y, z of s (6), and operating on P, is, plainly, a contravariant of u—or an invariant of L and u—of order 5 in the coefficients of u and order 3 in those of L: viz., it is, in fact, the "Quippian" Q, as may be verified by means of the canonical or other form of u [*Proc. Lond. Math. Soc.*, Vol. XIX., p. 491 (46)].

But it will also be shown (p. 259) that

$$-2\Sigma \frac{\partial P}{\partial l} \frac{\partial s}{\partial x} \equiv \Sigma (A'+20A)\, a + 2\Sigma (F'+20F) f \text{.........(iii.)};$$

and from (ii.), (iii.) it follows that

$$-2L\left(a\frac{\partial^2 P}{\partial l^2}+\dots\right) \equiv \Sigma (A'+28A)\, a + 2\Sigma (F'+28F) f \text{......(iv.)}.$$

Adding (iii.) to (24), multiplied by 2, a second form of the satellite line of L is obtained, viz.,

$$(A'+16A)\, a+\dots+2(F'+16F) f+\dots = 0 \text{............(25)};$$

the form given in the "Memoir" [*Phil. Trans.*, A. 1888 (41), p. 167, and (49) p. 170.] Finally, eliminating successively

$$\Sigma A'a+2\Sigma F'f$$

and

$$\Sigma Aa+2\Sigma Ff$$

from the above equation to the satellite line by means of (iv.); viz.,

$$\Sigma A'a+2\Sigma F'f \equiv -28\,(\Sigma Aa+2\Sigma Ff)-2LQ,$$

that equation takes the forms

$$6\,(\Sigma Aa+2\Sigma Ff)+QL=0 \text{ .... (26)}$$

and

$$3\,(\Sigma A'a+2\Sigma F'f)-8QL=0 \text{ ....(27)}.$$

Pencil of Six Lines.

7. The last two equations show that the satellite line—say L'—of L, the line L and the two lines K, K', where

$$K \equiv Aa+\dots+2Ff+\dots$$

$$K' \equiv A'a+\dots+2F'f+\dots$$

form a pencil of four lines which all meet in the point whose coordinates are determined by (3),

$$\big[(Aa_1+Bb_1+Cc_1+2Fe+2Ga_3+2Ha_2)\,x$$
$$+(Aa_2+Bb_2+Cc_2+2Fb_3+2Ge+2Hb_1)\,y$$
$$+(Aa_3+Bb_3+Cc_3+2Fc_2+2Gc_1+2He)\,z=0,\ i.e.\big].$$

$$\Theta_1x+\Theta_2y+\Theta_3z=0,\quad lx+my+nz=0,$$

Θ_1, Θ_2, Θ_3 being the invariants Θ of s and u_1, s and u_2, s and u_3, respectively.

The special values for those coordinates, in the case of the canonical u, have been given by Cayley in the "Memoir on Cubic Curves" (*Phil. Trans.*, 1857, § 33).

It will be remarked that, if L touches the Quippian of u, the three lines L', K, K' all coincide.

Equation (iii.), above, shows that the s-polar of the point

$$x:y:z=\frac{\partial P}{\partial l}:\frac{\partial P}{\partial m}:\frac{\partial P}{\partial n}$$

is another line passing through the intersection of K, K', L, L'. A sixth line passing through the same common intersection is the u-polar line of the s-pole of L.

For the s-pole of L is the point whose coordinates are proportional to (11)

$$\frac{\partial\sigma^*}{\partial l},\ \frac{\partial\sigma}{\partial m},\ \frac{\partial\sigma}{\partial n},$$

and its u-polar line is, therefore,

$$a\left(\frac{\partial\sigma}{\partial l}\right)^2+\ldots+2f\left(\frac{\partial\sigma}{\partial m}\frac{\partial\sigma}{\partial n}\right)+\ldots=0.$$

But, generally, since the discriminant of $s=P^2/4$,

$$\left(\frac{\partial\sigma}{\partial l}\right)^2=4A\sigma-P^2\,(\mathrm{b}n^2+\mathrm{c}m^2-2\mathrm{f}mn)\ldots,$$

$$\frac{\partial\sigma}{\partial m}\frac{\partial\sigma}{\partial n}=4F\sigma-P^2\,(\mathrm{a}mn+\mathrm{f}l^2-\mathrm{g}lm-\mathrm{h}nl)\ldots.$$

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* *i.e.*, $\frac{\partial\sigma}{\partial\xi}$ with ξ, η, ζ replaced by l, m, n.

By which substitutions the equation above becomes

$$4(Aa+\ldots+2Ff+\ldots)\,\sigma-P^2\left\{(\mathrm{b}n^2+\mathrm{c}m^2-2\mathrm{f}mn)\,a+\ldots\right.$$
$$\left.+2(-\mathrm{a}mn-\mathrm{f}l^2+\mathrm{g}lm+\mathrm{h}nl)\,f+\ldots\right\}.$$

