ON TWO THEOREMS OF F. CARLSON AND S. WIGERT.

BY

G. H. HARDY

NEW COLLEGE, OXFORD.

1. In this short note I have united a number of remarks relating to two theorems due in part to WIGERT and in part to CARLSON.[1] The theorems belong to the same region of the theory of functions, and it is natural to consider them together.

I.

2. I write $z = x + iy = re^{i\theta}$. Then the first theorem is as follows.[2]

---

[1] The manuscript of the note (then entitled 'On two theorems of Mr. S. Wigert') was sent to Prof. MITTAG-LEFFLER in 1917. I was at that time unaware of the existence of Mr. CARLSON's dissertation ('Sur une classe de séries de Taylor', Uppsala, 1914). This dissertation was given to me by Prof. MITTAG-LEFFLER in September 1919; and I found at once that Mr. CARLSON had anticipated not only Mr. WIGERT's theorem of 1916, referred to in § 2, but my own generalisation of this theorem and indeed the substance of most that I had to say.

The note, however, contains something in substance, and a good deal in presentation, that is new; and I have therefore agreed to Prof. MITTAG-LEFFLER's suggestion that it should still appear. Except as regards §§ 1—2, I have left it substantially in its original form.

[2] WIGERT ('Sur un théorème concernant les fonctions entières', *Arkiv för Matematik*, vol. 11, 1916, no. 22, pp. 1—5) proves a theorem which is less general in that (1) the angle is supposed to cover the whole plane and (2) $f(z)$ is supposed to vanish for all positive and negative integral values of z. CARLSON (*l.* c., p. 58) proves a theorem which contains the present theorem as a particular case (but is in fact substantially equivalent to it). His method of proof is similar to that of the first two proofs given here.

WIGERT (*l. c.*) refers to previous and only partially succesful attempts to prove his theorem, and gives a proof based on a theorem of PHRAGMÉN ('Sur une extension d'un théorème classique de la théorie des fonctions', *Acta Mathematica*, vol. 28, 1904, pp. 351—369). He deduces as a corollary a result relating to the case in which $f(z)$ vanishes only for positive integral values of z; in this the number π is replaced by the less favourable number $\frac{1}{2}\pi$. I may add that a similar result, in which $\frac{1}{2}\pi$ is replaced by the still less favourable number 1, was found independently by PÓLYA ('Über ganzwertige ganze Funktionen', *Rendiconti del Circolo Matematico*

If

(1) *$f(z)$ is regular at all points inside the angle $-\alpha \leqq \theta \leqq \alpha$, where $\alpha \geqq \frac{1}{2}\pi$;*

(2) $$|f(z)| < A e^{kr}, \text{ where } k < \pi, \text{ throughout this angle;}$$

(3) $$f(n) = 0 \text{ for } n = 1, 2, 3, \ldots;$$

then $f(z)$ is identically zero.

3. It seems most natural to deduce this theorem from those proved by PHRAGMÉN and LINDELÖF in Part III of their well-known memoir in Vol. 31 of the *Acta Mathematica*.[1] Let us suppose that $f(z)$ is not always zero, and write, with PHRAGMÉN and LINDELÖF,

$$h(\theta) = \limsup_{r \to \infty} \frac{\log |f(re^{i\theta})|}{r},$$

so that

$$h(\theta) \leqq k, \qquad (-\alpha \leqq \theta \leqq \alpha).$$

Then $h(\theta)$ is continuous for $-\alpha < \theta < \alpha$.[2]

Now let

$$F(z) = \frac{f(z)}{\sin \pi z},$$

so that $F(z)$ also is regular inside the angle of the theorem; and let $H(\theta)$ be formed from $F(z)$ as $h(\theta)$ is from $f(z)$. Then it is obvious that

(1) $$H(\theta) = h(\theta) - \pi |\sin \theta| \leqq k - \pi |\sin \theta|,$$

except possibly for $\theta = 0$.

