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Source: *The Mathematical Gazette*, Vol. 5, No. 84 (Mar., 1910), p. 207

Published by: Mathematical Association

Stable URL: <http://www.jstor.org/stable/3603146>

Accessed: 04-11-2015 11:36 UTC

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313. [D. 2.] The following proof that $\left(1 + \frac{1}{n}\right)^n$ tends to a finite limit as n (an integer) tends to infinity requires only the two simple inequalities :

(a) $\frac{a-x}{b-x} > \frac{a}{b}$, if $a > b > x$. a, b and x being positive.

(b) $(1+\alpha)^n > 1+n\alpha$, if n is a positive integer and $-1 < \alpha < 1$.

We have to show (i) that $\left(1 + \frac{1}{n}\right)^n$ increases as n increases, and (ii) that it is always finite.

(i) Consider $\frac{\left(1 + \frac{1}{n+1}\right)^{n+1}}{\left(1 + \frac{1}{n}\right)^n}$, which may be written

$$\left(1 + \frac{1}{n+1}\right)^{n+1} \left(1 - \frac{1}{n+1}\right)^n; \text{ i.e. } \left(1 + \frac{1}{n+1}\right) \left(1 - \frac{1}{(n+1)^2}\right)^n.$$

$$\begin{aligned} \text{Now } \left(1 + \frac{1}{n+1}\right) \left(1 - \frac{1}{(n+1)^2}\right)^n &> \left(1 + \frac{1}{n+1}\right) \left(1 - \frac{n}{(n+1)^2}\right), \dots\dots\dots (b) \\ &\text{i.e. } > \frac{(n+1)^3 + 1}{(n+1)^3}, \\ &\text{i.e. } > 1, \end{aligned}$$

which proves (i).

$$\begin{aligned} \text{(ii)} \quad \left(\frac{n+1}{n}\right)^n &< \frac{n+1}{n} \cdot \frac{n+\frac{1}{2}}{n-\frac{1}{2}} \cdot \frac{n}{n-1} \dots \frac{n+1-\frac{n-1}{2}}{n-\frac{n-1}{2}} \dots\dots\dots (a) \\ &< \frac{2\left(2+\frac{1}{n}\right)}{1+\frac{2}{n}} \\ &< 4. \end{aligned}$$

We may prove in a similar manner that $\left(1 - \frac{1}{n}\right)^{-n}$ decreases as n increases.

$$\begin{aligned} \text{For } \frac{\left(1 - \frac{1}{n+1}\right)^{-n-1}}{\left(1 - \frac{1}{n}\right)^{-n}} &= \left(1 + \frac{1}{n}\right)^{n+1} \left(1 - \frac{1}{n}\right)^n \\ &= \left(1 + \frac{1}{n}\right) \left(1 - \frac{1}{n^2}\right)^n \\ &< \frac{1 + \frac{1}{n}}{\left(1 + \frac{1}{n^2}\right)^n}, \text{ since } 1 - \frac{1}{n^2} < \frac{1}{1 + \frac{1}{n^2}}, \\ &< \frac{1 + \frac{1}{n}}{1 + \frac{1}{n^2}} \dots\dots\dots (b) \\ &< 1. \end{aligned}$$

Also, (ii) above will suggest a proof that $\left(1 - \frac{1}{n}\right)^{-n} > \frac{9}{4}$.

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