and (2) that referred to which the cubic surface $(P) = 0$ takes an equation of the form

$$(\alpha x+\beta y+\gamma z+\delta w)(x+y+z+w)^2+\alpha' yzw+\beta' xzw+\gamma' xyw+\delta' xyz = 0,$$

which enables us to reduce (11) to

$$(P) = x(A)+y(B)+z(C)+w(D)$$
$$+27(G''_a)\,yzw+(G''_b)\,xzw+(G''_c)\,xyw+(G''_d)\,xyz\}\;\ldots\ldots(14),$$

in which $27(G''_a) = 27(G_a)-9\{(B)+(C)+(D)\}$, &c.

9. The relative volume (P, Q) of the locus in the fixed space of one point $P\ (x, y, z, w)$ of the moving one with regard to another $Q(x', y', z', w')$ of it, is derived from (P) by the substitution for x, y, z, w of $x-x', y-y', z-z', w-w'$ in the general expression for it (11). Thus the locus of points giving, with regard to a specified one, a constant relative volume, is a cubic surface. Of the expressions (12), (13), (14), and others simplified by special reference, the terms below the third degree in x, y, z, w will have corresponding to them no terms in the corresponding expressions for (P, Q) for a like reason to that adduced in §6. The simplified form corresponding to (14) for relative volumes will assure us immediately of the fact, that there are six directions, not necessarily all real, of lines QP through any point Q of the moving space which sweep out zero relative volumes.

On certain Quartic Curves, which have a Cusp at Infinity, whereat the Line at Infinity is a Tangent. By Henry M. Jeffery.

[*Read Dec.* 14*th*, 1882.]

1. In a recent memoir, published in these *Proceedings* (Vol. xiii., No. 185), quartics were classified, which were met and touched by the line at infinity in four coincident points,

$$\kappa\alpha^4 = u_3.$$

I now proceed to the next division of quartics (Salmon's "Higher Plane Curves," p. 213), which are met and touched by the same line in three coincident and met in one other real point (see § 31), and may be thus denoted $\kappa\alpha^3\beta = u_3.$

Inasmuch as the three points may coincide in a cusp, triple point, or stapete-point, the title of the memoir is imperfect.

The quartics are usually unicusped ovals, pirum-shaped, but occasionally cardioidal, and are either unipartite or bipartite.

2. The satellite-conic ($u_2=0$) has double contact of the second order, real or imaginary, with the quartic in each case.

Two parameters κ, λ are used for the purposes of discrimination; while λ varies, the satellite-conics are concentric and coaxal, *i.e.* homothetic, $2\kappa x^3y = ax^2+2bxy+cy^2+2ex+2dy+\lambda.$

It will be convenient to consider three divisions of these quartics, elliptic, hyperbolic, and parabolic, according as $b^2 < > = ac$.

3. Classification in this memoir also depends on two discriminating curves, designated hereinafter Curve (D_1) and Curve (D_2).

The equation in (κ, λ) to Curve (D_1) expresses the mutual relation between κ and λ, when the quartics have double points, and is found as the eliminant of the equations of certain pole curves.

Curve (D_2) is determined by eliminating between the quartic and the conditions that two stationary points unite to form a folium-point or point of undulation.

4. By triple asymptote of the quartics is here meant any line, which meets the curve in two coincident points of the cusp at infinity, in whose direction the directions of three possible asymptotes are merged.

5. To determine the tangential equivalent equation to a group of quartics with triple and single asymptotes, as above defined.

Let the line ($x\xi+y\eta=1$) touch the quartic

$$2\kappa x^3y = ax^2+2bxy+cy^2+2ex+2dy+\lambda.$$

The condition of contact is $S^3 = 27T^2$,

if $$S = \tfrac{1}{12}\phi^2+\kappa\eta\,(2\lambda\xi\eta^2+3d\xi\eta+e\eta^2+c\xi+b\eta),$$

where ϕ denotes the ends of the asymptotes ($a\eta^2-2b\xi\eta+c\xi^2$), and

$$\begin{aligned}T = &-\tfrac{1}{216}\phi^3-\tfrac{1}{12}\kappa\eta\,(-4\lambda\xi\eta^2+3d\xi\eta+e\eta^2+c\xi+b\eta)\,\phi\\ &-\tfrac{1}{4}\kappa^2\,(\lambda\eta^4+2d\eta^3+c\eta^2)-\tfrac{1}{2}\kappa\xi\eta^3\,(d\xi-e\eta)^2\\ &+\kappa\xi\eta^3\,[(ce-bd)\,\xi+(ad-be)\,\eta]-\tfrac{1}{2}\kappa\,(b^2-ac)\,\xi\eta^3.\end{aligned}$$

The equivalent class-curve is defined by a nonic equation

$$S^3-27T^2=0;$$

since η^3 disappears as a factor, as was anticipated, for the cusp ($\eta=0$). If there are no other or higher singularities, the quartics in this group belong to Dr. Salmon's Genus III.