Now (15), $4\sigma=-v,$

and it will be shown below that

$$(\mathrm{b}n^2+\mathrm{c}m^2-2\mathrm{f}mn)\,a+\ldots\quad\equiv-PL\ldots\ldots\ldots\ldots\ldots\ldots\text{(v.)};$$

so that, finally, the u-polar line of the s-pole of L is

$$v(Aa+\ldots+2Ff+\ldots)+P^3L=0.$$

Proof of Equation (v.), *above.*

8. Since [*Lond. Math. Soc. Proc.*, Vol. xix., p. 494 (62), (63), (64)]

$$-\ lP\equiv(\mathrm{b}c_1+\mathrm{c}b_1-2\mathrm{f}e)\,l^2+\ldots$$
$$-mP\equiv(\mathrm{b}c_2+\mathrm{c}b_2-2\mathrm{f}b_3)\,l^2+\ldots$$
$$-\ nP\equiv(\mathrm{b}c_3+\mathrm{c}b_3-2\mathrm{f}c_2)\,l^2+\ldots,$$

with the notation of (1) ... (6) above,

$$\left.\begin{aligned}-(lx+my+nz)\,P&\equiv(\mathrm{b}c+\mathrm{c}b-2\mathrm{f}f)\,l^2+\ldots\\&\equiv(\mathrm{b}n^2+\mathrm{c}m^2-2\mathrm{f}mn)\,a+\ldots\\&\quad+2\,(-\mathrm{a}mn-\mathrm{f}l^2+\mathrm{g}lm+\mathrm{h}nl)\,f+\ldots\end{aligned}\right\}\ldots\ldots(28).$$

Proof of Equation (i.), p. 251.

9. Dividing both sides of this identity by

$$L\equiv lx+my+nz,$$

and operating with

$$\mathrm{a}\frac{\partial^2}{\partial l^2}+\ldots+2\mathrm{f}\frac{\partial^2}{\partial m\,\partial n}+\ldots,$$

the dexter being treated as the product of L^{-1}, and the dexter of (28), in which it is to be observed a, b, c, f, g, h are quadric functions of l, m, n;

$$-\left(\mathrm{a}\frac{\partial^2 P}{\partial l^2}+\mathrm{b}\frac{\partial^2 P}{\partial m^2}+\mathrm{c}\frac{\partial^2 P}{\partial n^2}+2\mathrm{f}\frac{\partial^2 P}{\partial m\partial n}+2\mathrm{g}\frac{\partial^2 P}{\partial n\partial l}+2\mathrm{h}\frac{\partial^2 P}{\partial l\partial m}\right)$$

$$\equiv 2L^{-3}s(-LP)+L^{-1}\left(\mathrm{a}\frac{\partial^2}{\partial l^2}+\ldots+2\mathrm{f}\frac{\partial^2}{\partial m\partial n}+\ldots\right)(-LP)$$

$$-4L^{-2}\left\{\begin{array}{l}(\mathrm{a}x+\mathrm{h}y+\mathrm{g}z)\{(\mathrm{bc}+\mathrm{c}b-2\mathrm{f}f)\,l+(fg+\ldots)\,m+(\mathrm{h}f+\ldots)\,n \\ \qquad\qquad +(\mathrm{b}_1c+\mathrm{c}_1b-2\mathrm{f}_1f)\,l^2+\ldots \qquad +\ldots\} \\ +(\mathrm{h}x+\mathrm{b}y+\mathrm{f}z)\{(fg+\ldots-\mathrm{hc})\,l+(ca+\ldots)\,m+(\mathrm{g}h+\ldots)\,n \\ \qquad\qquad +(\mathrm{b}_2c+\mathrm{c}_2b-2\mathrm{f}_2f)\,l^2+\ldots \qquad +\ldots\} \\ +(\mathrm{g}x+\mathrm{f}y+\mathrm{c}z)\{(\mathrm{h}f+\ldots-\mathrm{g}b)\,l+(\mathrm{g}h+\ldots)\,m+(\mathrm{a}b+\ldots)\,n \\ \qquad\qquad +(\mathrm{b}_3c+\mathrm{c}_3b-2\mathrm{f}_3f)\,l^2+\ldots \qquad +\ldots\}\end{array}\right\} \qquad \ldots\ldots\ldots\ldots(29),$$

the same notation being employed as above (9), viz.:

$$\mathrm{b}_1=\partial\mathrm{b}/2\partial l,\quad \mathrm{b}_2=\partial\mathrm{b}/2\partial m\;\ldots\;\mathrm{f}_3=\partial\mathrm{f}/2\partial n\;\ldots.$$

But (6)

$$\left(\mathrm{a}\frac{\partial^2}{\partial l^2}+\ldots\right)(LP)=L\left(\mathrm{a}\frac{\partial^2 P}{\partial l^2}+\ldots\right)+\frac{\partial P}{\partial l}\frac{\partial s}{\partial x}+\frac{\partial}{\partial m}\frac{\partial s}{\partial y}+\frac{\partial P}{\partial n}\frac{\partial s}{\partial z},$$

and the sum of the first, third, and fifth lines of the terms multiplied by $-4L^{-2}$, when multiplied out, will be found to be identically equal to [(2), (12)]

$$Aa+Bb+Cc+2Ff+2Gg+2Hh$$

$$-\{(Al+Hm+Gn)\,u_1+(Hl+Bm+Fn)\,u_2+(Gl+Fm+Cn)\,u_3\};$$

also [*Lond. Math. Soc. Proc.*, Vol. XIX., p. 493, (59), (65)],

$$\left.\begin{array}{l}(\mathrm{b}_1c+\mathrm{c}_1b-2\mathrm{f}_1f)\,l^2+\ldots+2\,(\mathrm{g}_1h+\mathrm{h}_1g-\mathrm{a}_1f-\mathrm{f}_1a)\,mn+\ldots=-xP, \\ (\mathrm{b}_2c+\mathrm{c}_2b-2\mathrm{f}_2f)\,l^2+\ldots+2\,(\mathrm{g}_2h+\mathrm{h}_2g-\mathrm{a}_2f-\mathrm{f}_2a)\,mn+\ldots=-yP, \\ (\mathrm{b}_3c+\mathrm{c}_3b-2\mathrm{f}_3f)\,l^2+\ldots+2\,(\mathrm{g}_3h+\mathrm{b}_3g-\mathrm{a}_3f-\mathrm{f}_3a)\,mn+\ldots=-zP,\end{array}\right\}$$

