ON CANONICAL FORMS

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[THIS paper consists of the second part of a dissertation "General Theorems and Canonical Forms," written by Mr. Wakeford in the spring of 1916. As it was almost complete in itself it has been thought best to print it separately, with a few words of explanation to enable the reader to grasp the principle on which the argument rests.

Suppose it is required to show that a general ternary cubic in x, y, z can, by a linear transformation, be reduced to the form

$$X^3+Y^3+Z^3+6mXYZ;$$

then we have to *identify* the given cubic with

$$(\alpha_1x+\alpha_2y+\alpha_3z)^3+(\beta_1x+\beta_2y+\beta_3z)^3+(\gamma_1x+\gamma_2y+\gamma_3z)^3$$
$$+6m(\alpha_1x+\alpha_2y+\alpha_3z)(\beta_1x+\beta_2y+\beta_3z)(\gamma_1x+\gamma_2y+\gamma_3z),$$

i.e. we have to show that the last expression is capable of representing the general cubic in x, y, z.

Writing it as $$\Sigma\, a_{pqr}\,x^py^qz^r,$$

it suffices to prove that the a's are independent functions of the ten variables consisting of the α's, β's, γ's and m.

This amounts to proving that the Jacobian does not vanish identically, but it will be seen (§ 2) that Mr. Wakeford does not use the Jacobian explicitly. He rather assumes the existence of a relation between the a's, and makes a deduction therefrom which is easily turned into a test to determine whether a proposed canonical form is possible or not.*

* The idea of using the Jacobian seems to be due to Kronecker: cf. Lasker, *Math. Annalen*, Vol. 58 (1904), pp. 434–446. The methods there used are the same as Mr. Wakeford's in principle, but the form is not so convenient. Several references being made to Elliott (*Algebra of Quantics*) and Richmond ("On Canonical Forms," *Quarterly Journal of Math.*, Vol. 33 (1902), pp. 331–340), they are quoted as "Quantics" and "Richmond" respectively.

The writer's thesis is to establish the possibility or otherwise of a given reduction not to find the reducing process or the number of solutions.—J. H. G.]

1. The problem of reducing a q-ary p-ic, that is to say a quantic of order p in q variables, to a canonical form, consists of expressing the quantic

$$\sum_{s=1}^{m} a_s x_1^{{}_sp_1} x_2^{{}_sp_2} \dots x_q^{{}_sp_q}$$

in the proposed form

$$F \equiv \sum_{s=1}^{m} x_1^{{}_sp_1} x_2^{{}_sp_2} \dots x_q^{{}_sp_q} f_s(l_1, l_2, \dots, l_n),$$

where $l_1, l_2, \dots, l_n$ are (usually) independent variables, and

$${}_sp_1 + {}_sp_2 + \dots + {}_sp_q = p \quad (s = 1, 2, \dots, m).$$

It will be noticed that F appears to differ from some proposed forms in that no indefinite linear forms occur in it. If such a form X should occur, write

$$X \equiv x_1\lambda_1 + x_2\lambda_2 + \dots + x_q\lambda_q,$$

and the coefficients λ take their place among the variables l. Thus any number of indefinite quantics of linear or higher order may be disposed of, and F obtained in the form above.

2. The proposed form F is or is not canonical according as the functions $f_1, f_2, \dots, f_m$ are or are not independent.

Suppose that the form is not canonical, so that a relation $\psi(f) = 0$ exists. Then

$$\frac{\partial f_1}{\partial l_r}\frac{\partial \psi}{\partial f_1} + \frac{\partial f_2}{\partial l_r}\frac{\partial \psi}{\partial f_2} + \dots + \frac{\partial f_m}{\partial l_r}\frac{\partial \psi}{\partial f_m} = 0 \quad (r = 1, 2, \dots, n). \qquad (1)$$

Now
$$\frac{\partial F}{\partial l_r} \equiv \sum_{s=1}^{m} \frac{\partial f_2}{\partial l_r} x_1^{{}_sp_1} x_2^{{}_sp_2} \dots x_q^{{}_sp_q}.$$

The contragredient quantic Φ defined by

$$\Phi \equiv \sum_{s=1}^{m} \frac{\partial \psi}{\partial f_1} u_1^{{}_sp} u_2^{{}_sp_2} \dots u_q^{{}_sp_q} \frac{p!}{{}_sp_1!\, {}_sp_2! \dots {}_sp_q!}$$

where $u_1, u_2, \ldots, u_q$ correspond with $x_1, x_2, \ldots, x_q$ respectively, is apolar to $\partial F/\partial l_r$, in virtue of relation (1). Hence if F is not canonical the quantics $\partial F/\partial l_r$ are all apolar to a certain contragredient quantic. Note that this apolar form must exist for all values of l. [If $\partial\psi/\partial f_s$ $(s = 1, 2, \ldots, m)$ vanishes for a certain set of values of $l_1, \ldots, l_n$, *two* apolar forms instead of one are obtained.] In order therefore to prove that F is canonical, it is only necessary to find a particular set of values of $l_1, l_2, \ldots, l_n$, so that the quantics $\partial F/\partial l_r$ have no form apolar to them.