6. To determine the first discriminating curve, or the mutual relation, which subsists between the parameters, when there are critical quartics in this group. Curve (D_1).

Let the previous equation to the group be resumed

$$-2\kappa x^3 y + ax^2 + 2bxy + cy^2 + 2ex + 2dy + \lambda = 0 \equiv \phi(x, y).$$

For critical values $\qquad \dfrac{d\phi}{dx} = 0, \quad \dfrac{d\phi}{dy} = 0,$

$$3\kappa x^2 y = ax + by + e \quad \ldots\ldots\ldots\ldots (1),$$

$$\kappa x^3 = bx + cy + d \quad \ldots\ldots\ldots\ldots (2);$$

whence $\qquad 0 = ax^2 + 2bxy + cy^2 + 3ex + 3dy + 2\lambda,$

or $\qquad 0 = ax^2 - cy^2 + 2ex + \lambda \ldots\ldots\ldots\ldots (3),$

since, from (1) and (2),

$$x(ax + by + e) = 3y(bx + cy + d) \ldots\ldots\ldots\ldots (4).$$

This last conic (4) is the locus of nodes of all quartics in the group. Curve (D_1) is the eliminant of (2), (3), and (4) and is of the tenth degree.

Cor. 1.—If $d = e = 0$, or the quartic asymptotes intersect in the centre of the satellite-conic, Curve (D_1) is resolved into two hyperbolæ. If $b^2 = ac$, it is represented by one hyperbola. See §§ 12—15.

Cor. 2.—If $c = 0$, or the quartics have a triple point at infinity, the Curve (D_1) is the quintic eliminant of the equations

$$\kappa x^3 = bx + d, \quad ax^2 + 2ex + \lambda = 0.$$

Cor. 3.—If $a = b = c = 0$, or the satellite is resolved into the lines

$$1 = 0, \quad 2ex + 2dy + \lambda = 0,$$

the Curve (D_1) is a quartic with a stapete-point and a folium-point

$$\kappa\lambda^3 + 8de^3 = 0.$$

7. On the locus of nodes, which may occur in quartics of this group. The discriminant of this conic

$$ax^2 - 2bxy - 3cy^2 + ex - 3dy = 0$$

is $\qquad 3ad^2 - 2bde - ce^2.$

If this be zero, the locus of nodes is resolved into two right lines,

$$(1)\ ex - 3dy = 0, \quad (2)\ \frac{ax}{e} + \frac{cy}{d} + 1 = 0.$$

Curve (D_1) is therefore resolved into three portions: a quintic, a parallel to the (κ) axis, and a quartic. For (1),

$$\kappa x^3 = \left(b + \frac{ce}{3d}\right)x + d, \quad \lambda + x^2\left(a - \frac{1}{9}\frac{ce^2}{d^2}\right) + 2ex = 0.$$

The eliminant is a bicusped quintic. To the cusp in a finite position there corresponds a cusped quartic.

For (2),
$$\kappa x^3 = bx + cy + d = \frac{x}{e}(be - ad),$$
$$a\lambda = e^2 + \left(a - \frac{ce^2}{d^2}\right) ay^2 = e^2 + \frac{2c}{d}(be - ad)\, y^2.$$

Either
$$x = 0, \quad cy + d = 0, \quad c\lambda = d^2,$$
or a bicusped quartic is derived as the eliminant of
$$\kappa e x^2 = be - ad, \quad 2c\,(be - ad)\, y^2 = d\,(a\lambda - e^2).$$

If the discriminant be not zero, this conic is an important guide in determining the several branches of Curve (D_1). In particular, the two or four limiting points should be obtained, at which the tangents are parallel to the coordinate axes of x and y. Such points often correspond to cusps in the group of quartics, inasmuch as points on the two sides of these limiting points are crunodes and acnodes in the quartics, which eventually blend in cusps. See §§ 19, 23, Figs. 3, 5.

8. On the double points or cusps, which occur in the first discriminating curve.

At such points
$$\frac{d\kappa}{dx} = 0, \quad \frac{d\lambda}{dx} = 0.$$

From (3) and (4) of § 6,
$$ax + e = cy\frac{dy}{dx}, \quad 2bx + 3cy + 3d = cx\frac{dy}{dx}.$$

The elimination of $\frac{dy}{dx}$ gives (4), the locus of nodes.

The actual position of the four singular points is found by differentiating (4), and combining the resulting equation with (4).

9. On the critical quartics in this group.

The cusp at infinity may be merged in a triple point, at which two of the branches may be imaginary; such quartics are trinodal in the nascent state.

The quartics may be also binodal, if (κ, λ) is a point on Curve (D_1); thus, besides the given cusp, quartics may have a crunode, an acnode, attached to, or detached from, the rest of the curve, or a second cusp, formed by the union of a crunode and acnode.

