On a theorem relating to the Multiple Thetafunctions.

By

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I propose — partly for the sake of the theorem itself, partly for that of the notation which will be employed — to demonstrate the general theorem (3'), p. 4, of Dr. Schottky's „Abriss einer Theorie der Abel'schen Functionen von drei Variabeln", (Leipzig, 1880), which theorem is there presented in the form:

$$(3')\quad e^{-\eta(u_1 \dots ;\mu', \nu')}\Theta(u_1+2\bar{\omega}_1' \cdots ; \mu, \nu) = e^{-2\pi i \Sigma \mu_\alpha \nu'_\alpha}\,\Theta(u_1 \cdots ; \mu+\mu', \nu+\nu'),$$

but which I write in the slightly different form

$$\exp[-\mathrm{H}(u;\mu',\nu')]\cdot\Theta(u+2\varpi';\mu,\nu) = \exp[-2\pi i\mu\nu']\cdot\Theta(u;\mu+\mu',\nu+\nu').$$

I remark that the theorem is given in the preliminary paragraphs the contents of which are, as mentioned by the Author, derived from Herr Weierstrass: and that the form of the thetafunction is a very general one, depending on the general quadric function

$$G(u_1, \cdots, u_\varrho;\ n_1, \cdots n_\varrho)$$

of 2ϱ variables, ϱ being the number of the arguments $u_1, \cdots u_\varrho$ (in fact the periods are not reduced to the normal form, but are arbitrary); and the characters $\nu_1, \cdots \nu_\varrho;\ \mu_1, \cdots \mu_\varrho$ instead, of having each of them the value 0 or 1, have each of them any integer or fractional value whatever. The meaning of the theorem (u denoting a set or row of ϱ letters $u_1, \cdots u_\varrho$, and so in other cases), is that the function $\Theta(u;\ \mu+\mu',\ \nu+\nu')$ with the new characters $\mu+\mu'$ and $\nu+\nu'$ is, save as to an exponential factor, equal to the function $\Theta(u+2\varpi';\mu,\nu)$ with the original characters μ, ν, but with the new arguments $u+2\varpi'$.

Notation.

This is in some measure a development of that employed in my „Memoir on the Theory of Matrices" Phil. Trans. t. CXLVIII (1858) pp. 17—37. I use certain single letters u etc. to denote sets or rows each of ϱ letters, $u = (u_1, \cdots u_\varrho)$: or if to fix the ideas $\varrho = 3$, then $u = (u_1, u_2, u_3)$, and so in other cases.

But I use certain other letters a, etc. to denote squares or matrices each of ϱ^2 letters; thus $\varrho = 3$ as before,

$$a = \begin{vmatrix} a_{11}, & a_{12}, & a_{13} \\ a_{21}, & a_{22}, & a_{23} \\ a_{31}, & a_{32}, & a_{33} \end{vmatrix},$$

and in any such case the transposed matrix is denoted by the same letter enclosed in parentheses

$$(a) = \begin{vmatrix} a_{11}, & a_{21}, & a_{31} \\ a_{12}, & a_{22}, & a_{32} \\ a_{13}, & a_{23}, & a_{33} \end{vmatrix}.$$

The sum $u+v$ of the row-letters $u, =(u_1, u_2, u_3)$ and $v, =(v_1, v_2, v_3)$ denotes the row $(u_1+v_1, u_2+v_2, u_3+v_3)$: and in like manner the sum $a+b$ of the two matrices or square-letters a and b, denotes the matrix

$$\begin{vmatrix} a_{11}+b_{11}, & a_{12}+b_{12}, & a_{13}+b_{13} \\ a_{21}+b_{21}, & a_{22}+b_{22}, & a_{23}+b_{23} \\ a_{31}+b_{31}, & a_{32}+b_{32}, & a_{33}+b_{33} \end{vmatrix}$$

and similarly for a sum of three or more terms.

The product $uv, = (u_1, u_2, u_3)\ (v_1, v_2, v_3)$, of the two row-letters u, v denotes the single term $u_1v_1 + u_2v_2 + u_3v_3$. We have $uv = vu$.

The product

$$au, = \begin{vmatrix} a_{11}, & a_{12}, & a_{13} \\ a_{21}, & a_{22}, & a_{23} \\ a_{31}, & a_{32}, & a_{33} \end{vmatrix} (u_1, u_2, u_3),$$

of a preceding square-letter a and a succeeding row-letter u, denotes the set or row

$$(a_{11}, a_{12}, a_{13})(u_1, u_2, u_3), (a_{21}, a_{22}, a_{23})(u_1, u_2, u_3), (a_{31}, a_{32}, a_{33})(u_1, u_2, u_3);$$

the notation ua is not employed.