If $\theta = 0$, $z = x$ is real. We write

$$f(x) = u(x) + iv(x), \quad F(x) = U(x) + iV(x).$$

Let us suppose that x is not an integer, and that n is the integer nearest to x. Then

---

di Palermo, vol. 40, 1915, pp. 1—16). It should be added that this result of PÓLYA appears only incidentally as a corollary of theorems of a somewhat different character and of the highest interest.

[1] E. PHRAGMÉN and E. LINDELÖF, 'Sur une extension d'un principe classique de l'analyse et sur quelques propriétés des fonctions monogènes dans le voisinage d'un point singulier', *Acta Mathematica*, vol. 31, 1908, pp. 381—406.

[2] This follows from the argument of pp. 404—405 of PHRAGMÉN and LINDELÖF's memoir. This argument presupposes that the value of $h(\theta)$ is not always $-\infty$, a possibility excluded by the theorem of p. 385.

$$U(x) = \frac{x-n}{\sin \pi x} u'(\xi_1), \quad V(x) = \frac{x-n}{\sin \pi x} v'(\xi_2),$$

where ξ_1 and ξ_2 are numbers between n and x. It follows that

$$|F(x)| < C\varphi_n,$$

where C is a constant and φ_n is the maximum of $|f'(x)|$ for $n - 1 \leqq x \leqq n + 1$. But

$$f'(x) = \frac{1}{2\pi i}\int \frac{f(z)}{(z-x)^2} dz,$$

where the contour of integration is the circle $|z - x| = 1$; and so

$$|f'(x)| < A e^{k(x+1)}, \quad \varphi_n < A e^{k(n+2)}.$$

Thus

$$|F(x)| < B e^{kx},$$

where B is a constant.

It follows that $H(0) \leqq k$. Hence $H(\theta)$ is continuous for $\theta = 0$, and (1) holds for all values of θ between $-\alpha$ und α. Thus $H(\theta) < 0$ for $\beta < \theta < \gamma$ and $-\gamma < \theta < -\beta$, β being the positive acute angle whose sine is $\frac{k}{\pi}$, and γ the lesser of α and $\pi - \beta$.

But it is easily proved that this is impossible. Suppose first that $\alpha > \frac{1}{2}\pi$. Then $H(\theta)$ cannot be negative for $-\alpha < \theta < \alpha$, since the length of this interval is greater than π.[1] There is therefore, inside this interval, an interval in which $H(\theta)$ is positive.[2] This last interval must form part of the interval $-\beta \leqq \theta \leqq \beta$; and its length is therefore less than π. And this, finally, is impossible.[3]

Secondly, suppose that $\alpha = \frac{1}{2}\pi$. If $H(\theta)$ is not negative for $-\frac{1}{2}\pi < \theta < \frac{1}{2}\pi$, we obtain a contradiction in the same way as before. If $H(\theta)$ is negative for $-\frac{1}{2}\pi < \theta < \frac{1}{2}\pi$, it must be of the form $L\cos\theta + M\sin\theta$.[4] It is plain that L must be negative and M zero, and that $H(\theta)$ must tend to zero as we approach an end of the interval, which is not the case.

---

[1] Phragmén and Lindelöf, *l. c.*, p. 400.
[2] *Ibid.*, p. 399.
[3] *Ibid.*, p. 399.
[4] *Ibid.*, p. 403.

We have therefore arrived in any case at a contradiction, and the theorem is proved. It is easy to see, by considering a function of the form $e^{-cz}\sin\pi z$, that it ceases to be true when $\alpha < \frac{1}{2}\pi$.

4. I shall now give an alternative proof of the theorem based on entirely different ideas. This proof is less elementary than the first, but seems to me to be of some instrinsic interest.

Suppose that w is positive, p a positive integer, $0 < \varkappa < 1$, and $p < \lambda < p+1$. Then it is clear that, under the conditions of the theorem, we have

$$\sum_1^p f(n)(-w)^n = \frac{1}{2\pi i}\int_{\lambda-i\infty}^{\lambda+i\infty}\frac{\pi}{\sin\pi z}f(z)w^z dz - \frac{1}{2\pi i}\int_{\varkappa-i\infty}^{\varkappa+i\infty}\frac{\pi}{\sin\pi z}f(z)w^z dz,$$

the paths of integration being rectilinear.