The quartics may be also trinodal. Besides the given cusp, there may be two nodes, or a node and a second cusp, as happens when (κ, λ) is the intersection of two branches of Curve (D_1), its position on each branch taking separate effect.

Two crunodes may unite in a tacnode, and two acnodes in a higher singularity.

10. To determine the second discriminating curve, or the mutual relation which subsists between the parameters, when quartics of this group possess points of undulation. Curve (D_2).

At a folium-point* or point of undulation, two stationary points are consecutive, so that both $\frac{d^2y}{dx^2}=0$, and $\frac{d^3y}{dx^3}=0$.

If these tests be applied to quartics of this group,

$$2\kappa x^3 y = ax^2+2bxy+cy^2+2ex+2dy+\lambda,$$

$$\kappa\left(3x^2y+x^3\frac{dy}{dx}\right) = ax+by+e+(bx+cy+d)\frac{dy}{dx},$$

$$\kappa\left(6xy+6x^2\frac{dy}{dx}\right) = a+2b\frac{dy}{dx}+c\left(\frac{dy}{dx}\right)^2,$$

$$y+3x\frac{dy}{dx}=0.$$

Curve (D_2) is defined by eliminating x, y from the given quartic, and

$$8\kappa x^3 y = 3ax^2+2bxy-cy^2+3ex-dy,$$

$$36\kappa x^3 y = 9ax^2-6bxy+cy^2.$$

It is of the twelfth degree in (κ, λ).

The locus of folium-points in the group of quartics is the conic

$$9ax^2+30bxy-11cy^2+27ex-9dy=0.$$

If its discriminant is zero, or

$$ad^2+10bde-11ce^2=0,$$

the conic is resolved into two straight lines

$$3ex-dy=0,\quad \frac{3ax}{e}+\frac{11cy}{d}+9=0.$$

Curve (D_2) is also resolved into two portions, a quartic and a sextic.

11. On the real inflexions which occur in quartics of this group.

It is established by Zeuthen (*Math. Annalen*, Band vii., p. 411), that a quartic can be at most quadrifolium, and therefore have at most eight inflexions.

Since the ordinary quartics of this group are unicusped ovals, and since two inflexions are merged in a cusp, there may be six inflexions in quartics of this group. According as the point (κ, λ) passes through the several branches of Curve (D_2), two inflexions are lost. There may be six, four, or two inflexions. See Figs. 2, 4.

But if the cusp at infinity becomes a triple point, only four inflexions are possible, and there may be two or none. See § 28, Fig. 10.

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* By "folium," Zeuthen understands the depression in the contour of a quartic oval, which is characterised by two inflexions and a bitangent. At a folium-point, or point of undulation, this depression is nascent.

To apply these processes, we will commence with simple cases.

12. To determine all the quartics in this group, when the asymptotes of the quartic meet in the centre of the satellite.

All these quartics are central, and are thus defined:—

$$2\kappa x^3 y = ax^2 + 2bxy + cy^2 + \lambda.$$

By § 6, the locus of nodes consists of two lines through the centre.

$$ax^2 - 2bxy - 3cy^2 = 0.$$

Curve (D_1) is defined by the elimination of x from the equations

$$(3\kappa x^2 - b)(\kappa x^2 - b) = ac,$$

$$\kappa^2 x^6 + (ac - b^2)\kappa x^2 + 2\kappa\lambda c = 0.$$

The locus of (κ, λ) consists of two hyperbolæ and the (κ) axis.

There are three divisions as $b^2 = \; < \; > ac$.

Parabolic Division.—If $b^2 = ac$,

$$(1)\ x = 0, \text{ and } \lambda = 0,$$

$$(2)\ 3\kappa x^2 = 4b,$$

and the sign of (κ) depends on that of (b).

Curve (D_1) is represented by one branch of the hyperbola

$$27\kappa\lambda + 32ab = 0.$$

13. Curve (D_2) also consists of two hyperbolæ.

By § 9, the folium-points of this sub-group (§ 12) lie in the two lines

$$9ax^2 + 30bxy - 11cy^2 = 0.$$

The eliminant of this equation, the given quartic, and the derived cubic

$$8\kappa x^3 y = 3ax^2 + 2bxy - cy^2$$

determines two hyperbolæ for the locus of (κ, λ).

Parabolic Division.—If $b^2 = ac$, the locus of folium-points is resolved into two lines

$$(1)\ 3ax = by, \quad (2)\ 3ax + 11by = 0.$$

The Curve (D_2) is thus resolved into the axis $(\lambda = 0)$, and a branch of the hyperbola

$$(11)^3 \kappa\lambda + 96ab = 0,$$

which is the eliminant of the two equations

$$\kappa x^3 = 4cy, \quad 9\lambda = 8cy^2 \text{ (which limits the sign of } \lambda).$$

14. Discussion of the quartics of this sub-group (§ 12) in the parabolic division.

(1) If $\lambda = 0$, there is a tacnode at the centre, if

$$2\kappa x^3 y = (ax - by)^2,$$

and a double acnode, if $2\kappa x^3 y = (ax + by)^2$.

