ON A CLASS OF CONDITIONALLY CONVERGENT INFINITE PRODUCTS

By J. E. LITTLEWOOD.

[Received May 10th, 1909.—Read May 13th, 1909.—Revised October 24th, 1909.*]

1. In a paper recently published in these *Proceedings*,† Mr. Hardy raises the problem: *to find a product* $\Pi(1+a_n)$, *such that* Σa_n^k *is always convergent* [*i.e.*, for positive integral values of k], *but never absolutely, and whose convergence, divergence, or oscillation is capable of proof.*

Mr. Hardy shows that the product $\Pi(1+a_n e^{n\theta\iota})$ is divergent when θ/π is rational, where a_n is a positive function of n, which tends steadily to zero as $n \to \infty$, and which is such that Σa_n^k is divergent for all positive integral values of k.‡ The question naturally arises: Is such a product ever convergent when θ/π is irrational? This question will be seen to admit of a comparatively simple answer, *i.e.*, an answer not cumbered with elaborate restrictions as to the nature of a_n. It will be shown that, *if a_n be any positive decreasing function of n which tends to zero as $n \to \infty$, the product $\Pi(1+a_n e^{n\theta\iota})$ is convergent for a certain class of irrational values of θ/π, which class is independent of the a's, and includes all algebraic numbers.*

2. It is well known that, if $\lim_{n\to\infty} |u_n| = 0$, the product

$$\prod_{n=1}^{\infty} \left[(1+u_n)\exp\left(-u_n+\frac{u_n^2}{2}-\frac{u_n^3}{3}+\ldots\pm\frac{u_n^n}{n}\right)\right]$$

is convergent, so that the same product, when taken from $n=1$ to ν,

* Shortly after the paper was communicated, and at the suggestion of the Council, I altered the original title, and gave a more explicit account of Abel's lemma, which is used in § 2. Some time later I discovered that the original discussion corresponding to §§ 2 and 3 was incomplete. The part of the paper then rewritten (October 24th) consists of § 3 and the part of § 2 which follows the result (3) of that article.

† Ser. 2, Vol. 7, p. 40, "On the Continuity or Discontinuity of a Function defined by an Infinite Product." See, in particular, pp. 47, 48.

‡ The simplest example of such a function is $\Pi(1+e^{n\theta\iota}/\log n)$.

tends to a definite limit as $\nu \to \infty$. Taking $u_n = a_n e^{n\theta\iota}$, we see then that

$$\prod_{n=1}^{\infty} (1+a_n e^{n\theta\iota})$$

is convergent, provided

$$\prod_{n=1}^{\nu} \exp\left[\sum_{m=1}^{n} \frac{(-)^m}{m} a_n^m e^{nm\theta\iota}\right]$$

tends to a definite non-zero limit, or (on taking the logarithm) provided

$$f(\nu) = \sum_{n=1}^{\nu} \sum_{m=1}^{n} \frac{(-)^m}{m} a_n^m e^{nm\theta\iota} \tag{1}$$

tends to a definite limit as $\nu \to \infty$.

Let $$\theta_m = 2\pi\left[\frac{m\theta}{2\pi}\right],$$

where $[x]$ denotes the difference between x and the nearest integer. Then, if θ/π is irrational, we evidently have

$$0 < |\theta_m| < \pi. \tag{2}$$

Since $m\theta$ differs from θ_m by a multiple of 2π, we obtain from (1)

$$f(\nu) = \sum_{n=1}^{\nu} \sum_{m=1}^{n} \frac{(-)^m}{m} a_n^m e^{n\theta_m\iota}.$$

If in this expression we change the order of summation, we obtain

$$f(\nu) = \sum_{m=1}^{\nu} \sum_{n=m}^{\nu} \frac{(-)^m}{m} a_n^m e^{n\theta_m\iota}$$

$$\left.\begin{aligned} &= \sum_{m=1}^{\nu} \frac{(-)^m}{m} \sigma_m(\nu) \\ \text{where} \quad \sigma_m(\nu) &= \sum_{n=m}^{\nu} a_n^m e^{n\theta_m\iota} \end{aligned}\right\}. \tag{3}$$

A necessary and sufficient condition that a function $f(\nu)$ of ν should tend to a definite limit is that, for an arbitrarily small positive ϵ, and for all values of $\nu' > \nu$, we have

$$|f(\nu')-f(\nu)| < \epsilon, \quad \text{when} \quad \nu > N,$$

where N depends on ϵ, but not on ν'.

In the present case we have

$$f(\nu')-f(\nu) = \sum_{m=1}^{\nu} \frac{(-)^m}{m} \{\sigma_m(\nu')-\sigma_m(\nu)\} + \sum_{m=\nu+1}^{\nu'} \frac{(-)^m}{m} \sigma_m(\nu'). \tag{4}$$

Now the expressions

$$\sigma_m(\nu')-\sigma_m(\nu) = \sum_{n=\nu+1}^{\nu'} a_n^m e^{n\theta_m \iota},$$

and

$$\sigma_m(\nu') = \sum_{n=m}^{\nu'} a_n^m e^{n\theta_m \iota},$$

are sums of the type considered in the theorem known as Abel's lemma. The theorem is as follows.*