Let us suppose now that $\lambda = p+\frac{1}{2}$ and that $p\to\infty$. Then

$$\left|\int_{\lambda-i\infty}^{\lambda+i\infty}\frac{f(z)}{\sin\pi z}w^z dz\right| \leqq w^{p+\frac{1}{2}}\int_{-\infty}^{\infty}\frac{\left|f\left(p+\frac{1}{2}+iy\right)\right|}{\cosh\pi y}dy.$$

Also

$$\left|f\left(p+\frac{1}{2}+iy\right)\right| < A\,exp\left[k\sqrt{\left(p+\frac{1}{2}\right)^2+y^2}\right] < Be^{k(p+|y|)},$$

where B is a constant; and so

$$\left|\int_{\lambda-i\infty}^{\lambda+i\infty}\frac{f(z)}{\sin\pi z}w^z dz\right| < C(we^k)^p,$$

where C is another constant. Thus the integral tends to zero if w is sufficiently small. We have therefore

$$(1)\qquad \Phi(w) = wf(1) - w^2 f(2) + w^3 f(3) - \cdots = \frac{1}{2\pi i}\int_{\varkappa-i\infty}^{\varkappa+i\infty}\frac{\pi}{\sin\pi z}f(z)w^z dz,$$

for sufficiently small positive values of w. This formula is of course well-known, and shows that $\Phi(w)$ is an analytic function regular for all positive values of w.

Suppose now that $-1 < s < 0$. Then we can choose μ and ν so that $0 < \nu < -s < \mu < 1$. And we have[1]

$$(2) \qquad \int_0^1 w^{s-1} \Phi(w)\, dw = \frac{1}{2\pi i} \int_0^1 w^{s-1}\, dw \int_{\mu - i\infty}^{\mu + i\infty} \frac{\pi}{\sin \pi z} f(z) w^z\, dz,$$

$$(3) \qquad \int_1^\infty w^{s-1} \Phi(w)\, dw = \frac{1}{2\pi i} \int_1^\infty w^{s-1}\, dw \int_{\nu - i\infty}^{\nu + i\infty} \frac{\pi}{\sin \pi z} f(z) w^z\, dz,$$

provided only that these integrals are convergent.

The double integral

$$\int_0^1 \int_{\mu - i\infty}^{\mu + i\infty} \left| \frac{\pi}{\sin \pi z} f(z) w^{z+s-1}\, dw\, dz \right|$$

is convergent, as may be seen at once by comparison with the integral

$$\int_0^1 \int_{-\infty}^{\infty} w^{\mu+s-1} e^{-(\pi - k)|y|}\, dw\, dy.$$

Hence the integral (2) is convergent, and may be calculated by inversion of the order of integration. The same arguments may be applied to the integral (3). Inverting the order of integration, and combining the results, we obtain

$$\int_0^\infty w^{s-1} \Phi(w)\, dw = \frac{1}{2\pi i} \int_{\mu - i\infty}^{\mu + i\infty} \frac{\pi}{\sin \pi z} \frac{f(z)}{z + s}\, dz - \frac{1}{2\pi i} \int_{\nu - i\infty}^{\nu + i\infty} \frac{\pi}{\sin \pi z} \frac{f(z)}{z + s}\, dz,$$

or

$$(4) \qquad \int_0^\infty w^{s-1} \Phi(w)\, dw = -\frac{\pi}{\sin \pi s} f(-s).$$

This formula has been proved for $-1 < s < 0$. It is equivalent to

[1] This artifice is due to Mellin, from whose work the ideas of the proof are borrowed. See Hj. Mellin, 'Über die fundamentale Wichtigkeit des Satzes von Cauchy für die Theorien der Gamma und der Hypergeometrischen Funktionen; *Acta Societatis Fennicae,* vol. 21, no. 1. 1896, pp. 1—111 (pp. 37 *et seq.*).

$$(5) \qquad \int_0^\infty w^{t-1}(a_0 - a_1 w + a_2 w^2 - \cdots)\, dw = \frac{\pi}{\sin \pi t} a_{-t},$$

where $0 < t < 1$ and $a_z = a(z)$ is an analytic function of z subject to certain conditions. In this form the formula was communicated to me some years ago by Mr S. RAMANUJAN, in ignorance of MELLIN's work.