If κ be negative, the converse is true. The class of all is the fifth; they retain two inflexions, four having been merged at the centre, and two at the cusp.

(2) $$2\kappa x^3y = ax^2+2bxy+\frac{1}{a}b^2y^2+\lambda.$$

If a, b, c be all positive, crunodes can correspond to points (κ, λ) in the second quadrant only, since for critical values $3\kappa x^2 = 4b$, and $27\kappa\lambda+32ab = 0$. Curve (D_1).

But, since the quartics are central, they are trinodal for critical values of κ, λ. For values of (κ, λ) above Curve (D_1), the curve is unipartite; for values below it, bipartite.

Points of undulation occur for values of (κ, λ) in the fourth quadrant. For values of (κ, λ) below Curve (D_3), the quartics have two inflexions; for all others, six.

15. Discussion of the quartics of this sub-group in the elliptic and hyperbolic divisions.

(1) If $\lambda = 0$, for this sub-group,

$$2\kappa x^3y = ax^2+2bxy+cy^2.$$

The origin is a biflecnode or an acnode; and the class is the seventh. Two inflexions are merged at the centre, four inflexions remain.

Folium-points exist, if the following relations are compatible:

$$\begin{aligned} -\ 2\kappa x^3y+\ ax^2+2bxy+cy^2 &= 0,\\ -\ 8\kappa x^3y+3ax^2+2bxy-cy^2 &= 0,\\ -36\kappa x^3y+9ax^2-6bxy+cy^2 &= 0. \end{aligned}$$

For this condition $\qquad 49ac = 81b^2.$

(2) $$2\kappa x^3y = ax^2+2bxy+cy^2+\lambda.$$

Critical values depend on the reality of the lines of nodes (§ 11),

$$ax^2-2bxy-3cy^2 = 0.$$

If $b^2+3ac>0$, there are further singularities: hence, in the elliptic division, crunodes must be possible; in the hyperbolic division, they may or may not exist.

For critical values, the quartics are trinodal; they have a pair of crunodes, since they are central, besides the cusp at infinity.

For values of (κ, λ) above or below the Curve (D_3), the quartics have six or two inflexions; these last disappear, at the folium-point, when (κ, λ) is on Curve (D_3), the discriminating hyperbola.

If there are no critical values, when $b^2+3ac<0$, the quartics have four inflexions. Folium-points, however, are possible, if

$$25b^2+33ac<0.$$

16. To determine the two discriminating curves, when the satellite-conic may be resolved into two parallel lines. (Parabolic Division.)

Let this sub-group be thus denoted (Figs. 1, 2):—

$$2\kappa x^3 y = (ax+by+c)^2+\lambda.$$

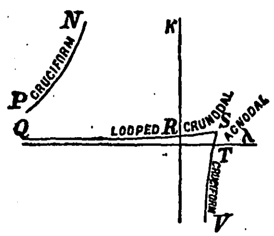

Fig. 1.

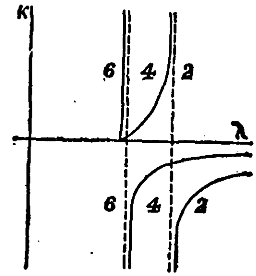

Fig. 2.

Curve (D_1) is, by § 6, the quintic eliminant of the quadratic and cubic

$$8b^2y^2+6bcy+\lambda+c^2=0, \quad 27\kappa b^2y^3-4a^3by-a^3c=0$$

The locus of nodes is resolved into two lines,

$$ax-3by=0, \quad ax+by+c=0.$$

The first line determines the preceding quintic, the second the axis $(\lambda=0)$.

In the quintic a cusp is found at the point $8ax=24by=-9c$, determined by the conditions

$$\frac{d\kappa}{dy}=0, \quad \frac{d\lambda}{dy}=0.$$

To Curve (D_2), the equation is found by elimination. Since the locus of folium-points is defined by

$$3ax+11by+9c=0,$$

after rejecting the factor $(3ax-by)$,

$$2\kappa x^3y=(ax+by+c)^2+\lambda=\tfrac{1}{16}(3ax-by)^2.$$

A cusp is at the intersection of $(ax+by+c=0)$ and $(3ax=by)$. In this point both curves (D_1) and (D_2) meet, when $\kappa=0$, $\lambda=0$, and the quartic is resolved into two pairs of coincident lines.

17. To obtain all the quartics which occur, when the satellite is resolved into two parallel straight lines.

(1) If $\lambda=0$, or the satellite conic consists of two coincident lines, the quartics of § 15 are bicusped, and bifolium.