If the sequence $(v_1, v_2, \ldots)$ of positive terms never increases, then

$$\left| \sum_{n=1}^{p} a_n v_n \right| \leqslant H v_1,$$

where H is the upper limit of the expressions

$$|a_1|, \ |a_1+a_2|, \ |a_1+a_2+a_3|, \ \ldots, \ |a_1+a_2+\ldots+a_p|.$$

In $\sigma_m(\nu')-\sigma_m(\nu)$, we may take

$$(a_{\nu+1}^m, a_{\nu+2}^m, \ldots, a_{\nu'}^m), \quad (e^{(\nu+1)\theta_m \iota}, e^{(\nu+2)\theta_m \iota}, \ldots, e^{\nu'\theta_m \iota}),$$

respectively for the sequences of v's and a's, and for $\sigma_m(\nu')$, we have the sequences

$$(a_m^m, a_{m+1}^m, \ldots, a_{\nu'}^m), \quad (e^{m\theta_m \iota}, e^{(m+1)\theta_m \iota}, \ldots, e^{\nu'\theta_m \iota}).$$

Now the sum $\sum_{n=p}^{q} e^{n\theta_m \iota}$ of any number of consecutive terms of the sequence $e^{\theta_m \iota}, e^{2\theta_m \iota}, e^{3\theta_m \iota}, \ldots$, has a modulus less than $2\pi|\theta_m|^{-1}$.† Hence the number H corresponding to each of the expressions $\sigma_m(\nu')-\sigma_m(\nu)$, $\sigma_m(\nu')$, is less than $2\pi|\theta_m|^{-1}$. By the theorem, then, we have

$$|\sigma_m(\nu')-\sigma_m(\nu)| < 2\pi a_{\nu+1}^m |\theta_m|^{-1},$$

$$|\sigma_m(\nu')| < 2\pi a_m^m |\theta_m|^{-1}.$$

* The theorem in Abel's original form is given in Bromwich's "Infinite Series," pp. 54, 55. The theorem for complex a's given above follows by a trifling modification of the argument: it is also a particular case of the result to be found at the bottom of p. 205.

† We have $\quad \left| \sum_{n=p}^{q} e^{n\iota\theta_m} \right| = \left| \dfrac{e^{(q+1)\iota\theta_m} - e^{p\iota\theta_m}}{1-e^{\iota\theta_m}} \right| < \dfrac{2}{|1-e^{\iota\theta_m}|}.$

Now, if $|\theta_m| < \frac{1}{2}\pi$, then

$$|1-e^{\iota\theta_m}|^{-1} < |\sin\theta_m|^{-1} < [\,|\theta_m|/(\tfrac{1}{2}\pi)]^{-1} < \tfrac{1}{2}\pi\,|\theta_m|^{-1};$$

and, if $\frac{1}{2}\pi \leqslant |\theta_m| < \pi$, then

$$|1-e^{\iota\theta_m}|^{-1} \leqslant |1-\cos\theta_m|^{-1} \leqslant 1 < \pi\,|\theta_m|^{-1}.$$

It follows that in any case $\quad \left| \sum_{n=p}^{q} e^{n\iota\theta_m} \right| < 2\pi\,|\theta_m|^{-1}.$

Hence, from (4),

$$|f(\nu')-f(\nu)| < 2\pi \sum_{m=1}^{\nu} \frac{1}{m} a_{\nu+1}^{m} |\theta_m|^{-1} + 2\pi \sum_{m=\nu+1}^{\nu'} \frac{1}{m} a_m^m |\theta_m|^{-1}$$

$$< 2\pi \sum_{m=1}^{\nu} a_\nu^m |\theta_m|^{-1} + 2\pi \sum_{m=\nu+1}^{\nu'} a_\nu^m |\theta_m|^{-1}$$

$\left(\text{since } \frac{1}{m} \leqslant 1,\ a_{\nu+1} \leqslant a_\nu, \text{ and } a_m \leqslant a_\nu, \text{ when } m > \nu\right)$

$$< 2\pi \sum_{m=1}^{\nu'} a_\nu^m |\theta_m|^{-1}. \tag{5}$$

3. Let us now suppose that for *all* values of m we have

$$|\theta_m| > K^{-m},$$

where K is some positive constant. Then, from (5) above,

$$|f(\nu')-f(\nu)| < 2\pi \sum_{m=1}^{\nu'} (Ka_\nu)^m. \tag{1}$$

Let ν be chosen so large that

$$Ka_n < \tfrac{1}{2}.$$

Then we shall have, taking the geometric series in (1) to infinity,

$$|f(\nu')-f(\nu)| < 4\pi a_\nu.$$

This last expression is independent of ν', and may be made less than ϵ by choosing ν sufficiently large. The function $f(\nu)$ will consequently tend to a definite limit.

4. We conclude, then, that $\Pi(1+a_n e^{n\theta i})$ is convergent, provided we can find a constant K, such that

$$|\theta_m| > K^{-m}, \quad \text{when} \quad m \geqslant 1.$$

This condition will certainly be fulfilled if θ/π is an algebraic number. For it is well known that, if x is an algebraic number, and r/m an arithmetical fraction differing from x by less than (say) unity, then

$$|x-r/m| > hm^{-q},$$

where q is the degree of the equation with rational coefficients of which x is a root, and where h depends only on x.* Hence, in the case we are considering, if

$$m\theta/(2\pi) = \theta_m/(2\pi)+r,$$

* See, for example, Borel, *Leçons sur la Théorie des Fonctions*, p. 27.

so that r/m differs from $\theta/(2\pi)$ by less than unity, we have

$$|\theta/(2\pi)-r/m| > hm^{-q},$$

or

$$|\theta_m| > hm^{1-q}.$$

We shall then have, for all values of $m \geqslant 1$,

$$|\theta_m| > K_1^{-1}e^{-(m-1)},$$

where K_1 is a constant, for the expression

$$e^{-(m-1)}/(hm^{1-q})$$

has, for positive values of m, a finite maximum value when $m=q-1$. If now we choose K to be the greater of e and K_1, we have

$$|\theta_m| > K^{-m}.$$