so that the sum of the second, fourth, and sixth lines of the terms in (29), multiplied by $-4L^{-2}$, is simply equal to

$$-Ps.$$

Making these substitutions, (29) becomes

$$-L^{-1}\Sigma\frac{\partial P}{\partial l}\frac{\partial s}{\partial x}-4L^{-2}\{L(Aa+\ldots+2Ff+\ldots)$$

$$-(Al+\ldots)u_1-(Hl+\ldots)u_2-(Gl+\ldots)u_3\}+2L^{-2}Ps\equiv 0,$$

which, multiplied by $L^2/2$, is the identity to be proved.

Proof of Identity (ii.), p. 251.

10. Returning to the identity (28),

$$-LP\equiv(\mathrm{b}c+\mathrm{c}b-2\mathrm{f}f)\,l^2+\ldots+2\,(\mathrm{g}h+\mathrm{h}g-\mathrm{a}f-\mathrm{f}a)\,mn+\ldots$$

$$\equiv(\mathrm{b}n^2+\mathrm{c}m^2-2\mathrm{f}mn)\,a+(\mathrm{c}l^2+\mathrm{a}n^2-2\mathrm{g}nl)\,b+(\mathrm{a}m^2+\mathrm{b}l^2-2\mathrm{h}lm)\,c$$

$$+2\,(-\mathrm{a}mn-\mathrm{f}l^2+\mathrm{g}lm+\mathrm{h}nl)f+2\,(-\mathrm{b}nl+\ldots)\,g+2\,(-\mathrm{c}lm+\ldots)\,h \qquad \ldots\ldots\ldots(30),$$

and, operating on this with

$$\mathrm{a}\frac{\partial^2}{\partial l^2}+\ldots+2\mathrm{f}\frac{\partial^2}{\partial m\partial n}+\ldots,$$

the result on the sinister is

$$-L\left(\mathrm{a}\frac{\partial^2 P}{\partial l^2}+\mathrm{b}\frac{\partial^2 P}{\partial m^2}+\mathrm{c}\frac{\partial^2 P}{\partial n^2}+2\mathrm{f}\frac{\partial^2 P}{\partial m\partial n}+2\mathrm{g}\frac{\partial^2 P}{\partial n\partial l}+2\mathrm{h}\frac{\partial^2 P}{\partial l\partial m}\right)$$

$$-\left(\frac{\partial P}{\partial l}\frac{\partial s}{\partial x}+\ldots\right) \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(31),$$

if (6) $\qquad s\equiv \mathrm{a}x^2+\ldots+2\mathrm{f}yz+\ldots.$

For the result on the dexter side there is the choice of three forms (*Lond. Math. Soc. Proc.*, Vol. xx., p. 111), of which that ("γ," *ibid.* and p. 116) most suited to the present case is as follows :—

If v, w are two quadric forms in l, m, n [as are b, n^2 ... in (30)],

then $$\left(\mathrm{a}\frac{\partial^2}{\partial l^2}+\ldots 2\mathrm{f}\frac{\partial^2}{\partial m\,\partial n}+\ldots\right)vw$$

$$=12\,\{\mathrm{a}v_1w_1+\ldots \qquad +\mathrm{f}\,(v_2w_3+v_3w_2)+\ldots\}$$

$$+2\,\{(\mathrm{b}C_1+\mathrm{c}B_1-2\mathrm{f}F_1)\,l^2+\ldots+2\,(\mathrm{g}H_1+\mathrm{h}G_1-\mathrm{a}F_1-\mathrm{f}A_1)\,mn+\ldots\} \qquad \ldots\ldots\ldots\ldots(\text{vi.}).$$

where

$$v_1 = \partial v/2\partial l, \quad v_2 = \partial v/2\partial m \ldots w_3 = \partial w/2\partial n,$$

$$A_1 = \tfrac{1}{4}\left(\frac{\partial^2 v}{\partial m^2}\frac{\partial^2 w}{\partial n^2} + \frac{\partial^2 v}{\partial n^2}\frac{\partial^2 w}{\partial m^2} - 2\frac{\partial^2 v}{\partial m\partial n}\frac{\partial^2 w}{\partial m\partial n}\right),$$

$$F_1 = \tfrac{1}{4}\left(\frac{\partial^2 v}{\partial n\partial l}\frac{\partial^2 w}{\partial l\partial m} + \frac{\partial^2 w}{\partial n\partial l}\frac{\partial^2 v}{\partial l\partial m} - \frac{\partial^2 v}{\partial l^2}\frac{\partial^2 w}{\partial m\partial n} - \frac{\partial^2 w}{\partial l^2}\frac{\partial^2 v}{\partial m\partial n}\right),$$

with analogous values for v_3, w_1, w_2, B_1, C_1, G_1, H_1.