The product

$$auv = \begin{vmatrix} a_{11}, & a_{12}, & a_{13} \\ a_{21}, & a_{22}, & a_{23} \\ a_{31}, & a_{32}, & a_{33} \end{vmatrix} (u_1, u_2, u_3)(v_1, v_2, v_3)$$ of a preceding square-letter a followed by the two row-letters u et v, denotes the single term

$$(a_{11}, a_{12}, a_{13})(u_1, u_2, u_3)v_1 + (a_{21}, a_{22}, a_{23})(u_1, u_2, u_3)v_2 + (a_{31}, a_{32}, a_{33})(u_1, u_2, u_3)v_3.$$

Observe that auv is not in general $= avu$; but it is easy to verify that $auv = (a)vu$; and hence if $(a) = a$, that is if the matrix a be symmetrical, then $auv = avu$.

A product of two matrices

$$ab, = \begin{vmatrix} a_{11}, & a_{12}, & a_{13} \\ a_{21}, & a_{22}, & a_{23} \\ a_{31}, & a_{32}, & a_{33} \end{vmatrix} \cdot \begin{vmatrix} b_{11}, & b_{12}, & b_{13} \\ b_{21}, & b_{22}, & b_{23} \\ b_{31}, & b_{32}, & b_{33} \end{vmatrix}$$

denotes a matrix

	$(b_{11},\ b_{21},\ b_{31})$,	$(b_{12},\ b_{22},\ b_{32})$,	$(b_{13},\ b_{23},\ b_{33})$
$(a_{11},\ a_{12},\ a_{13})$	„	„	„
$(a_{21},\ a_{22},\ a_{23})$	„	„	„
$(a_{31},\ a_{32},\ a_{33})$	„	„	„

viz. the top-line of the compound matrix is

$$(a_{11}, a_{12}, a_{13})(b_{11}, b_{21}, b_{31}),\ (a_{11}, a_{12}, a_{13})(b_{12}, b_{22}, b_{32}),\ (a_{11}, a_{12}, a_{13})(b_{13}, b_{23}, b_{33})$$

and so for the other lines: or expressing this in words, we say that any *line* of the compound matrix is obtained by compounding the corresponding *line* of the first or further component matrix with the several columns of the second or nearer component matrix.

Clearly ab is not in general $= ba$. We may easily verify that $(ab) = (b)(a)$, that is, the transposed matrix (ab) is that obtained by the composition of the transposed matrix (b) as first or further matrix, with the transposed matrix (a) as second or nearer matrix. Even if a and b are each symmetrical, we do not in general have $ab = ba$, but only $(ab) = ba$, or what is the same thing $ab = (ba)$.

In a symbol such as $abuv$, we first combine a, b into a single matrix ab, and then regard the expression as a combination such as auv: the expression denotes therefore a single term. The theory might be explained in greater detail; but the mode of working with row- and square-letters will be readily understood from what precedes.

In all that follows $u, \mu, \nu, \mu', \nu', n, \varpi', \zeta$ are row-letters; $a, b, h, \omega, \omega', \eta, \eta'$ are square-letters: a and b are symmetrical, viz. $a = (a)$, $b = (b)$.

And I write

$$\begin{aligned}(*)(u, \nu)^2, &= (a, h, b)(u, \nu)^2 \\ &= au^2 + 2hu\nu + b\nu^2 \\ &= \begin{vmatrix} a_{11}, & a_{12}, & a_{13} \\ a_{21}, & a_{22}, & a_{23} \\ a_{31}, & a_{32}, & a_{33} \end{vmatrix}(u_1,\ u_2,\ u_3)^2 \\ &+ 2\begin{vmatrix} h_{11}, & h_{12}, & h_{13} \\ h_{21}, & h_{22}, & h_{23} \\ h_{31}, & h_{32}, & h_{33} \end{vmatrix}(u_1, u_2, u_3)(\nu_1, \nu_2, \nu_3) \\ &+ \begin{vmatrix} b_{11}, & b_{12}, & b_{13} \\ b_{21}, & b_{22}, & b_{23} \\ b_{31}, & b_{32}, & b_{33} \end{vmatrix}(\nu_1, \nu_2, \nu_3)^2.\end{aligned}$$

to denote the general quadric function of the 2ϱ letters u, v, with $\frac{1}{2}\varrho(\varrho+1)+\varrho^2+\frac{1}{2}\varrho(\varrho+1), =\varrho(2\varrho+1)$ coefficients. It is assumed that the determinant formed with the $\frac{1}{2}\varrho(\varrho+1)$ coefficients b is negative: this is the necessary and sufficient condition for the convergence of the series.

