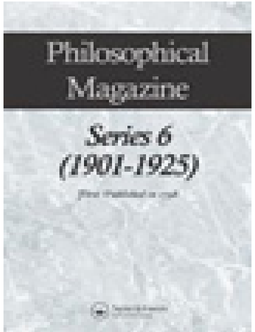

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LXXVI. A method for the summation of a type of infinite series

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in the expression obtained the real from the imaginary part, we finally have

$$= \sum_{\lambda=0}^{m} \binom{m}{\lambda} a^{m-\lambda} g^{\lambda} \frac{1}{h^{\lambda+1} 2^{p+1}} \sum_{\mu=0}^{p} \binom{p}{\mu} \sum_{\zeta=1}^{\lambda} \frac{1}{\lfloor\underline{\zeta}} \sum_{\eta=0}^{\zeta-1} (-1)^{\eta} \binom{\zeta}{\eta} (\zeta-\eta)^{\lambda}$$

$$\sum_{\beta=0}^{\zeta} \binom{\zeta}{\beta} (-1)^{\zeta-\beta} \frac{1}{r^{\frac{b-\beta}{h}}} \binom{b+\zeta-\beta-1}{\zeta-\beta} \lfloor\underline{\zeta-\beta} \Big[\sum_{\kappa=1}^{h} \Big\{ \epsilon^{r^{1/h} \cos \frac{2\kappa\pi+N}{h}} \cos$$

$$\Big[(b-\beta) \frac{2\kappa\pi - N}{h} + M + r^{1/h} \sin \frac{2\kappa\pi + N}{h} \Big] + \epsilon^{r^{1/h} \cos \frac{2\kappa\pi - N}{h}} \cos$$

$$\Big[(b-\beta) \frac{2\kappa\pi + N}{h} - M + r^{1/h} \sin \frac{2\kappa\pi - N}{h} \Big] \Big\} - 2h \sum_{\kappa=1}^{q} \frac{\cos (M - \kappa N)}{b - \kappa h - \beta} r^{\frac{b-\kappa h-\beta}{h}} \Big]$$

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LXXVI. *A Method for the Summation of a Type of Infinite Series.* *By* I. J. Schwatt *.

To find

$$S = \sum_{n=0}^{\infty} \frac{(\pm 1)^n r^n}{\prod_{m=1}^{p} (tn+m)}, \quad r > 0.$$

Now

$$\frac{1}{\prod_{m=1}^{p} (tn+m)} = \frac{1}{\lfloor\underline{p-1}} \sum_{\kappa=0}^{p-1} (-1)^{\kappa} \binom{p-1}{\kappa} \frac{1}{tn+\kappa+1}. \quad (1)$$

Let $r = x^t$, then

$$S = \frac{1}{\lfloor\underline{p-1}} \sum_{\kappa=0}^{p-1} (-1)_{\kappa} \binom{p-1}{\kappa} \frac{1}{x^{\kappa+1}} S_{\kappa},$$

wherein

$$S_{\kappa} = \sum_{n=0}^{\infty} (\pm 1)^n \frac{x^{tn+\kappa+1}}{tn+\kappa+1}$$

$$= \int_0^x \frac{x^{\kappa}}{1 \mp x^t} dx + C_{\kappa}. \quad \ldots \ldots \quad (2)$$

Since κ can be either greater or less than t, we can write

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$t\alpha+\beta$ for κ ($\beta=0, 1, 2, \ldots, t-1$), and have

$$S'_\kappa = S'_{t\alpha+\beta} = \int \frac{x^{t\alpha+\beta}}{1-x^t}\,dx$$

$$= -\int \sum_{\gamma=0}^{\alpha-1} x^{t\gamma+\beta}\,dx + \int \frac{x^\beta}{1-x^t}\,dx + C'_\kappa. \quad . \quad . \quad . \quad . \quad (3)$$

and

$$S''_\kappa = S''_{t\alpha+\beta} = \int \frac{x^{t\alpha+\beta}}{1+x^t}\,dx$$

$$= \int \sum_{\gamma=0}^{\alpha-1} (-1)^{\alpha+\gamma-1}\, x^{t\gamma+\beta}\,dx + (-1)^\alpha \int \frac{x^\beta}{1+x^t}\,dx + C''_\kappa. \quad . \quad (4)$$

Now

$$\frac{x^\beta}{1-x^t} = -\frac{1}{t}\Bigg[\frac{1}{x-1} + \frac{\frac{1}{2}(-1)^\beta\,[1+(-1)^t]}{x+1}$$

$$+ \sum_{\lambda=1}^{\frac{t-2}{2} \text{ or } \frac{t-1}{2}} \frac{\left(2x-2\cos\frac{2\lambda\pi}{t}\right)\cos\frac{2\lambda(\beta+1)\pi}{t}}{x^2-2x\cos\frac{2\lambda\pi}{t}+1}$$

$$+ \sum_{\lambda=1}^{\frac{t-2}{2} \text{ or } \frac{t-1}{2}} \frac{2\cos\frac{2\lambda\pi}{t}\cos\frac{2\lambda(\beta+1)\pi}{t} - 2\cos\frac{2\lambda\beta\pi}{t}}{x^2-2x\cos\frac{2\lambda\pi}{t}+1}\Bigg]. \quad (5)$$