The result of the operation on the dexter of (30) is, then,

$$\begin{aligned}
&12\Sigma a\{2(\mathrm{bc}-\mathrm{f}^2)\\
&-l(\mathrm{bc}_1+\mathrm{cb}_1-2\mathrm{ff}_1)-m(\mathrm{fg}_1+\mathrm{gf}_1-\mathrm{ch}_1-\mathrm{hc}_1)-n(\mathrm{hf}_1+\mathrm{fh}_1-\mathrm{bg}_1-\mathrm{gb}_1)\}\\
&+12\Sigma f\{4(\mathrm{gh}-2\mathrm{f})\\
&-l(\mathrm{fg}_3+\mathrm{gf}_3-\mathrm{ch}_3-\mathrm{hc}_3)-m(\mathrm{ca}_3+\mathrm{ac}_3-2\mathrm{gg}_3)-n(\mathrm{gh}_3+\mathrm{hg}_3-\mathrm{af}_3-\mathrm{fa}_3)\\
&-l(\mathrm{hf}_2+\mathrm{fh}_2-\mathrm{bg}_2-\mathrm{gb}_2)-m(\mathrm{gh}_2+\mathrm{hg}_2-\mathrm{af}_2-\mathrm{fa}_2)-n(\mathrm{ab}_2+\mathrm{ba}_2-2\mathrm{hh}_2)\}\\
&+\ 2\Sigma a\{(\mathrm{bc}_{11}+\mathrm{cb}_{11}-2\mathrm{ff}_{11})l^2+\ldots+2(\mathrm{gh}_{11}+\ldots)mn+\ldots\}\\
&+\ 4\Sigma f\{(\mathrm{bc}_{23}+\mathrm{cb}_{23}-2\mathrm{ff}_{23})l^2+\ldots+2(\mathrm{gh}_{23}+\ldots)mn+\ldots\} \quad \ldots\ldots\ldots(32).
\end{aligned}$$

[In forming the above result, consider the multiplier of a,

$$\mathrm{b}n^2+\mathrm{c}m^2-2\mathrm{f}mn,$$

in the operand (30). For it, plainly,

$$\begin{aligned}
\mathrm{a}v_1w_1+\ldots+\mathrm{f}(v_2w_3+v_3w_2)+\ldots &= \mathrm{b}(m\mathrm{c}_2-n\mathrm{f}_2)+\mathrm{c}(n\mathrm{b}_3-m\mathrm{f}_3)\\
&\quad+\mathrm{f}(n\mathrm{b}_2+m\mathrm{c}_3-m\mathrm{f}_2-n\mathrm{f}_3)+\mathrm{g}(n\mathrm{b}_1-m\mathrm{f}_1)+\mathrm{h}(m\mathrm{c}_1-n\mathrm{f}_1);
\end{aligned}$$

i.e., (9),

$$\begin{aligned}
&= \mathrm{b}(\mathrm{c}-l\mathrm{c}_1+n\mathrm{g}_1)+\mathrm{c}(\mathrm{b}-l\mathrm{b}_1+m\mathrm{h}_1)\\
&\quad+\mathrm{f}(-2\mathrm{f}+2l\mathrm{f}_1-n\mathrm{h}_1-m\mathrm{g}_1)+\mathrm{g}(n\mathrm{b}_1-m\mathrm{f}_1)+\mathrm{h}(m\mathrm{c}_1-n\mathrm{f}_1)\\
&= 2(\mathrm{bc}-\mathrm{f}^2)\\
&-l(\mathrm{bc}_1+\mathrm{cb}_1-2\mathrm{ff}_1)-m(\mathrm{gf}_1+\mathrm{fg}_1-\mathrm{ch}_1-\mathrm{hc}_1)-n(\mathrm{hf}_1+\mathrm{fh}_1-\mathrm{bg}_1-\mathrm{gb}_1),
\end{aligned}$$

which is the sum of terms multiplied by a in the first two lines of (32); and

$$A_1 = \mathrm{b}_{22}+\mathrm{c}_{33}+2\mathrm{f}_{23} = (9)-\mathrm{h}_{12}-\mathrm{g}_{31} = \mathrm{a}_{11},\ B_1 = \mathrm{b}_{11},\ C_1 = \mathrm{c}_{11},\ F_1 = \mathrm{f}_{11},$$

$$G_1 = -\mathrm{f}_{12}-\mathrm{c}_{31} = \mathrm{g}_{11},\quad H_1 = -\mathrm{b}_{12}-\mathrm{f}_{23} = \mathrm{h}_{11};$$

whence, (vi.),

$$(\mathrm{b}C_1+\mathrm{c}B_1-2\mathrm{f}F_1)l^2+\ldots=(\mathrm{bc}_{11}+\mathrm{cb}_{11}-2\mathrm{ff}_{11})l^2+(\mathrm{ca}_{11}+\mathrm{ac}_{11}-2\mathrm{gg}_{11})m^2+\ldots$$
$$+2\,(\mathrm{gh}_{11}+\mathrm{hg}_{11}-\mathrm{af}_{11}-\mathrm{fa}_{11})\,mn+\ldots,$$

agreeing with the sixth line of (32).

Again, for the part which multiplies f in (30), viz.,

$$-\mathrm{a}mn-\mathrm{f}l^2+\mathrm{g}lm+\mathrm{h}ln,$$

$$\begin{aligned}\mathrm{a}v_1w_1+\ldots &= \mathrm{a}\,(-2l\mathrm{f}_1+m\mathrm{g}_1+n\mathrm{h}_1)+\mathrm{b}\,(-n\mathrm{a}_2+l\mathrm{g}_2)+\mathrm{c}\,(-m\mathrm{a}_3+l\mathrm{h}_3)\\ &\quad +\mathrm{f}\,(-m\mathrm{a}_2-n\mathrm{a}_3+l\mathrm{g}_3+l\mathrm{h}_2)\\ &\quad +\mathrm{g}\,(-m\mathrm{a}_1-2l\mathrm{f}_3+m\mathrm{g}_3+n\mathrm{h}_3+l\mathrm{h}_1)\\ &\quad +\mathrm{h}\,(-n\mathrm{a}_1-2l\mathrm{f}_2+l\mathrm{g}_1+m\mathrm{g}_2+n\mathrm{h}_2);\end{aligned}$$