If $\lambda=0$, and $27\kappa c^2=8a^3b$, which is a point on the quintic Curve (D_1), the quartic is bicusped and crunodal, with two inflexions.

(2) If the case, when λ is zero, is excluded, the Curve (D_1) (Fig. 1) will be a guide to the singular and non-singular quartics of the group.

If (κ, λ) be a point in the first quadrant, and on the portion RS, the corresponding quartic is crunodal, with a loop.

This curve separates into an ampullate or bipartite form (the oval replacing the loop), as (κ, λ) is above or below RS.

If (κ, λ) be a point in TS, the oval shrinks to a detached acnode. At the cusp S of the quintic (D_1), the corresponding quartic is bicusped, since the crunode is united with the acnode to form a cusp.

In the second quadrant, if (κ, λ) is on QR, the continuation of RS, the quartics are still crunodal with a loop, and the companion-curves are ampullate, until (κ, λ) reaches PN, when the quartics become cruciform, and if (κ, λ) is above PN, the branches separate. If (κ, λ) is below QR, the quartics are bipartite.

In the third quadrant, (κ, λ) has no critical value.

In the fourth quadrant, if (κ, λ) is on TV, the quartics are crunodal and cruciform; for values of (κ, λ) within or without TV, the branches separate in different ways.

Curve (D_2) is confined to the first and fourth quadrants, and meets Curve (D_1) only in the cusp $(0, 0)$.

There is, therefore, great simplicity in the classification of the inflexions. All quartics of this sub-group have six, four, or two inflexions, according as (κ, λ) lies to the left, between, or to the right of the branches of Curve (D_2), as is shown in Fig. 2.

18. In the parabolic division, Curve (D_1) is resolved into two portions, if the discriminant of the locus of nodes (§ 7) be zero.

$$3ad^2-2bde-ce^2=0.$$

Since $b^2=ac$, (1) $ad-be=0$, (2) $3ad+be=0$.

The case (1) has been discussed in §§ 16, 17.

In case (2), the quartics in this sub-group are thus denoted:

$$2\kappa x^3y = ax^2+2bxy+\frac{b^2}{a}y^2-\frac{bad}{b}x+2dy+\lambda.$$

The locus of nodes (§ 7) is resolved into two lines,

$$ax+by=0,\quad x=\frac{3b}{a}y+\frac{3d}{b}.$$

From the first line there is derived one quartic portion of Curve (D_1) with a stapete-point and a folium-point,

$$b^3\kappa\lambda^3=216a^3d^4,\ \text{since}\ \kappa x^3=d,\ b\lambda=6adx.$$

From the second line, the other portion is a bicuspidal quartic

$$2a\sqrt{\left(\frac{b}{3\kappa}\right)}+e=\frac{3}{2\sqrt{2}}\sqrt{(e^2-a\lambda)},$$

since, by § 6,

$$a\lambda - e^2 + 8b^2y^2 = 0, \quad \kappa x^3 = bx + \frac{b^2}{a}y + d = \frac{4}{3}bx.$$

A third portion of Curve (D_1) is found, when $x = 0$, $b^2\lambda = ad^2$.

The figures are readily seen to be limiting forms of Fig. 3. Curve (D_2) is resolved into two portions, when $ad = be$, but not when

$$3ad + be = 0.$$

19. In the parabolic division, to determine the first discriminant curve in the most general case.

For the quartics in this sub-group,

$$2\kappa x^3 y = ax^2 + 2bxy + \frac{b^2}{a}y^2 + 2ex + 2dy + \lambda.$$

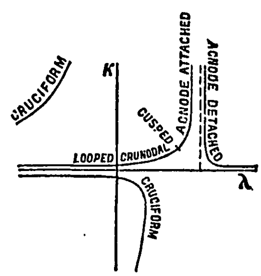

Fig. 3.

As in § 6, $\kappa x^3 = bx + \dfrac{b^2}{a}y + d,$

$$0 = ax^2 - \frac{b^2}{a}y^2 + 2ex + \lambda,$$

$$x(ax + by + e) = 3y\left(bx + \frac{b^2}{a}y + d\right) \quad \text{.....................(A).}$$

The elimination of x, y determines the locus of (κ, λ), or Curve (D_1). Curve (D_1) has three asymptotes, when

$$x = 0, \quad y = 0, \quad \kappa = \infty, \quad \lambda = 0,$$
$$x = 0, \quad by + e = 0, \quad \kappa = \infty, \quad a\lambda = e^2,$$
$$x = \infty, \quad y = \infty, \quad \kappa = 0, \quad \lambda = \infty.$$

Two limiting values of (x, y) (§ 7), in the locus of nodes (A), should be noticed, since for values of (κ, λ) on either side of one of the two corresponding points (κ, λ) of Curve (D_1) there may be a change from acnode to crunode through a cusp. There is no change which corresponds to the other limiting value of (x, y), in the cruciform portion of the (κ, λ) value.

