Review

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$NAB$ of $V$ with respect to $j$. Let a circle with centre $N$ orthogonal to $j$ meet $CV$ in $W$. $AWB$ is the angle required. H. HILTON.

169. [K. 21. b.] *An approximate construction for the trisection of an angle.*

Let $ABC$ be the angle to be trisected.
Draw $YHMO$ parallel to the bisector of $ABC$ intersecting $AB$ in $H$.
Draw $BY$ perpendicular to $YO$.
Find $L$ on $AB$ and $M$ on $YO$, equidistant from $H$, so that $LM=2BY$.
With centre $B$, radius $BL$, describe a circle cutting $YO$ in $O$.
Bisect the angle $MBO$ by $BX'$.
Bisect the angle $ABX'$ by $BR$ cutting $YO$ in $Z$.
Draw $ZT$ perpendicular to $BR$ and equal to $YB$.
Draw $TX$ parallel to $BR$.
This construction has reached me from a correspondent.
He thinks that $BX$ is one of the lines of trisection. The error is $\frac{1}{3}XBX'$.

3rd April, 1905. W. H. H. HUDSON.

## REVIEWS.

**Lectures on the Calculus of Variations.** By OSKAR BOLZA. Pp. xv + 271. The University of Chicago Press, 1904. [Decennial Publications of the University of Chicago, Second Series, volume 14.]

The methods of the Calculus of Variations have been almost entirely recast during the last 33 years: in this subject, as in most other branches of analysis, the first advances were deficient in rigour. The critical revision of the foundations was undertaken first by Weierstrass, who gave ten courses of lectures on the subject (1865-1889); and, from statements in Professor Bolza's book (§§ 24, 28), it appears that Weierstrass's chief discoveries were given in his courses of 1872 and 1879. In the former, the parametric method was used;* in the latter, it was shewn that the three conditions previously known to be *necessary* for a minimum must be supplemented by a fourth condition; and these four conditions were proved to be *sufficient*, a conclusion which could not be obtained from the older methods (see Bolza, § 17). Professor Bolza had the privilege of hearing the first exposition of these discoveries, which mark an epoch in the history of the subject.

For some 16 years Weierstrass's work remained generally unknown in this country; but in 1895-6, Professor Forsyth inaugurated a course at Cambridge on the subject. Since that date, Kneser has published his text-book (in 1900), and convenient summaries are now available;† besides which, a number of articles have appeared in the American Mathematical Journals. Last year, too, Professors Hancock and Bolza published courses of lectures, one of which is the subject of this review.

According to Professor Bolza's preface, his book originated in lectures delivered at the 1901 summer meeting of the American Mathematical Society; although the work has been somewhat modified to make it

* It does not appear to be certain whether Weierstrass had used this method in any earlier courses; according to the table of lectures given in Bd. 3 of Weierstrass's works, the course in 1872 was his fourth on the subject.

† *Encyclopaedia Brit.*, vol. 33 (by Professor Love); and *Encyclo. d. Math. Wiss.*, Bd. 2 (by Kneser, Zermelo and Hahn).

suitable for a text-book. Turning now to details, the theory of minimising the integral

$$(1) \qquad \int_{x_0}^{x_1} F(x, y, y')\,dx, \qquad (y' = dy/dx).$$

is contained in chapters 1-3; and here I would direct special attention to the careful tabulation (pp. 101, 102) of the *sufficient* conditions in contrast to the *necessary* conditions.* Chapters 4-6 give the corresponding theory, in Weierstrass's parametric form, for the integral

$$(2) \qquad \int^{t_1} F(x, y, x', y')\,dt, \qquad (x' = dx/dt,\ y' = dy/dt).$$

Chapter 5 contains Kneser's theory, which depends on an extension of Gauss's theorems on geodesics; chapter 6 the theory of isoperimetric (or *relative*) problems. Chapter 7 (the last) gives an account of Hilbert's theorem on the existence of solutions of minimum problems, under certain restrictions on the function $F$.