$$\begin{aligned}\textit{i.e.}, (9), \quad &= \mathrm{a}\,(-2\mathrm{f}-m\mathrm{c}_3-n\mathrm{b}_2+m_1\mathrm{f}_2+m\mathrm{f}_3)+\mathrm{b}\,(\ldots)+\mathrm{c}\,(\ldots)\\ &\quad +\mathrm{f}\,(-2\mathrm{a}-l\mathrm{h}_2-l\mathrm{g}_3+m\mathrm{a}_2+n\mathrm{a}_3)\\ &\quad +\mathrm{g}\,(2\mathrm{h}+l\mathrm{b}_2-l\mathrm{f}_3+2m\mathrm{g}_3-m\mathrm{h}_2-n\mathrm{h}_3)\\ &\quad +\mathrm{h}\,(2\mathrm{g}+l\mathrm{c}_3-l\mathrm{f}_2-m\mathrm{g}_2+2n\mathrm{h}_2-n\mathrm{g}_3)\\ &= 4\,(\mathrm{gh}-\mathrm{af})-l\,(\mathrm{gf}_3+\mathrm{fg}_3+\mathrm{hf}_2+\mathrm{fh}_2-\mathrm{ch}_3-\mathrm{hc}_3-\mathrm{bg}_2-\mathrm{gb}_2)\\ &\quad -m\,(\mathrm{ca}_3+\mathrm{ac}_3+\mathrm{gh}_2+\mathrm{hg}_2-2\mathrm{gg}_3-\mathrm{af}_2-\mathrm{fa}_2)\\ &\quad -n\,(\mathrm{ab}_2+\mathrm{ba}_2+\mathrm{gh}_3+\mathrm{hg}_3-2\mathrm{hh}_2-\mathrm{af}_3-\mathrm{fa}_3);\end{aligned}$$

and for the same,

$$A_1=2\mathrm{a}_{23},\quad B_1=-2\,(\mathrm{h}_{31}+\mathrm{f}_{33})=2\mathrm{b}_{23},\quad C_1=-2\,(\mathrm{g}_{12}+\mathrm{f}_{22})=2\mathrm{c}_{23},$$
$$F_1=2\mathrm{f}_{23}+\mathrm{a}_{11}+\mathrm{h}_{12}+\mathrm{g}_{31}=2\mathrm{f}_{23},\quad G_1=\mathrm{g}_{23}-\mathrm{a}_{12}-\mathrm{h}_{22}=2\mathrm{g}_{23},$$
$$H_1=\mathrm{h}_{23}-\mathrm{a}_{31}-\mathrm{g}_{33}=2\mathrm{h}_{23}.]$$

But [*Lond. Math. Soc. Proc.*, Vol. XIX., p. 502 (95), (97)] the coefficient of a, in (32), is equal to [(12), (14), p. 249]

$$24A+3\,(A'+4A)/2-(A'+12A)/2$$
$$=A'+24A;$$

and, similarly, the coefficient of f

$$=2\,(F'+24F);$$

viz., (31), (32),

$$-L\left(\mathrm{a}\,\frac{\partial^2P}{\partial l^2}+\ldots+2\mathrm{f}\,\frac{\partial^2P}{\partial m\,\partial n}+\ldots\right)-\Sigma\left(\frac{\partial P}{\partial l}\,\frac{\partial s}{\partial x}\right)$$
$$=\Sigma\left\{(A'+24A)\,a+2\,(F'+24F)\,f\right\}\ldots\ldots\ldots\ldots\ldots(33).$$

Corollary.

11. If the "crude" [*Proc.*, Vol. xx., p. 115 (15)] form of the result of the operation on the dexter of (30) is used, it is (as above, pp. 256–8),

$$4\{(\mathrm{bc}-\mathrm{f}^2)a+\ldots+2(\mathrm{gh}-\mathrm{af})f+\ldots\}+8(av_1w_1+\ldots)$$
$$+2\Sigma\mathrm{a}\{(\mathrm{b}_{11}c+\mathrm{c}_{11}b-2\mathrm{f}_{11}f)l^2+\ldots\}$$
$$+4\Sigma\mathrm{f}\{(\mathrm{b}_{23}c+\mathrm{c}_{23}b-2\mathrm{f}_{23}f)l^2+\ldots\}$$

i.e., $4(Aa+\ldots+2Ff+\ldots)+(20A+A')a+\ldots+2(20F+F')f+\ldots$
$$+2\Sigma\mathrm{a}\{(\mathrm{b}_{11}c+\ldots)l^2+\ldots\}+4\Sigma\mathrm{f}\{(\mathrm{b}_{23}c+\ldots)l^2+\ldots\}$$

or [(14), p. 249] $\Sigma(A'+24A)a+2\Sigma(F'+24F)f+2\Sigma\mathrm{a}\{\ldots\}+4\Sigma\mathrm{f}\{\ldots\}$,

and comparing this result with (33), it appears that

$$\Sigma\mathrm{a}\{(\mathrm{b}_{11}c+\mathrm{c}_{11}b-2\mathrm{f}_{11}f)l^2+\ldots\}+2\Sigma\mathrm{f}\{(\mathrm{b}_{23}c+\mathrm{c}_{23}b-2\mathrm{f}_{23}f)l^2+\ldots\}\equiv 0 \quad \ldots\ldots\ldots\ldots(\text{vii.});$$