So far we have made no assumption as to the values of $f(z)$ for integral values of z. It is plain that, if $f(n) = 0$ for $n = 1, 2, 3, \ldots$, we obtain

$$f(-s) = 0 \qquad (-1 < s < 0),$$

so that $f(z)$ is always zero.

II.

5. Suppose that $f(z)$ is an integral function of z such that

$$(1) \qquad |f(z)| < e^{(\gamma+\varepsilon) r}$$

for every positive ε and all sufficiently large values of r,

$$(2) \qquad |f(z)| > e^{(\gamma-\varepsilon) r}$$

for every positive ε and for a corresponding sequence of values of r whose limit is infinity. Then, following PRINGSHEIM,[1] I shall call $f(z)$ an integral function of order 1 and type γ. This being so, the second theorem which I wish to discuss may be stated as follows.

The necessary and sufficient condition that

$$f(z) = a_0 + a_1 z + a_2 z^2 + \cdots$$

should be an integral function of $\frac{1}{1-z}$ *is that there should be an integral function* $a(z)$, *of order* 1 *and type* 0, *which takes the values* $a_1, a_2, a_3, \ldots$ *for* $z = 1, 2, 3, \ldots$

I may insert here a few references to the rather extensive literature connected with this theorem. The complete theorem is due to WIGERT[2]; but the second half of it, asserting the *sufficiency* of the condition, was discovered almost simultaneously, and by a quite different method, by LE ROY.[3] Another proof of this part of the

[1] A. PRINGSHEIM, 'Elementare Theorie der ganzen transcendenten Funktionen von endlicher Ordnung', *Mathematische Annalen*, vol. 58, 1904, pp. 257—342.

[2] S. WIGERT, 'Sur les fonctions entières', *Öfversikt af K. Vet.-Ak. Förhandlingar*, Årg. 57, 1900, pp. 1001—1011.

[3] E. LE ROY, 'Sur les séries divergentes et les fonctions données par un développement de Taylor', *Annales de la Faculté des Sciences de Toulouse*, ser. 2, vol. 2, 1900, pp. 317—430 (pp 350—353).

theorem has been given by LINDELÖF,[1] who deduces it, along with many other important theorems, from the '*formules sommatoires*' of the calculus of residues. The whole theorem was rediscovered at a slightly later date by FABER,[2] whose proof does not differ in principle from WIGERT's. It has been further discussed by PRINGSHEIM,[3] who presents the whole proof in a particularly simple and elementary form. And, as explained in the footnote to § 1, it is a special case of much more general theorems to be found in CARLSON's dissertation.

6. The proof which I give here stands in closest connection with the ideas of LE ROY. I begin by proving the following lemma.

In order that $a(z)$ should be an integral function of order 1 and type 0, it is necessary and sufficient that $a(z)$ should be of the form

$$a(z) = \frac{1}{2\pi i}\int_C e^{zu}\varphi\left(\frac{1}{u}\right)du, \tag{1}$$

where C is a simple contour enclosing the origin, and $\varphi\left(\frac{1}{u}\right) = \varphi(w)$ is an integral function of w.

This lemma is extremely easy to prove and very useful, but I do not remember having seen it stated explicitly. In the first place the condition is sufficient. For we may replace the contour by a circle whose centre is the origin and whose radius is ε, and then

$$|a(z)| = |a(re^{i\theta})| \leqq \varepsilon M e^{\varepsilon r},$$

where M is the maximum of $\left|\varphi\left(\frac{1}{u}\right)\right|$ on the circle.