That Professor Bolza is a pupil of Weierstrass's may be seen from every page of this work; but he is not merely content to reproduce Weierstrass's results, he revises them where necessary.† But personally I am inclined to regret that Professor Bolza did not see his way to make more use of Hilbert's invariant-property (§§ 21, 37) that the integral

$$\int_{x_0}^{x_1} [F(x, y, p) + (y' - p)F_p(x, y, p)]\,dx, \qquad (y' = dy/dx).$$

is independent of the path; ‡ for this property leads to very attractive proofs of the leading theorems of the subject. In particular the complicated analysis belonging to the theory of the second variation may be avoided; for details I refer to Hedrick's article, quoted above.

It may seem strange to English readers, accustomed to the text-books of Todhunter and Williamson, that Bolza restricts the theory to integrals of the types (1), (2), which contain no derivates beyond the first. There are two reasons for this; first, hardly any practical problems contain higher derivates; secondly, the theory for such cases is not so complete. Perhaps, too, some may be sceptical about the weakness of the older methods; I therefore refer briefly to one objection. Up to 1879, it was invariably assumed that a small displacement of a curve involved a small change in shape also; but

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* Compare E. R. Hedrick, *Bull. Amer. Math. Soc.*, vol. 9, 1902, p. 23. Note that necessary *and* sufficient conditions are known for the integral (2), but not for (1).

† I quote from a review by Stäckel in the *Archiv der Math. u. Phys.*, Bd. 8, 1904, p. 259: "Nicht der ist ein treuer Schüler von Weierstrass, wer kritiklos hinnimmt, was dieser gelehrt hat, sondern wer bemüht ist, nach seinen Kräften die Teile der mathematischen Wissenschaft zu fördern, die dem Meister am Herzen lagen."

‡ Here $p(x, y)$ is a one-valued function of $x$, $y$, such that $y' = p(x, y)$ is a first integral of the differential equation of the problem arising from the integral (1).

this assumption really limits the variations permissible.* Thus the old theory admits, as a variation of the axis of $x$, a curve of small ordinates, *only if the slope and curvature are also small*; but it excludes the curve obtained by drawing a succession of small semi-circles on alternate sides of the axis.

After becoming acquainted with the Weierstrassian parametric method, one is tempted to regard the methods based on the integral (1) as out-of-date and incomplete. But, as Bolza remarks in a footnote on p. 115, the two methods really apply to *different* problems; the parametric being specially adapted to geometrical problems, and the other to analytical problems, where $y$ is to be found as a function of $x$. I may illustrate this contrast by Weierstrass's theorem (p. 142) that *the integral* (2) *has no minimum if $F$ is rational in $x'$, $y'$*; but, on the contrary, *the integral* (1) *may have a minimum, even though $F$ is rational in $y'$, provided $y$ is restricted to be a one-valued function of $x$.* For example, the integral $\int_0^1 (dy/dx)^2 dx$ has a minimum given by $y=x$, if the terminal conditions are $y=0$ for $x=0$, and $y=1$ for $x=1$. But the equivalent parametric integral $\int (y'^2/x')dt$, taken from (0, 0) to (1, 1), has no minimum.†

The last example recalls a more celebrated one which might have been mentioned by Bolza in connection with his account of Hilbert's existence-theorem; I refer to the integral $\int_0^1 x^2(dy/dx)^2 dx$, with the same end-conditions. As Weierstrass shewed in 1870, the differential equation derived from this integral admits no continuous solution with

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* Expressed analytically, the old assumption amounts to taking variations of the type $\epsilon\phi(x)$, where $\epsilon$ is small and $\phi(x)$ is arbitrary, *but does not contain* $\epsilon$. This would exclude $\epsilon \sin(x/\epsilon)$. Variations such as $\epsilon\phi(x)$ are now called *weak variations.*