a result which will presently be of use, and may be verified independently, with a moderate amount of work, by expressing the formula as (*Lond. Math. Soc. Proc.*, Vol. IX., pp. 232, 3)

$$\Sigma x\left[\Sigma\{u_1D^2u_1-(Du_1)^2\}\{s_{11}D^2u_1+u_1D^2s_{11}-2Du_1Ds_{11}\}\right.$$
$$\left.+\Sigma\{u_2D^2u_3+u_3D^2u_2-2Du_2Du_3\}\{s_{23}D^2u_1+u_1D^2s_{23}-2Du_1Ds_{23}\}\right]/\Delta^2,$$

where $\qquad s_{11}=\tfrac{1}{2}\dfrac{\partial^2 s}{\partial l^2}\ \ldots\ldots,\quad s_{23}=\tfrac{1}{2}\dfrac{\partial^2 s}{\partial m\,\partial n}\ \ldots\ldots.$

Proof of Identity (iii.), p. 252.

12. Differentiating the triad of equations, p. 255, end, with respect to l, then multiplying by a, h, g, respectively; with respect to m, then multiplying by h, b, f, respectively; with respect to n, then multiplying by g, f, c, respectively; adding the nine results, and having regard for the identity with zero (vii.) just proved:

$$-2\Sigma\frac{\partial P}{\partial l}\frac{\partial s}{\partial x}$$
$$=8\mathrm{a}\{(\mathrm{b}_1c+\mathrm{c}_1b-2\mathrm{f}_1f)l+(\mathrm{f}_1g+\mathrm{g}_1f-\mathrm{c}_1h-\mathrm{h}_1c)m$$
$$+(\mathrm{h}_1f+\mathrm{f}_1h-\mathrm{b}_1g-\mathrm{g}_1b)n\}$$

$$+8\mathrm{b}\left\{(\mathrm{f}_2 g+\mathrm{g}_2 f-\mathrm{c}_2 h-\mathrm{h}_2 c)\,l+(\mathrm{c}_2 a+\mathrm{a}_2 c-2\mathrm{g}_2 g)\,m\right.$$
$$\left.+(\mathrm{g}_2 h+\mathrm{h}_2 g-\mathrm{a}_2 f-\mathrm{f}_2 a)\,n\right\}$$
$$+8\mathrm{c}\left\{(\mathrm{h}_3 f+\mathrm{f}_3 h-\mathrm{b}_3 g-\mathrm{g}_3 b)\,l+(\mathrm{g}_3 h+\mathrm{h}_3 g-\mathrm{a}_3 f-\mathrm{f}_3 a)\,m\right.$$
$$\left.+(\mathrm{a}_3 b+\mathrm{b}_3 a-2\mathrm{h}_3 h)\,n\right\}$$
$$+8\mathrm{f}\left\{(\mathrm{f}_3 g+\mathrm{g}_3 f-\mathrm{c}_3 h-\mathrm{h}_3 c)\,l+(\mathrm{c}_3 a+\mathrm{a}_3 c-2\mathrm{g}_3 g)\,m\right.$$
$$+(\mathrm{g}_3 h+\mathrm{h}_3 g-\mathrm{a}_3 f-\mathrm{f}_3 a)\,n$$
$$+(\mathrm{h}_2 f+\mathrm{f}_2 h-\mathrm{b}_2 g-\mathrm{g}_2 b)\,l+(\mathrm{g}_2 h+\mathrm{h}_2 g-\mathrm{a}_2 f-\mathrm{f}_2 a)\,m$$
$$\left.+(\mathrm{a}_2 b+\mathrm{b}_2 a-2\mathrm{h}_2 h)\,n\right\}$$
$$+8\mathrm{g}\left\{(\mathrm{h}_1 f+\mathrm{f}_1 h-\mathrm{b}_1 g-\mathrm{g}_1 b)\,l+(\mathrm{g}_1 h+\mathrm{h}_1 g-\mathrm{a}_1 f-\mathrm{f}_1 a)\,m\right.$$
$$+(\mathrm{a}_1 b+\mathrm{b}_1 a-2\mathrm{h}_1 h)\,n$$
$$+(\mathrm{b}_3 c+\mathrm{c}_3 b-2\mathrm{f}_3 f)\,l+(\mathrm{f}_3 g+\mathrm{g}_3 f-\mathrm{c}_3 h-\mathrm{h}_3 c)\,m$$
$$\left.+(\mathrm{h}_3 f+\mathrm{f}_3 h-\mathrm{b}_3 g-\mathrm{g}_3 b)\,n\right\}$$
$$+8\mathrm{h}\left\{(\mathrm{b}_2 c+\mathrm{c}_2 b-2\mathrm{f}_2 f)\,l+(\mathrm{f}_2 g+\mathrm{g}_2 f-\mathrm{c}_2 h-\mathrm{h}_2 c)\,m\right.$$
$$+(\mathrm{h}_2 f+\mathrm{f}_2 h-\mathrm{b}_2 g-\mathrm{g}_2 b)\,n$$
$$+(\mathrm{f}_1 g+\mathrm{g}_1 f-\mathrm{c}_1 h-\mathrm{h}_1 c)\,l+(\mathrm{c}_1 a+\mathrm{a}_1 \mathrm{c}-2\mathrm{g}_1 g)\,m$$
$$\left.+(\mathrm{g}_1 h+\mathrm{h}_1 g-\mathrm{a}_1 f-\mathrm{f}_1 a)\,n\right\}$$
$$=8\Sigma a\left\{\mathrm{b}\,(\mathrm{c}_2 m-\mathrm{f}_2 n)+\mathrm{c}\,(\mathrm{b}_3 n-\mathrm{f}_3 m)+\mathrm{f}\,(\mathrm{b}_2 n+\mathrm{c}_3 m-\mathrm{f}_2 m-\mathrm{f}_3 n)\right.$$
$$\left.+\mathrm{g}\,(\mathrm{b}_1 n-\mathrm{f}_1 m)+\mathrm{h}\,(\mathrm{c}_1 m-\mathrm{f}_1 n)\right\}$$
$$+8\Sigma f\left\{\mathrm{a}\,(-2\mathrm{f}_1 l+\mathrm{g}_1 m+\mathrm{h}_1 n)+\mathrm{b}\,(\mathrm{g}_2 l-\mathrm{a}_2 n)+\mathrm{c}\,(\mathrm{h}_3 l-\mathrm{a}_3 m)\right.$$
$$+\mathrm{f}\,(\mathrm{g}_3 l-\mathrm{a}_3 n+\mathrm{h}_2 l-\mathrm{a}_2 m)+\mathrm{g}\,(\mathrm{h}_1 l-\mathrm{a}_1 m-2\mathrm{f}_3 l+\mathrm{g}_3 m+\mathrm{h}_3 n)$$
$$\left.+\mathrm{h}\,(\mathrm{g}_1 l-\mathrm{a}_1 n-2\mathrm{f}_2 l+\mathrm{g}_2 m+\mathrm{h}_2 n)\right\},$$