By means of (5) we obtain from (3)

$$= -\sum_{\gamma=0}^{\alpha-1} \frac{x^{t\gamma+\beta+1}}{t\gamma+\beta+1}$$

$$-\frac{1}{t}\Bigg[\log(1-x) + \frac{1}{2}(-1)^\beta\,[1+(-1)^t]\log(1+x)$$

$$+ \sum_{\lambda=1}^{\frac{t-2}{2} \text{ or } \frac{t-1}{2}} \cos\frac{2\lambda(\beta+1)\pi}{t}\log\left(x^2-2x\cos\frac{2\lambda\pi}{t}+1\right)$$

$$+ \sum_{\lambda=1}^{\frac{t-2}{2} \text{ or } \frac{t-1}{2}} \frac{2\cos\frac{2\lambda\pi}{t}\cos\frac{2\lambda(\beta+1)\pi}{t} - 2\cos\frac{2\lambda\beta\pi}{t}}{\sin\frac{2\lambda\pi}{t}} \tan^{-1}\frac{x\sin\frac{2\lambda\pi}{t}}{1-x\cos\frac{2\lambda\pi}{t}}\Bigg]$$

$$. \quad . \quad . \quad . \quad . \quad . \quad (6)$$

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The last summation includes

$$C'_\kappa = -\frac{1}{t}\sum_{\lambda=1}^{\frac{t-2}{2}\text{ or }\frac{t-1}{2}}\left[2\cos\frac{2\lambda\pi}{t}\cos\frac{2\lambda(\beta+1)\pi}{t}-2\cos\frac{2\lambda\beta\pi}{t}\right].$$

We finally obtain for positive values of r

$$S=\frac{1}{t\,|\underline{p-1}}\sum_{\kappa=0}^{p-1}(-1)^{\kappa+1}\binom{p-1}{\kappa}\frac{1}{r^{\frac{\kappa+1}{t}}}$$

$$\times\left\{\log(1-r^{1/t})+\frac{1}{2}(-1)^{\kappa-t\left[\frac{\kappa}{t}\right]}[1+(-1)^t]\log(1+r^{1/t})\right.$$

$$+\sum_{\lambda=1}^{\frac{t-2}{2}\text{ or }\frac{t-1}{2}}\cos\frac{2\lambda\left(\kappa+1-t\left[\frac{\kappa}{t}\right]\right)\pi}{t}\log\left(r^{2/t}-2r^{1/t}\cos\frac{2\lambda\pi}{t}+1\right)$$

$$+\sum_{\lambda-1}^{\frac{t-2}{2}\text{ or }\frac{t-1}{2}}\frac{2\cos\frac{2\lambda\pi}{t}\cos 2\lambda\left(\kappa+1-t\left[\frac{\kappa}{t}\right]\right)\pi-2\cos\frac{2\lambda\left(\kappa-t\left[\frac{\kappa}{t}\right]\right)\pi}{t}}{\sin\frac{2\lambda\pi}{t}}$$

$$\left.\times\tan^{-1}\frac{r^{1/t}\sin\frac{2\lambda\pi}{t}}{1-r^{1/t}\cos\frac{2\lambda\pi}{t}}\right\}$$

$$-\frac{1}{|\underline{p-1}}\sum_{\kappa=t}^{p-1}(-1)^\kappa\binom{p-1}{\kappa}\sum_{\gamma=0}^{\left[\frac{\kappa}{t}\right]-1}\frac{r^{\gamma-\left[\frac{\kappa}{t}\right]}}{t\left(\gamma-\left[\frac{\kappa}{t}\right]\right)+\kappa+1}, \quad . \quad . \quad (7)$$

wherein $\left[\frac{\kappa}{t}\right]$ is the greatest integer in $\frac{\kappa}{t}$.

By means of

$$\frac{x^\beta}{1+x^t}=\frac{1}{t}\left[\sum_{\lambda=1}^{\frac{t-1}{2}\text{ or }\frac{t}{2}}\frac{2\cos\left[(2\lambda+1)\frac{\beta\pi}{t}\right]-2x\cos(2\lambda+1)\frac{\beta+1}{t}\pi}{x^2-2x\cos\left[(2\lambda+1)\frac{\pi}{t}\right]+1}\right.$$

$$\left.+\frac{\frac{1}{2}(-1)^{\beta-1}[1-(-1)^t]}{x+1}\right]$$

we obtain from (4) for negative values of r an expression for S similar to (7).

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The following example will illustrate the method of work.