20. To determine the various singular values of the quartics in the parabolic division, according as the parameters κ, λ are mutually related, as points on Curve (D_1). (Fig. 3.)

In the first quadrant, if (κ, λ) be on a branch of Curve (D_1) near the origin, the corresponding quartic is crunodal with a loop; but if (κ, λ) be distant, the quartics are acnodal, with an acnode attached to, or detached from, the quartics, as it lies on the further or nearer branch. At an intermediate position, where (x, y) is a limiting position of the locus of nodes, as above defined, the acnode and crunode unite in a cusp. In the second quadrant, if (κ, λ) be on the lower branch of Curve (D_1), which is a continuation of the former branch, the quartics are crunodal and looped. If (κ, λ) be on the upper branch, the quartics are cruciform.

To points (κ, λ) situated in the third and fourth quadrants, the critical forms of the quadric are cruciform, without any alteration for the limiting value of (x, y) in the locus of nodes.

The non-singular companion-curves have unipartite forms near the cruciform quartics, and bipartite when (κ, λ) lies between parts of a branch, which yield looped crunodal and detached acnodal quartics.

21. To determine the various folium-points, in quartics of the parabolic division, as the parameters κ, λ are mutually related as points on Curve (D_2).

Let the equation be resumed for such quartics,

$$2\kappa x^3 y = ax^2 + 2bxy + \frac{b^2}{a} y^2 + 2ex + 2dy + \lambda$$

by § 10 $= \frac{1}{18a}(3ax - by)^2$.

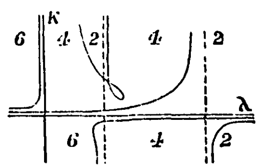

Fig. 4.

For the locus of folium-points,

$$9ax^2 + 30bxy - \frac{11}{a} b^2 y^2 + 27ex - 9dy = 0.$$

The elimination of x, y determines the locus of (κ, λ), or Curve (D_2). Curve (D_2) has four asymptotes, when

$$x = 0, \quad y = 0, \quad \kappa = \infty, \quad \lambda = 0.$$

$$x = 0, \quad y = -\frac{9ad}{11b^2}, \quad \kappa = \infty, \quad 242b^2\lambda = 243ad^2.$$

$$ax = -3e, \quad y = 0, \quad \kappa = \infty, \quad 2a\lambda = 3e^2.$$

$$x = \infty, \quad y = \infty, \quad \kappa = 0, \quad \lambda = \infty.$$

The Curves (D_1) and (D_2) do not intersect.

22. To determine the number of real inflexions in the quartics of the parabolic division, as the parameters (κ, λ) are mutually related with reference to the branches of Curve (D_2). (Fig. 4.)

The changes from six to four and two inflexions are best seen on the diagram (Fig. 4), where the figures 6, 4, 2 denote the number of inflexions which depend on the mutual values of (κ, λ). No change has been observed to take place as (κ, λ) passes through any branch of Curve (D_1), which passes between adjacent branches of Curve (D_2), as has been noticed in other divisions of Quartics. A change is noticed on either side of the (λ) axis.

23. To determine the singular values of quartics in the elliptic division, by the aid of the first discriminating curve—Curve (D_1).

The quartics of this sub-group are thus denoted: $2\kappa x^3 y = ax^2 + 2bxy + cy^2 + 2ex + 2dy + \lambda$, when $b^2 < ac$. The equations to Curve (D_1) are found in § 6. Four branches of Curve (D_1) meet in a double cusp, and two others in a single cusp. Three asymptotes occur, as in § 18. For related values of κ, λ, which correspond to points in Curve (D_1), the corresponding quartics are crunodal, acnodal, and (for two limiting values of x, y in the locus of nodes, § 7) cuspidal.

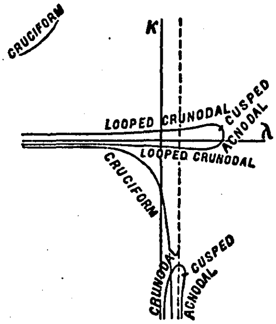

Fig. 5.

Corresponding to points (κ, λ) in the outer branches, the crunodal quartics are cruciform; for points in the two inner loops, the quartics are crunodal and looped or acnodal. The crunodes and acnodes unite in cusps at the two points above defined. In the lower loop of Curve (D_1) the acnodes are attached to their quartics, in the upper loop they are partly attached and partly detached. For points (κ, λ) within the upper loop the quartics are bipartite, but not for points (κ, λ) within the lower loop.

For the node (κ, λ) in Curve (D_1), in which two of its branches intersect, the corresponding quartic is trinodal, since the crunodal effect of each branch of Curve (D_1) is distinct.

The companion curves to the singular quartics are discussed in § 17.

24. To determine the number of points of inflexion of quartics in the elliptic division, by the aid of the second discriminating curve—Curve (D_2).