In the second place the condition is necessary. For if

$$a(z) = \sum_0^\infty c_k z^k$$

is a function of the required type, then[4]

---

[1] E. LINDELÖF, *Le calcul des résidus*, Paris, 1905, p. 127. See also 'Quelques applications d'une formule sommatoire générale', *Acta Societatis Fennicae*, vol. 31, no. 3, 1902, pp. 1—46.

[2] G. FABER, 'Über die Fortsetzbarkeit gewisser Taylorscher Reihen', *Mathematische Annalen*, vol. 57, 1903, pp. 369—388.

[3] A. PRINGSHEIM, 'Über einige funktionentheoretische Anwendungen der Eulerschen Reihen-Transformation', *Münchener Sitzungsberichte*, 1912, pp. 11—92 (pp. 40—45).

[4] See, *e. g.*, PRINGSHEIM, *l. c.*, p. 38.

$$\sqrt[n]{n!\,|c_n|} \to 0;$$

so that

$$\varphi(w) = \sum_0^\infty n!\, c_n w^{n+1}$$

is an integral function of w. Thus

$$a(z) = \sum_0^\infty n!\, c_n \frac{1}{2\pi i} \int_C \frac{e^{zu}}{u^{n+1}}\, du = \frac{1}{2\pi i} \int_C e^{zu} \varphi\left(\frac{1}{u}\right) du.$$

7. We have now to show that $f(z)$ is an integral function of $\frac{1}{1-z}$ if and only if there exists a function of the form (1) which assumes the values $a_1, a_2, \ldots$ for $z = 1, 2, \ldots$

In the first place, if such a function exists, we have

$$(1) \qquad f_1(z) = f(z) - a_0 = \sum_1^\infty a_n z^n = \frac{1}{2\pi i} \int_C \frac{z e^u}{1 - z e^u} \varphi\left(\frac{1}{u}\right) du,$$

if C is a contour enclosing the origin and $|z e^u| < 1$ at all points of C. These conditions will be satisfied if $|z| < 1$ and C lies entirely to the left of the line

$$\Re(u) = \log \left|\frac{1}{z}\right|.$$

The only singularities of the integrand, other than the origin and infinity, are the various values of $\log \frac{1}{z}$; and it follows, by a familiar argument due in principle to HADAMARD,[1] that the only possible singularities of $f_1(z)$ are the values of z for which $\log \frac{1}{z}$ is zero or infinite, that is to say the values 0, 1, and ∞.

Let us draw a cut in the plane of z from 1 to ∞, say along the positive real axis. Then there is a branch $\mathfrak{f}_1(z)$ of $f_1(z)$, the so-called 'principal' branch, which is one-valued and regular in the cut plane and vanishes at the origin. If finally we can show that $f_1(z)$ *is one-valued in the neighbourhood of* $z = 1$, it will follow that $\mathfrak{f}_1(z)$ is the *only* branch of $f_1(z)$, and so that $f_1(z)$ is a one-valued

[1] J. HADAMARD, 'Essai sur l'étude des fonctions données par leur développement de Taylor', *Journal de mathématiques*, ser. 4, vol. 8, 1892, pp. 101—186.

function with $z = 1$ as its sole finite singularity, that is to say an integral function of $\frac{1}{1-z}$.

Suppose then, to fix our ideas, that z tends to 1 through positive values less than 1, that C is a circle whose centre is the origin and whose radius is less than $\log\left|\frac{1}{z}\right|$, and that C' is a concentric circle whose radius is greater than $\log\left|\frac{1}{z}\right|$. Then[1]

$$\mathfrak{f}_1(z) = \frac{1}{2\pi i}\int_C \frac{ze^u}{1-ze^u}\varphi\left(\frac{1}{u}\right)du$$

$$= \varphi\left(\frac{1}{\log\frac{1}{z}}\right) + \frac{1}{2\pi i}\int_{C'} \frac{ze^u}{1-ze^u}\varphi\left(\frac{1}{u}\right)du,$$

where $\log\frac{1}{z}$ denotes, of course, the value of the logarithm which vanishes for $z = 1$. It is plain that the last integral representents a function regular for $z = 1$. As

$$\varphi\left(\frac{1}{\log\frac{1}{z}}\right)$$

is one-valued in the neighbourhood of $z = 1$, so also is $f_1(z)$. Thus one half of the theorem is proved.