To find

$$S = \sum_{n=0}^{\infty} \frac{r^n}{\prod_{m=1}^{5} (4n+m)}.$$

Then

$$S = \frac{1}{\lfloor 4} \sum_{\kappa=0}^{4} (-1)^\kappa \binom{4}{\kappa} \sum_{n=0}^{\infty} \frac{r^n}{4n+\kappa+1}.$$

Let $r = x^4$, then

$$S = \frac{1}{\lfloor 4} \sum_{\kappa=0}^{4} (-1)^\kappa \binom{4}{\kappa} \frac{1}{x^{\kappa+1}} S_\kappa, \quad . \quad . \quad . \quad . \quad (1)$$

wherein

$$S_\kappa = \sum_{n=0}^{\infty} \frac{x^{4n+\kappa+1}}{4n+\kappa+1} = \int \frac{x^\kappa}{1-x^4} dx + C_\kappa.$$

We have

$$S_0 = \frac{1}{2} \tan^{-1} x + \frac{1}{4} \log \frac{1+x}{1-x},$$

$$S_1 = \frac{1}{4} \log \frac{1+x^2}{1-x^2},$$

$$S_2 = \frac{1}{4} \log \frac{1+x}{1-x} - \frac{1}{2} \tan^{-1} x,$$

$$S_3 = \frac{1}{4} \log (1-x^4),$$

$$S_4 = \frac{1}{2} \tan^{-1} x + \frac{1}{4} \log \frac{1+x}{1-x} - x.$$

By means of these results (1) becomes, after replacing x by $r^{1/4}$,

$$S = \frac{1}{\lfloor 4} \Bigg[\frac{r-6r^{1/2}+1}{2r} \frac{\tan^{-1} r^{1/4}}{r^{1/4}} + \frac{r+6r^{1/2}+1}{4r^{5/4}} \log \frac{1+r^{1/4}}{1-r^{1/4}}$$

$$- \frac{1+r^{1/2}}{r} \log (1+r^{1/2}) - \frac{1-r^{1/2}}{r} \log (1-r^{1/2}) - \frac{1}{r} \Bigg]. \quad (2)$$

If

$$S = \sum_{n=0}^{\infty} \frac{(-1)^n r^n}{\prod_{m=1}^{5} (4n+m)}, \quad r>0,$$

$$S = \frac{1}{\lfloor 4} \sum_{\kappa=0}^{4} (-1)^\kappa \binom{4}{\kappa} \frac{1}{x^{\kappa+1}} \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+\kappa+1}}{4n+\kappa+1}, \quad . \quad (3)$$

wherein $x = r^{1/4}$.

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Designating the second summation by S_κ, we obtain

$$S_0 = \frac{1}{4\sqrt{2}}\left[\log\frac{x^2+x\sqrt{2}+1}{x^2-x\sqrt{2}+1} + 2\tan^{-1}\frac{x\sqrt{2}}{1-x^2}\right],$$

$$S_1 = \frac{1}{2}\tan^{-1}x^2,$$

$$S_2 = \frac{1}{4\sqrt{2}}\left[-\log\frac{x^2+x\sqrt{2}+1}{x^2-x\sqrt{2}+1} + 2\tan^{-1}\frac{x\sqrt{2}}{1-x^2}\right],$$

$$S_3 = \frac{1}{4}\log(1+x^4),$$

$$S_4 = x - \frac{1}{4\sqrt{2}}\left[\log\frac{x^2+x\sqrt{2}+1}{x^2-x\sqrt{2}+1} + 2\tan^{-1}\frac{x\sqrt{2}}{1-x^2}\right].$$

By means of these results we obtain

$$S = \frac{1}{4}\left[\frac{\sqrt{2}}{4r^{5/4}}(r+6r^{1/2}-1)\tan^{-1}\frac{r^{1/4}\sqrt{2}}{1-r^{1/2}} - \frac{2}{r^{1/2}}\tan^{-1}r^{1/2}\right.$$
$$\left. + \frac{\sqrt{2}}{8r^{5/4}}(r-6r^{1/2}-1)\log\frac{r^{1/2}+r^{1/4}\sqrt{2}+1}{r^{1/2}-r^{1/4}\sqrt{2}+1} - \frac{1}{r}\log(1+r) + \frac{1}{r}\right].$$

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LXXVII. *Low Potential Discharges in High Vacua.* *By* F. Horton, *D.Sc., M.A.**

IN the 'Proceedings' of the Royal Society (No. A. 607, 1913) Professor Strutt has described an interesting investigation into the origin of a peculiar form of low potential discharge produced in high vacua by the application of a magnetic field. With an apparatus in which the electrodes were two coaxial cylinders, and the gas pressure very low, Professor Strutt found that a difference of potential of many thousands of volts can be applied without a discharge passing, but that if a magnetic field parallel to the axis of the cylinders is created, a luminous discharge occurs with a potential difference of 300 or 400 volts.

An effect of a similar nature to that observed by Professor Strutt is obtained when the negative discharge from a

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