By reference to the diagram, the quartics are seen to have 6, 4, or 2 inflexions, as (κ, λ) lies with respect to the several branches of Curve (D_2). As (κ, λ) passes through a branch of Curve (D_2), two inflexions are merged in a folium-point. If (κ, λ) fall on any of the branches of Curve (D_1), which lie near or between branches of Curve (D_2), two inflexions are merged in a node; but, on either side of such a position, Curve (D_2) is the guide for any diversity.

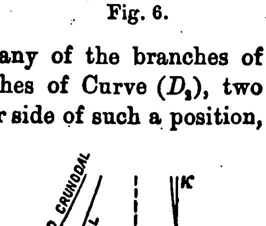

Fig. 6.

25. To determine the singular values of quartics in the hyperbolic division, by the aid of the first discriminating curve—Curve (D_1).

In this division, if the quartic equation be assumed as usual,

$$2\kappa x^3 y = ax^2 + 2bxy + cy^2 + 2ex + 2dy + \lambda,$$

then $$b^2 > ac.$$

The equations to the Curve (D_1) in (κ, λ) are

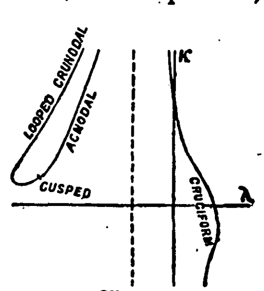

Fig. 7.

implicitly found in § 6. The conic locus of nodes in this sub-group is

$$ax^2 - 2bxy - 3cy^2 + ex - 3dy = 0.$$

In the case considered, this conic is supposed to be an ellipse, that is, (b^2+3ac) is negative, so that x, y, and therefore κ, are limited; there are only two asymptotes parallel to the (κ) axis. The singular quartics are crunodal with loops, acnodal, and cuspidal, if (κ, λ) be a point on the loop of Curve (D_1). The cuspidal quartic corresponds to one of the two limiting values of (x, y) in the locus of nodes, and therefore also of (κ, λ) in Curve (D_1).

The quartics are crunodal and cruciform, which correspond to points (κ, λ) on the serpentine portion of Curve (D_1).

26. To determine the number of points of inflexion of quartics in the hyperbolic division, by the aid of the second discriminating curve—Curve (D_2).

The implicit equations in (κ, λ) to Curve (D_2) are given in § 10.

The number of inflections is 6, 4, or 2, according to the position of (κ, λ) with respect to the several branches of Curve (D_2).

The corresponding quartic has a folium-point, or point of undulation, if (κ, λ) is on a branch.

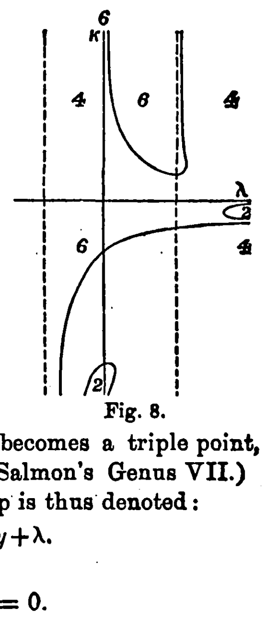

Fig. 8.

27. In the special case, when an asymptote of the satellite hyperbola is parallel to a triple asymptote of the quartic, the cusp at infinity becomes a triple point, and the quartics are of the sixth class. (Dr. Salmon's Genus VII.)

In this case $c = 0$, and the quartic sub-group is thus denoted:

$$2\kappa x^3 y = ax^2 + 2bxy + 2ex + 2dy + \lambda.$$

For a critical value,

$$\kappa x^3 = bx + d, \quad ax^2 + 2ex + \lambda = 0.$$

Hence y is given by a vanishing fraction, and a linear equation in x must be a factor of the quartic.

All singular quartics in this special sub-group are resoluble into cubics and lines parallel to the (y) axis.

For all non-singular quartics, the three asymptotes, real or imaginary, are given by the cubic $\kappa x^3 = bx + d$.

The equations to Curve (D_1), which is unipartite, are given above implicitly. (Fig. 9.)

28. To determine the number of inflexions of quartics in this special case, by the aid of the second discriminating curve.

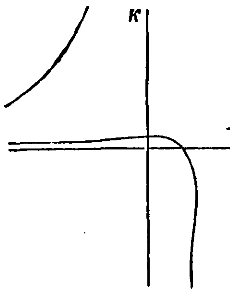

Fig. 9.

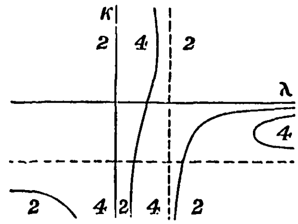

Fig. 10.

The numbers on either side of the branches of Curve (D_2) will make the sequence of inflexions clear, as previously explained.