8. Secondly, let us suppose that $f(z)$ is a function of the type prescribed. We have

$$a_n = \frac{1}{2\pi i}\int \frac{f(z)}{z^{n+1}}dz, \tag{1}$$

the path of integration being a closed contour surrounding the origin, but excluding the point $z = 1$. It is evident that, if $n \geqq 1$,[2] we may deform this contour into that formed by (1) the right hand half of the circle

$$|z| = e^{-\gamma},$$

[1] Compare G. H. Hardy, 'A method for determining the behaviour of a function represented by a power series near a singular point on the circle of convergence', *Proc. London Math. Soc.*, ser. 2, vol. 3, 1905, pp. 381—389.

[2] The argument fails for $n = 0$ unless $f(\infty) = 0$. Compare Wigert's paper.

γ being any positive number, and (2) the parts of the imaginary axis which stretch from the ends of this semicircle to infinity.

Let us now effect the transformation $z = e^{-u}$, $u = \log \frac{1}{z}$, where that value of the logarithm is chosen whose imaginary part lies between $-\pi$ and π. The contour in the z-plane becomes a contour in the u-plane formed by three sides of an infinite rectangle whose vertices are

$$\gamma + \frac{1}{2}\pi i, \quad -\infty + \frac{1}{2}\pi i, \quad -\infty - \frac{1}{2}\pi i, \quad -\gamma - \frac{1}{2}\pi i$$

and we have

$$a_n = \frac{1}{2\pi i}\int e^{nu} f(e^{-u})\,du,$$

the integration being effected along this contour. It is obvious that the contour may now be deformed into any simple closed contour which encloses the origin but lies entirely inside the circle $|u| = 2\pi$. Finally, $f(e^{-u})$ is plainly regular except for $u = 0$ and $u = \pm 2k\pi i$ ($k = 1, 2, 3, \ldots$), and is therefore of the form

$$f(e^{-u}) = \varphi\left(\frac{1}{u}\right) + \psi(u),$$

where $\varphi(w)$ is an integral function of w and $\psi(u)$ is a power series whose radius of convergence is at least 2π. And

$$a_n = \frac{1}{2\pi i}\int e^{nu}\varphi\left(\frac{1}{u}\right) du,$$

which proves the theorem.

9. The preceding proof of Wigert's theorem is of course less elementary than (for example) Pringsheim's. It seems to me interesting none the less on account of its almost intuitive character. It has the further advantage of lending itself very readily to generalisation, as I shall proceed to show.

In the first place, the lemma of § 6 may be at once generalised as follows:

In order that $a(z)$ should be an integral function of order 1 and type γ, it is necessary and sufficient that $a(z)$ should be of the form

$$a(z) = \frac{1}{2\pi i}\int_C e^{zu}\varphi\left(\frac{1}{u}\right) du,$$

where C is a contour which includes the circle $|u| = \gamma$, *and* $\varphi(w)$ *is a function regular for* $|w| < \frac{1}{\gamma}$ $\left(\text{but not for } |w| \leqq \frac{1}{\gamma}\right)$.

To prove this we observe that $a(z) = \Sigma c_n z^n$ will be a function of the type required if, and only if,

$$\limsup_{n \to \infty} \sqrt[n]{n!\,|c_n|} = \gamma,^{1}$$

so that the radius of convergence of $\Sigma n!\, c_n z^n$ is precisely γ. The proof is then practically the same as that of § 6.