which, pp. 257, 258, has been shown to be equal to

$$8\Sigma a\left\{2\,(\mathrm{bc}-\mathrm{f}^2)\right.$$
$$\left.-l\,(\mathrm{bc}_1+\mathrm{cb}_1-2\mathrm{ff}_1)-m\,(\mathrm{fg}_1+\mathrm{gf}_1-\mathrm{ch}_1-\mathrm{hc}_1)-n\,(\mathrm{hf}_1+\mathrm{fh}_1-\mathrm{bg}_1-\mathrm{gb}_1)\right\}$$
$$+8\Sigma f\left\{4\,(\mathrm{gh}-\mathrm{af})\right.$$
$$-l\,(\mathrm{fg}_3+\mathrm{gf}_3-\mathrm{ch}_3-\mathrm{hc}_3)-m\,(\mathrm{ca}_3+\mathrm{ac}_3-2\mathrm{gg}_3)-n\,(\mathrm{gh}_3+\mathrm{hg}_3-\mathrm{af}_3-\mathrm{fa}_3)$$
$$\left.-l(\mathrm{hf}_2+\mathrm{fh}_2-\mathrm{bg}_2-\mathrm{gb}_2)-m(\mathrm{gh}_2+\mathrm{hg}_2-\mathrm{af}_2-\mathrm{fa}_2)-n(\mathrm{ab}_2+\mathrm{ba}_2-2\mathrm{hh}_2)\right\}$$

$$[(12),\ \text{p. }249]=\Sigma a\left(16A-4l\frac{\partial A}{\partial l}-4m\frac{\partial H}{\partial l}-4n\frac{\partial G}{\partial l}\right)$$
$$+2\Sigma f\left(16F-2l\frac{\partial H}{\partial n}-2m\frac{\partial B}{\partial n}-2n\frac{\partial F}{\partial n}\right.$$
$$\left.-2l\frac{\partial G}{\partial m}-2m\frac{\partial F}{\partial m}-2n\frac{\partial C}{\partial m}\right)$$

$$[(16),\text{p. }249]=\Sigma a\,(A'+20A)+2\Sigma f\,(F'+20F).$$

On the factor (v) *of the Eliminant* (23), p. 251.

13. It may now be shown that

$$v = (Al^2 + \dots)\,u + L^2\,(Aa + \dots)$$
$$-L\left[2\left\{(Al + \dots)\,u_1 + (Hl + \dots)\,u_2 + (Gl + \dots)\,u_3\right\} - Ps\right]$$

is a cubic—the "Cotesian" of the "Memoir"—having a node at the s-pole of L. For [(1), (3), p. 248]

$$\frac{\partial v}{\partial x} = 3\,(Al^2 + \dots)\,u_1 + L^2\,(Aa_1 + Bb_1 + Cc_1 + 2Fe + 2Ga_3 + 2Ha_2)$$
$$+L\Big[2l\,(Aa + \dots + 2Ff + \dots)$$
$$-4\left\{(Al + \dots)\,a + 4\,(Hl + \dots)\,h + 4\,(Gl + \dots)\,g\right\} + P\frac{\partial s}{\partial x}\Big]$$
$$+l\left[-2\left\{(Al + \dots)\,u_1 + (Hl + \dots)\,u_2 + (Gl + \dots)\,u_3\right\} + Ps\right].$$

Substituting the coordinates of the s-pole of L, viz.,

$$\left.\begin{array}{l} Al + Hm + Gn \text{ for } x, \\ Hl + Bm + Fn \text{ for } y, \\ Gl + Fm + Cn \text{ for } z, \end{array}\right\} \quad \dots\dots\dots\dots\dots\dots\dots(34),$$

$$L \text{ becomes } Al^2 + \dots + 2Fmn + \dots,$$

and

$$(Al + \dots)\,a + (Hl + \dots)\,h + (Gl + \dots)\,g \text{ becomes identical with } u_1,$$

while u_1 becomes, identically,

$$(Al^2 + \dots)\,(Aa_1 + \dots + 2Ha_2) - D\left\{(\mathrm{b}c_1 + \mathrm{c}b_1 - 2\mathrm{f}_1 e)\,l^2 + \dots\right\},$$

where D is the discriminant of s or $P^2/4$, and [*Proc.*, Vol. XIX., p. 494, (62)]

$$(\mathrm{b}c_1 + \mathrm{c}b_1 - 2\mathrm{f}e)\,l^2 + \dots + 2\,(\mathrm{g}a_2 + \mathrm{h}a_3 - \mathrm{a}e - \mathrm{f}a_1)\,mn + \dots = -lP.$$