Definition of $\Theta(u;\ \mu,\ \nu)$

$\Theta(u;\ \mu,\ \nu)$, the general thetafunction with ϱ arguments u, and 2ϱ characters μ, ν is the sum of a ϱ-tuple series of exponentials

$$\Theta(u;\ \mu,\ \nu) = \Sigma \exp.\ [(\bigstar)(u,\ n+\nu)^2 + 2\pi i\mu(n+\nu)]$$

where each of the letters n, $=(n_1, \cdots n_\varrho)$, has all integer values (zero included) from $-\infty$ to $+\infty$.

The general theorem in regard to $\Theta(u;\ \mu,\ \nu)$.

This is

$$\exp.[-H(u;\mu',\ \nu')]\cdot\Theta(u+2\varpi';\mu,\nu)$$
$$= \exp.\ [-2\pi i\mu\nu']\cdot\Theta(u;\mu+\mu',\ \nu+\nu'),$$

establishing a relation between the function $\Theta(u;\mu+\mu',\ \nu+\nu')$, with arbitrary character-increments μ', ν', and the function $\Theta(u+2\varpi';\mu,\nu)$ with the original characters, but with new arguments $u+2\varpi'$. $H(u;\ \mu',\ \nu')$ denotes a function, linear as regards the arguments u, but quadric as regards μ' et ν'; $-2\pi i\mu\nu'$ is a single term depending only on μ et ν'; and the theorem thus is that the two functions differ only by an exponential factor. The relations between the constants will be obtained in the course of the investigation.

Demonstration.

The truth of the theorem depends on the equality of corresponding exponentials on the two sides of the equation: viz. substituting for the thetafunctions their values, and comparing the exponents or arguments of the exponentials: writing also for convenience

$$G(u+2\varpi',\ n+\nu)$$

to denote the quadric function $(\bigstar)(u+2\varpi',\ n+\nu)^2$; we ought to have

$$-H(u;\mu',\nu')+G(u+2\varpi',n+\nu)+2\pi i\mu(n+\nu)$$
$$=-2\pi i\mu\nu'+G(u,n+\nu+\nu')+2\pi i(\mu+\mu')(n+\nu+\nu'),$$

or say

$$H(u;\mu',\nu')=G(u+2\varpi',n+\nu)-G(u,n+\nu+\nu')-2\pi i(n+\nu+\nu')\mu'.$$

In this equation, if true at all, the terms containing n must destroy eachother, and assuming that they do so, the equation becomes

$$H(u;\mu',\nu')=G(u+2\varpi',\nu)-G(u,\ \nu+\nu')-2\pi i(\nu+\nu')\mu'.$$

Consider first the terms in n: the right hand side is

$$= a(u+2\varpi')^2 + 2h(u+2\varpi')(n+\nu) + b(n+\nu)^2$$
$$- au^2 \qquad - 2hu(n+\nu+\nu') - b(n+\nu+\nu')^2 - 2\pi i n\mu'$$

and the terms herein which contain n thus are

$$\begin{array}{lll} 2h(u+2\varpi')n + bn^2 + 2bn\nu & & \\ -2hun & - bn^2 - 2bn(\nu+\nu') - 2\pi i n\mu', & \\ =4h\varpi' n & \qquad - 2bn\nu' & - 2\pi i n\mu' \end{array}$$

which, b being symmetrical, may be written

$$= 2(2h\varpi' - b\nu' - \pi i\mu')n$$

and these terms will vanish if, and only if

$$2h\varpi' - b\nu' - \pi i\mu' = 0,$$

a system of ϱ equations connecting ϖ', μ', ν'.