We can now express

$$f_1(z) = f(z) - a_0 = \sum_1^\infty a_n z^n$$

in the form (1) of § 7. The possible singularities of $f_1(z)$ are now 0, ∞, and the values of z for which $\left|\log \frac{1}{z}\right| \leqq \gamma$. The latter values cover the interior of the curve defined by the equation

$$\left|\log \frac{1}{z}\right| = \gamma. \tag{1}$$

We suppose that $0 < \gamma < \pi$ and $-\pi < \theta < \pi$. Then the curve defined by (1) has the polar equation

$$\left(\log \frac{1}{r}\right)^2 = \gamma^2 - \theta^2, \tag{2}$$

and consists of a single loop, enclosing the point $r = 1$, $\theta = 0$, and cutting the unit circle where $\theta = \pm \gamma$. The function $f(z)$ is regular outside this curve, and has a branch regular at the origin and at infinity. The curve may be the boundary of existence of the function: in this case the function is one-valued. But in other circumstances the function may have other branches of which 0 or ∞ are singular points.[2]

10. I shall now suppose that $f(z)$ is a branch of an analytic function, one-valued and regular throughout the region exterior to the curve (2), including

---

[1] See pp. 337—342 of PRINGSHEIM's paper in the *Mathematische Annalen* quoted above.

[2] The substance of these results is contained in the work of LE ROY and LINDELÖF. Cf. LE ROY, *l. c.*, and LINDELÖF, *Le calcul des residus*, pp. 135—136. A less complete result is given by PRINGSHEIM: see p. 46 of his paper in the *Münchener Sitzungsberichte* already referred to.

infinity; and I shall show that in these circumstances there is an integral function $a(z)$, of oder 1 and type γ, which assumes the values $a_1, a_2, a_3, \dots$ for $z = 1, 2, 3, \dots$

We start from the formula (1) of § 8, and deform the path of integration into one of the same general character as that used in § 8, but so constructed as to leave the curve (2) entirely on its right. We may take the contour, for example, to be formed by part of the circle $r = e^{-\delta}$, where $\delta > \gamma$, and parts of the radii $\theta = \pm\lambda$, where $\gamma < \lambda < \pi$. This contour transforms, as in § 8, into a quasi-rectangular contour, which now lies entirely outside the circles $|u| = \gamma$ and $|u \pm 2k\pi i| = \gamma$ $(k = 1, 2, \dots)$. We thus obtain

$$a_n = \frac{1}{2\pi i}\int e^{nu} f(e^{-u})\,du,$$

where $f(e^{-u})$ is regular in the region exterior to the circles just referred to. We can express $f(e^{-u})$ in the form of a LAURENT's series

$$\sum_{-\infty}^{\infty} b_n u^n = \sum_{-\infty}^{-1} b_n u^n + \sum_{0}^{\infty} b_n u^n = \varphi\left(\frac{1}{u}\right) + \psi(u),$$

the first series being convergent for $|u| > \gamma$ and the second for $|u| < 2\pi - \gamma$; and plainly

$$a_n = \frac{1}{2\pi i}\int e^{nu}\varphi\left(\frac{1}{u}\right)du,$$

where now the path of integration is any closed contour at all points of which $|u| > \gamma$.

We have thus proved the following theorem.

The necessary and sufficient condition that $f(z) = \Sigma a_n z^n$ should be a one-valued branch of an analytic function, regular in the region exterior to the curve

$$\left(\log\frac{1}{r}\right)^2 = \gamma^2 - \theta^2 \qquad (0 < \gamma < \pi),$$

and including infinity, but not in any more extensive region of the same character, is that there should be an integral function $a(z)$, of order 1 and type γ, which assumes the values $a_1, a_2, \dots$ for $z = 1, 2, \dots$

The function $a(z)$ is, in virtue of WIGERT's first theorem, unique. It is plain that the theorem ceases to be true if $\gamma \geqq \pi$. The critical curve then is no longer a simple loop surrounded by an open infinite region, and there are infinitely many different functions which have the properties specified.

The proof of the theorem which I have given seems to be that which is most in conformity with the general ideas of this note. But it can also be proved by an argument more on the lines of WIGERT's note and depending upon the properties of the functions

$$\psi_n(z) = 1^n z + 2^n z^2 + 3^n z^3 + \cdots = \left(z\frac{d}{dz}\right)^n \frac{1}{1-z}.$$

Chelsea, London, August 1917.