Thus, in $\partial v/\partial x$,

$$3\,(Al^2 + \dots)\,u_1 - 4L\left\{(Al + \dots)\,a + (Hl + \dots)\,h + (Gl + \dots)\,g\right\}$$
$$+L^2\,(Aa_1 + \dots + 2Fe + \dots)$$

becomes $$-l\,(Al^2 + \dots)\,DP \dots\dots\dots\dots\dots\dots\dots\dots(35).$$

Again [(i.), p. 251], for all values of x, y, z,

$$2L\,(Aa + \dots + 2Ff + \dots) - 2\left\{(Al + \dots)\,u_1 + (Hl + \dots)\,u_2 + (Gl + \dots)\,u_3\right\}$$
$$= Ps - \frac{L}{2}\,\Sigma\,\frac{\partial P}{\partial l}\,\frac{\partial P}{\partial s},$$

so that the residue of terms in $\partial v/\partial x$ is, for all values of x, y, z,

$$LP\frac{\partial s}{\partial x}+2lPs-l\frac{L}{2}\Sigma\frac{\partial P}{\partial l}\frac{\partial P}{\partial x} \quad\text{.................... (36).}$$

For the present values (34) of x, y, z,

$$\frac{\partial s}{\partial x}=2Dl,$$

$$\Sigma\frac{\partial P}{\partial l}\frac{\partial s}{\partial x}=6DlP,$$

$$s=(Al^2+\ldots)D.$$

Thus the residue (36) becomes

$$2(Al^2+\ldots)lDP+2(Al^2+\ldots)lDP-3(Al^2+\ldots)lDP,$$

or

$$l(Al^2+\ldots)DP;$$

whence, and from (35), it appears that, for the values (34),

$$\frac{\partial v}{\partial x}=0.$$

Similarly, it may be proved that $\partial v/\partial y$, $\partial v/\partial z$ vanish for the same alues (33) of x, y, z.

Thursday, April 3rd, 1890.

J. J. Walker, Esq., F.R.S., President, in the Chair.

Mr. E. J. Brooksmith, M.A., LL.M., late Scholar and Law Student of St. John's College, Cambridge, Instructor in Mathematics at the Royal Academy, Woolwich, was elected a member.

The following communications were made:—

On the Properties of some Circles connected with a Triangle formed by Circular Arcs: R. Lachlan, M.A.

Some Properties of Numbers: Mr. R. W. D. Christie.

The Modular Equations for $n = 17$, 29: R. Russell, M.A. (Communicated by Professor Greenhill.)

The following presents were received:—

"Proceedings of the Royal Society," Vol. XLVII., No. 287.

"Proceedings of the Cambridge Philosophical Society," Vol. VII., Part I.

"Educational Times," for April.

"Bulletin des Sciences Mathématiques," Tome XIV.; March, 1890.

"Beiblätter zu den Annalen der Physik und Chemie," Band XIV., Stück 3.

"Memorias de la Sociedad Cientifica—'Antonio Alzate,'" Tomo III., No. 3; Mexico, 1889.

"Bollettino delle Pubblicazioni Italiane, ricevute per Diritto di Stampa," No. 101.

"Jornal de Sciencias Mathematicas e Astronomicas," Vol. IX., No. 4; Coimbra, 1889.

"Atti della Reale Accademia dei Lincei—Rendiconti," Vol. VI., Fasc. 1, 2, 3.

"Annali di Matematica pura ed applicata," Tome XVII., Fasc. 4.

"Sitzungsberichte der Königlich Preussischen Akademie der Wissenschaften zu Berlin," XXXIX.–LIII.

"Bulletin International de l'Academie des Sciences de Cracovie—Comptes Rendus des Séances de l'Année 1890," Février, 1890.

On the Properties of some Circles connected with a Triangle formed by Circular Arcs. By R. LACHLAN, M.A.

[*Read April 3rd*, 1890.]

Introduction, §§ 1, 2.

1. Let ABC be a triangle formed by three given circular arcs, and let the complete circles be drawn intersecting again in A', B', C'; we obtain three triangles $A'BC$, $AB'C$, and ABC' which may be called the associated triangles of ABC, and four triangles $A'B'C'$, $AB'C'$, $A'BC'$, $A'B'C$ which are the inverse triangles with respect to the circle, which cuts the given circles orthogonally, of ABC and its associated triangles respectively.

Each of these triangles has a circum-circle, an inscribed circle, and a Hart-circle, which last corresponds to the nine-point circle of an ordinary triangle. In the present paper I propose to exhibit the properties of these groups of circles in as complete a form as possible, but as some of these properties have already been investigated in a paper published in the *Quarterly Journal*, Vol. XXI., pp. 1–59, and in my Memoir on "Circles and Spheres," in the *Phil. Trans.*, Vol. CLXXVII., Part II., pp. 481–625, I shall merely quote results to be found in those papers.*

* References to these papers will be indicated by the letters *Q.*, *T.*, respectively.