Assuming them to be satisfied, the remaining relation,

$$H(u;\mu',\nu') = G(u+2\varpi',\nu) - G(u,\nu+\nu') - 2\pi i(\nu+\nu')\mu',$$

becomes

$$H(u;\mu',\nu') = a(u+2\varpi')^2 + 2h(u+2\varpi')n + b\nu^2$$
$$- au^2 \qquad -2hu(\nu+\nu') - b(\nu+\nu')^2 - 2\pi i(\nu+\nu')\mu'.$$

Here a and b being symmetrical, we have $a(u+2\varpi')^2 = au^2 + 4a\varpi' u + 4a\varpi'^2$, $b(\nu+\nu')^2 = b\nu^2 + 2b\nu'\nu + b\nu'^2$, and the value therefore is

$$= 4a(\varpi' u + \varpi'^2) + 2h(2\varpi'\nu - u\nu') - b(2\nu'\nu + \nu'^2) - 2\pi i(\nu+\nu')\mu'.$$

On the right hand side putting the term in h under the form

$$-2h(u+\varpi')\nu' + 2h\varpi'(2\nu+\nu'), = -2(h)\nu'(u+\varpi') + 2h\varpi'(2\nu+\nu')$$

and the last term under the form $-\pi i\mu'(2\nu+\nu') - \pi i\mu'\nu'$, the equation becomes

$$H(u;\mu',\nu') = (4a\varpi' - 2(h)\nu')(u+\varpi') - \pi i\mu'\nu'$$
$$+ (2h\varpi' - b\nu' - \pi i\mu')(2\nu+\nu'),$$

where the second line vanishes in virtue of the foregoing equation $2h\varpi' - b\nu' - \pi i\mu' = 0$; the equation thus is

$$H(u;\mu',\nu') = (4a\varpi' - 2(h)\nu')(u+\varpi') - \pi i\mu'\nu'$$

which equation, regarding therein ϖ' as a linear function of μ', ν' shows that $H(u;\mu',\nu')$ is a function linear as regards u, (and containing this only through $u+\varpi'$) but quadric as regards μ', ν'.

Introducing the new row-letter ζ', we may write

$$H(u;\mu',\nu') = 2\zeta'(u+\varpi') - \pi i\mu'\nu'$$

viz. the expression on the right hand side is here assumed as the value of the function

$$H(u;\mu',\nu'),\ = G(u+2\varpi',\nu) - G(u,\nu+\nu') - 2\pi i(\nu+\nu')\mu';$$

and the theorem then is

$$\begin{aligned}\exp.[-H(u;\mu',\nu')]\cdot\Theta(u+2\varpi';\mu,\nu)&\\ =\exp.[-2\pi i\mu\nu']\cdot\Theta(u;\mu+\mu',\nu+\nu'),&\end{aligned}$$

where by what precedes

$$\begin{aligned}2h\varpi'-b\nu'-\pi i\mu'&=0,\\ 2a\varpi'-(h)\nu'-\zeta'&=0,\end{aligned}$$

2ϱ equations for determing the 2ϱ functions ϖ', ζ' as linear functions of μ', ν': which equations depend on the $\varrho(2\varrho+1)$ constants a, b, h.

Suppose that the resulting values of ϖ', ζ' are

$$\begin{aligned}\varpi'&=\omega\mu'+\omega'\nu',\\ \zeta'&=\eta\mu'+\eta'\nu',\end{aligned}$$

where $\omega,\omega',\eta,\eta'$ are square-letters; then, regarding a, b, h as arbitrary, the $4\varrho^2$ new constants $\omega,\omega',\eta,\eta'$ cannot be all of them arbitrary, but must be connected by $4\varrho^2-\varrho(2\varrho+1),=\varrho(2\varrho-1)$ equations.

We may regard $\omega,\omega',\eta,\eta'$ as satisfying these $\varrho(2\varrho-1)$ equations, but as being otherwise arbitrary; the foregoing equations then are

$$\begin{aligned}2h\varpi'-b\nu'-\pi i\mu'&=0,\\ 2a\varpi'-(h)\nu'-\zeta'&=0,\\ \varpi'&=\omega\mu'+\omega'\nu',\\ \zeta'&=\eta\mu'+\eta'\nu',\end{aligned}$$

which lead to the equations connecting a, b, h with $\omega,\omega',\eta,\eta'$.

