The Teaching of Euclid
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THE TEACHING OF EUCLID.

It has been customary when Euclid, considered as a text-book, is attacked for his verbosity or his obscurity or his pedantry, to defend him on the ground that his logical excellence is transcendent, and affords an invaluable training to the youthful powers of reasoning. This claim, however, vanishes on a close inspection. His definitions do not always define, his axioms are not always indemonstrable, his demonstrations require many axioms of which he is quite unconscious. A valid proof retains its demonstrative force when no figure is drawn, but very many of Euclid's earlier proofs fail before this test.

The first proposition assumes that the circles used in the construction intersect—an assumption not noticed by Euclid because of the dangerous habit of using a figure. We require as a lemma, before the construction can be known to succeed, the following: If A and B be any two given points, there is at least one point C whose distances from A and B are both equal to AB. This lemma may be derived from an axiom of continuity. The fact that in elliptic space it is not always possible to construct an equilateral triangle on a given base, shows also that Euclid has assumed the straight line to be not a closed curve—an assumption which certainly is not made explicit. When these facts are taken account of, it will be found that the first proposition has a rather long proof, and presupposes the fourth. We require the axiom: on any straight line there is at least one point whose distance from a given point on or off the line exceeds a given distance.

The fourth proposition is a tissue of nonsense. Superposition is a logically worthless device; for if our triangles are spatial, not material, there is a logical contradiction in the notion of moving them, while if they are material, they cannot be perfectly rigid, and when superposed they are certain to be slightly deformed from the shape they had before. What is presupposed, if anything analogous to Euclid's proof is to be retained, is the following very complicated axiom: Given a triangle ABC and a straight line DE, there are two triangles, one on either side of DE, having their vertices at D, and one side along DE, and equal in all respects to the triangle ABC. (This axiom presupposes the definition of the two sides of a line, for which see below.) When the existence of a triangle thus equal in all respects to ABC is assured, we can prove that the triangle considered in the fourth proposition is this triangle.

The sixth proposition requires an axiom which may be stated as follows: If OAA', OBB', OCC' be three lines in a plane, meeting two transversals in A, B, C, A', B', C' respectively; and if O be not between A and A', nor B and B', nor C and C', or be between in all three cases; then, if B be between A and C, B' is between A' and C'. This axiom is the basis of the measurement of angles by distances, and is required for proving that if D be on AB, and BD be less than BA, the triangle DBC is less than the triangle ABC.

The seventh proposition is so thoroughly fallacious that Euclid would have done better not to attempt a proof. In the first place, it uses an undefined term in the enunciation, namely, *on the same side*. The definition requires an axiom, and may be set forth as follows: Given a line AB and a point C, with regard to any point D in the plane ABC, three cases may arise; (1) the straight line CD does not meet AB; (2) CD meets AB, produced if necessary, in a point not between C and D; (3) CD meets AB in a point between C and D. In cases (1) and (2), C and D are said to be on the same side of AB; in case (3), on opposite sides. The above very complicated axiom is better replaced by the following two: (1) Given three points A, B, C, a point D between B and C, and a point G between A and D, BG produced meets AC in a point between A and C; (2) A, B, C, D being as before, and E being between A and C, AD and BE meet in a point between A and D and also between B and E.* (The definition of *between* is long, and I omit it here for want of space.) The proof of I. 7 further assumes that if C and D be on the same side of AB, then if CB is between CA and CD, DA is between DC and DB; while if CB is between CD and AC produced, then AD produced is between DC and DB. This is a very complicated assumption, of which Euclid is to all appearance completely ignorant. The assumption may be stated more simply as follows: Of three lines in a plane starting from a point, either there is one which is

* Cf. Pasch, *Vorlesungen über neuere Geometrie*, Leipzig, 1882; Peano, *I Principii di Geometria*, Turin, 1889.

between the other two, or else any one of them produced is between the other two. But in this statement, the meaning of *between* has to be very carefully defined.

I. 8 involves the same fallacy as I. 4, and requires the same axiom as to the existence of congruent triangles in different places. In the following propositions, we require the equality of all right angles, which is not a true axiom, since it is demonstrable.* I. 12 involves the assumption that a circle meets a line in two points or in none, which has not been in any way demonstrated. Its demonstration requires an axiom of continuity, by the help of which the circle can be dispensed with as an independent figure.

I. 16 is false in elliptic space, although Euclid does not explicitly employ any assumption which fails for that space. Implicitly, he uses the following: If ABC be a triangle, and E the middle point of AC; and if BE be produced to F so that $BE=EF$, then CF is between CA and BC produced. In spaces where the straight line is not a closed series, this follows from the axioms mentioned in connection with I. 6 and I. 7. No other points of interest, except that I. 26 involves the same fallacy as I. 4 and I. 8, arise until we come to parallels; and the treatment of parallels in Euclid is, so far as I know, wholly free from logical defects.

Many more general criticisms might be passed on Euclid's methods, and on his conception of Geometry; but the above definite fallacies seem sufficient to show that the value of his work as a masterpiece of logic has been very grossly exaggerated.

B. RUSSELL.

THE COMMITTEE ON GEOMETRY.

THE following report, preliminary and subject to revision, has been drawn up by a Committee of the Mathematical Association, consisting of representatives from a large number of public schools, especially of those near London. It is the outcome of many meetings, and of prolonged deliberation on the more drastic of the changes proposed. It affords striking evidence, if any were needed, of the fact that mathematical teachers are neither unconscious of, nor indifferent to, the condition of things so forcibly depicted at the last meeting of the British Association. It is gratifying to note, *en passant*, that the counsels of the Committee were on the whole pervaded by singular unanimity.

Similar reports will shortly be issued on the teaching of Arithmetic and Algebra. The reports taken as a whole will represent a body of opinion which cannot be ignored, and should have a wholesome effect upon the future of mathematical teaching in this country.

It is hoped that the readers of the *Gazette* will make a serious study of the reports. All criticisms and suggestions will receive careful consideration at the hands of the Committee. They should, in the first instance, be sent to the Secretary, Mr. A. W. Siddons, Harrow School, Middlesex. It is very desirable that mathematical masters and others should fully avail themselves of this opportunity of placing on record their views as to the proposed changes; and, it is hardly necessary to add, that the hands of the Committee

* Cf. Hilbert, *Grundlagen der Geometrie*, Leipzig, 1899, p. 16.