† For, put $y=x+\xi$, where $\xi$ is a continuous one-valued function of $x$, which is zero for $x=0$ and $x=1$; then, after integration, we find that the first integral becomes

$$\int_0^1 (dy/dx)^2 dx = 1 + \int_0^1 (d\xi/dx)^2 dx,$$

shewing that $\xi=0$ makes the integral a minimum. But in the parametric form the same transformation gives

$$\int_0^1 (y'^2/x')\,dt = 1 + \int_0^1 (\xi'^2/x')\,dt;$$

and, as $x'$ may now be *negative*, the minimum property no longer holds. In particular, consider the "curve" of zig-zag straight lines given by

$$\left.\begin{array}{lll} x=\dfrac{n+1}{n-1}t, \; y=0, & \text{from } t=0 \text{ to } \dfrac{n-1}{2n} \\ x=1-t, \; y=nt-\tfrac{1}{2}(n-1), & \text{from } t=\dfrac{n-1}{2n} \text{ to } \dfrac{n+1}{2n} \\ x=\dfrac{n+1}{n-1}t-\dfrac{2}{n-1}, \; y=1, & \text{from } t=\dfrac{n+1}{2n} \text{ to } 1 \end{array}\right\} n>1.$$

This curve makes the value of the parametric integral $-n$; and hence, as $n$ is not limited, the integral can have no minimum. Of course, on this curve, $y$ is *not* a one-valued function of $x$.

the given end-conditions; further, by taking* $y\theta = \text{arc} \tan (x \tan \theta)$, where $\theta$ is slightly less than $\frac{1}{2}\pi$, the integral may be made arbitrarily small, leading to the inference that it has no minimum (because $y = \text{const.}$ does not satisfy the end-values). This example shews that Hilbert's existence-theorem for the integral (1) fails if $\partial^2 F/\partial y'^2$ is zero, even once, in the range of the integral. From a historical standpoint, this example is also of interest, for it first exposed the fallacy on which "Dirichlet's principle" was based by Lord Kelvin and Dirichlet;† and, in a sense, it may be said that this example led to Hilbert's theory, as he seems to have been guided by the hope of resuscitating the Kelvin-Dirichlet method.‡

In conclusion, it may be said that an excellent feature of Bolza's book is the systematic use of special problems to illustrate general theorems; but the addition of an index to these problems (as in Kneser's book) would have rendered the book more useful to one who needs to look up all the details of a particular problem. I heartily commend the book to any readers who wish for a connected account of modern results in the Calculus of Variations. T. J. I'A. B.

**Rational Geometry, based on Hilbert's Foundations.** By G. B. HALSTED. (New York: John Wiley & Sons, 1904, pp. 285.)

Mr. Halsted has added one more to the list of the classics which he has made accessible. The present volume is more than a translation. The earlier part is an adaptation of Hilbert's *Grundlagen der Geometrie* (also translated into French by M. Laugel and into English by Mr. E. T. Townsend).

It may be said at once, and would doubtless be admitted by Mr. Halsted, that this book has little or nothing to do with school work. School geometry must be based upon intuition, which Hilbert and Mr. Halsted exclude. At the same time it is well that some schoolmasters

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*I note, incidentally, that this value of $y$ makes $\int_0^1 (x^2+\cot^2\theta)(dy/dx)^2 dx$ a minimum, if $y$ is restricted to be a one-valued function of $x$. In fact, if we write

$$y = \frac{1}{\theta} \text{arc} \tan (x \tan \theta) + \xi,$$

$\xi$ being subject to the same conditions as before, we find

$$\int_0^1 (x^2+\cot^2\theta)(dy/dx)^2 dx = \frac{1}{\theta}\cot\theta + \int_0^1 (x^2+\cot^2\theta)(d\xi/dx)^2 dx.$$

† Their argument is that the integral

$$\iiint \left[\left(\frac{\partial V}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial y}\right)^2 + \left(\frac{\partial V}{\partial z}\right)\right] dx\, dy\, dz,$$

taken through a volume on the boundary of which $V$ is to have assigned values, *must* have a minimum because the subject of integration is never negative. Granting this assumption, the existence of a solution of the potential equation $\triangle V = 0$, with given boundary-values can be deduced. But the same argument applies to Weierstrass's example, for which the corresponding conclusions are *wrong*; because the integral has no minimum, neither has the differential equation a solution (with the given end-conditions).

‡ In his first account of the theory (*Jahresbericht d. D. Math. Verein.*, Bd. 8, 1900, p. 184), Hilbert says: "Das folgende ist ein Versuch der Wiederbelebung des Dirichlet'schen Princips."