29. If $c = 0$ and $d = 0$, Curve (D_1) is a bicusped quartic, and Curve (D_2) is modified, and the sequence of inflections altered.

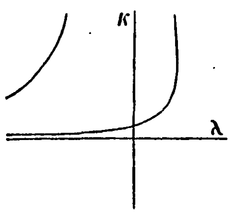

Fig. 11.

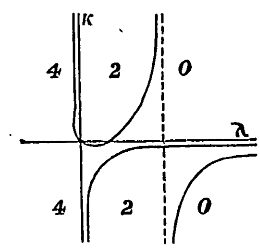

Fig. 12.

If $a = c = d = e = 0$, there is no singular quartic, and no folium-point in any curve of this sub-group.

30. In the special case, when the satellite is resolved into two lines, which are triple asymptotes of the quartic, the cusp is merged in a stápete-point at infinity, and the quartics are of the fourth class.

Let the equation to the sub-group be thus denoted:

$$2\kappa x^3 y = ax^2 + 2ex + \lambda, \quad \text{and} \quad b = c = d = 0 \text{ in § 5.}$$

$$S = \tfrac{1}{12} a^2\eta^4 + \kappa\eta^3 (2\lambda\xi + e),$$

$$T = -\tfrac{1}{216} a^3\eta^6 - \tfrac{1}{12}\kappa a\eta^5 (e - 4\lambda\xi) - \tfrac{1}{4}\kappa^2\lambda\eta^4 - \tfrac{1}{2}\kappa e^2 \xi\eta^5.$$

For the equivalent class-curve $S^3 = 27T^2$.

The factor η^8 is rejected for the stapete-point, at which two cusps and a node disappear; and a class quartic remains (Salmon's Div. IX.) There is no further singularity; but a line of folium points $(2a\lambda + 3e^2 = 0)$ represents Curve (D_2), and such quartics are denoted $\quad 2\kappa ax^3 y = (ax + 3e)(ax - e).$

31. Quartics, which have a cusp at infinity, and the line at infinity for a tangent thereat, may be written

$$(x^3 + 3bx^2)\, y + u_2 = 0.$$

By changing the origin to the point $(-b, 0)$, they may be reduced to the form considered in this memoir,

$$x^3y+u_2=0,$$

in which ($x=0$) is the determinate line passing through the cusp and the two points of osculation of the osculating conic.

The general form of quartics, which are simply met by the line at infinity in three coincident points and another point and which have a triple and a single asymptote ($x^3y+u_3+u_2=0$), is not here considered.

[The assumed form $\kappa\alpha^3\beta=u_2$, or, as this is afterwards written, $2\kappa x^3y=ax^2+2bxy+cy^2+2ex+2dy+\lambda$, is, I think, introduced without a proper explanation. Say, the form is $x^3y=z^2\,(*\between x,y,z)^2$, it ought to be shown how for a cuspidal quartic we arrive at this form; viz., taking the cusp to be at the point ($x=0$, $z=0$), $z=0$ for the tangent at the cusp, and $x=0$ an arbitrary line through the cusp; then the line $z=0$ besides intersects the curve in a single point, and, if $y=0$ is taken as the tangent at that point, the equation of the curve must, it can be seen, be of the form

$$(x^3+\theta x^2z)\,y=z^2\,(a,b,c,f,g,h\between x,y,z)^2.$$

The conic $(a,b,c,f,g,h\between x,y,z)^2=0$ touches the quartic at each of the two intersections of the quartic with the arbitrary line $x=0$; and we cannot, so long as the line remains arbitrary, find a conic which shall osculate the quartic at the two points in question; but, for the particular line $x+\frac{1}{3}\theta z=0$, there exists such a conic, viz., writing x instead of $x+\frac{1}{3}\theta z$, the form is $x^3y=z^2\,(a',b',c',f',g',h'\between x,y,z)^2$, and the new conic $(a',\ldots\between x,y,z)^2=0$ has the property in question. This is the adopted form, and it thus appears that in it the line $x=0$ is a determinate line, viz., the line passing through the cusp and the two points of osculation of the osculating conic. It thus appears that in the assumed form the lines $x=0$, $y=0$, $z=0$ are determinate lines.—A. C.]

Note on an Exceptional Case in which the Fundamental Postulate of Professor Sylvester's Theory of Tamisage fails. By Mr. J. Hammond, M.A.

[*Read Dec. 14th*, 1882.]

1. The expansion of the generating function for the binary seventhic is

$$\begin{aligned}1+ax^7+a^2\,(x^2+x^6+x^{10}+x^{14})\\+a^3\,(x^3+x^5+x^7+2x^9+x^{11}+x^{13}+x^{15}+x^{17}+x^{21})\\+a^4\,(\ldots+x^0+\ldots)+a^5\,(\ldots+4x^{13}+\ldots)+\ldots;\end{aligned}$$