The first and second equations, substituting for ϖ', ζ' their values, become

$$\begin{aligned}(2h\omega-\pi i)\mu'+(2h\omega'-b)\nu'&=0,\\ (2a\omega-\eta)\mu'+(2h\omega'-\eta'-(h))\nu'&=0,\end{aligned}$$

or μ', ν' being arbitrary, we thus obtain the $4\varrho^2$ equations

$$\begin{aligned}2a\omega-\eta&=0,\\ 2h\omega-\pi i&=0,\\ 2a\omega'-\eta'-(h)&=0,\\ 2h\omega'-b&=0,\end{aligned}$$

which are the equations in question. It is to be observed that πi is like the other symbols a matrix, viz. it is regarded as containing the matrix unity; or what is the same thing it denotes

$$\pi i \begin{vmatrix} 1, & 0, & 0 \cdots \\ 0, & 1, & 0 \\ \vdots & & \end{vmatrix},$$

We can from these equations eliminate a, b, h and thus obtain the $\varrho(2\varrho - 1)$ equations before referred to, which connect the $4\varrho^2$ constants ω, ω', η, η'. I give, but without a complete explanation, the steps of the elimination.

The equation $2a\omega - \eta = 0$, may be written in the form

$$2(a\omega) - (\eta) = 0,$$

that is

$$2(\omega)(a) - (\eta) = 0,$$

or since

$$(a) = a, \quad 2(\omega)a - (\eta) = 0;$$

from the original form, and the new form respectively, we find

$$2(\omega)a\omega - (\omega)\eta = 0, \quad 2(\omega)a(\omega) - (\eta)\omega = 0;$$

and comparing these

$$(\omega)\eta - (\eta)\omega = 0, \quad \text{(first result)}.$$

The equation $2a\omega' + \eta' - (h) = 0$, or say $(h) = -\eta' + 2a\omega'$ may be written in the form $h = -(\eta') + 2(a\omega')$, that is, since $a = (a)$

$$h = -(\eta') + 2(\omega')a,$$

and we thence deduce

$$h\omega = -(\eta')\omega + 2(\omega')a\omega.$$

But from the equation $2a\omega - \eta = 0$, we have $2(\omega')a\omega - (\omega')\eta = 0$, and the equation thus becomes $h\omega = -(\eta')\omega + (\omega')\eta$; which in virtue of $2h\omega - \pi i = 0$, becomes

$$\tfrac{1}{2}\pi i = -(\eta')\omega + (\omega')\eta, \quad \text{(second result)}.$$

From the equation above obtained, $h = -(\eta') + 2(\omega')a$, we have

$$h\omega' = -(\eta')\omega' + 2(\omega')a\omega';$$

in virtue of $2h\omega' - b = 0$, this becomes $-2(\eta')\omega' + 4(\omega')a\omega' = b$; an equation which may also be written $-2((\eta')\omega') + 4((\omega')a\omega') = (b)$, or what is the same thing $-2(\omega')\eta' + 4(\omega')(a)\omega' = (b)$; or since $(a) = a$ and $(b) = b$, this is $-2(\omega')\eta' + 4(\omega')a\omega' = b$: and comparing with the original equation $-2(\eta')\omega' + 4(\omega')a\omega' = b$, we obtain

$$(\omega')\eta' - (\eta')\omega' = 0, \quad \text{(third result)}.$$

We have thus the three systems

$$(\omega)\eta - (\eta)\omega = 0, \quad \tfrac{1}{2}\varrho(\varrho - 1) \text{ equations}$$

$$(\omega')\eta - (\eta')\omega = \frac{1}{2}\pi i, \quad \varrho^2 \qquad \text{equations}$$

$$(\omega')\eta' - (\eta')\omega' = 0 \quad , \quad \frac{1}{2}\varrho(\varrho-1) \quad ,,$$

in all $\varrho(2\varrho - 1)$ equations. As to these systems, observe that $(\omega)\eta$, $(\eta)\omega$, etc. are all of them matrices of ϱ^2 terms; each of the three systems denotes therefore in the first instance ϱ^2 equations, viz. the equations obtained by equating to zero the several terms of such a matrix: but in the first system each diagonal term so equated to zero gives the identity $0 = 0$; and equating to zero the terms which are symmetrical in regard to the diagonal we obtain twice over, in the forms $P = 0$, and $-P = 0$, one and the same equation; the number of equations is thus diminished from ϱ^2 to $\frac{1}{2}\varrho(\varrho - 1)$; and similarly in the third system the number of equations is $= \frac{1}{2}\varrho(\varrho-1)$: but for the second system the number of equations is really $= \varrho^2$. It is hardly necessary to remark that in this second system $\frac{1}{2}\pi i$ is as before regarded as a matrix.

The foregoing three systems of equations are in fact the equations (6) p. 4 of Dr. Schottky's work.

Cambridge, 12. July 1880.