Again, if such an apolar form Φ exists for general values of l, the relations following hold good for suitable values of λ,

$$\frac{\partial f_1}{\partial l_1}\lambda_1+\frac{\partial f_2}{\partial l_1}\lambda_2+\ldots+\frac{\partial f_m}{\partial l_1}\lambda_m=0,$$

$$\frac{\partial f_1}{\partial l_2}\lambda_1+\frac{\partial f_2}{\partial l_2}\lambda_2+\ldots+\frac{\partial f_m}{\partial l_2}\lambda_m=0,$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

$$\frac{\partial f_1}{\partial l_n}\lambda_1+\frac{\partial f_2}{\partial l_n}\lambda_2+\ldots+\frac{\partial f_m}{\partial l_n}\lambda_m=0.$$

The Jacobian of the m functions with respect to any m of the n variables l must therefore vanish in general, and so vanish always. Hence the functions are not independent, and the form is not canonical. If there is no apolar form, then

$$\lambda_1\frac{\partial F}{\partial l_1}+\lambda_2\frac{\partial F}{\partial l_2}+\ldots+\lambda_n\frac{\partial F}{\partial l_n}$$

can be made to represent any q-ary p-ic for suitable values of $\lambda_1, \lambda_2, \ldots, \lambda_n$, and conversely.

3. It has hitherto been required that all implicit parameters in F, such as those contained in a linear form X, should be written out explicitly: *e.g.*

$$X \equiv \lambda_1 x_1+\lambda_2 x_2+\ldots+\lambda_q x_q.$$

This may be avoided, for if

$$\frac{\partial F}{\partial X_r} \equiv x_r\frac{\partial F}{\partial X} \quad (r = 1, 2, \ldots, q)$$

is apolar to Φ, the quantic $\partial F/\partial X$ of order $p-1$ is apolar to Φ: and conversely if $\partial F/\partial X$ is apolar to Φ, then $x_r . \partial F/\partial X$ is also apolar to Φ.

Thus suppose the quantic is the binary quartic, and

$$F \equiv X^4+6mX^2Y^2+Y^4.$$

Then
$$\tfrac{1}{4}\frac{\partial F}{\partial X} \equiv X^3+3mXY^2,$$

$$\tfrac{1}{4}\frac{\partial F}{\partial Y} \equiv 3mX^2Y+Y^3,$$

and
$$\tfrac{1}{6}\frac{\partial F}{\partial m} \equiv X^2Y^2,$$

are apolar to Φ. Put $m = 0$, and it is clear that Φ does not exist, and the form is canonical, since no binary quartic has X^3, Y^3, and X^2Y^2 apolar to it.

A similar method may be adopted in case an indefinite quantic of the second or higher order occurs in F. Thus in order to justify

$$F \equiv c_2c_3-c_1^2,$$

for the ternary quartic, where c_1, c_2, c_3 are conics, it is only necessary to consider that

$$-\tfrac{1}{2}\frac{\partial F}{\partial c_1} \equiv c_1,$$

$$\frac{\partial F}{\partial c_2} \equiv c_3,$$

$$\frac{\partial F}{\partial c_3} \equiv c_2\,;$$

and there is not always a class quartic to which those conics are apolar, as may be seen by considering the case

$$c_1 \equiv x_1^2, \quad c_2 \equiv x_2^2, \quad c_3 \equiv x_3^2.$$

Now take the case when the proposed form is the sum of n perfect p-th powers of linear form, that is to say

$$F \equiv X_1^p+X_2^p+\ldots+X_n^p.$$

If Φ exists X_r^{p-1} ($r = 1, 2, \ldots, n$) is apolar to it. It will now be convenient to reciprocate the problem. Suppose S is a quantic of order p in

$x_1, x_2, \ldots, x_q$, and U^{p-1} is apolar to S. Considering S as denoting a hyper-surface, and U a point, it follows that the $(p-1)$-th polar of the hyper-surface with respect to the point vanishes, so that the point is a double point on the hyper-surface. Hence, if in the original problem Φ exists, there must be a hyper-surface of order p having n arbitrary double points. If for one set of points there is no such hyper-surface, F is canonical. If, on the other hand, a method of finding such a surface for the general case can be given, F is not canonical. A number of well known results follow immediately from this geometrical aspect of the problem, which often enables a solution to be obtained without putting pen to paper.

(1) The binary $(2n-1)$-ic can be expressed as the sum of n form X_r^{2n-1}, for no $(2n-1)$-ic can have double points at n different points.

(2) The ternary quadratic cannot be expressed as the sum of two squares, for the square of the line joining two points is a conic with double points at both of them.

(3) The ternary quartic cannot be expressed as the sum of five fourth powers, for the square of the conic through five points is a quartic with double points at all of them. [*Quantics*, § 230 ; *Richmond*, § 9.]

(4) The ternary quintic can be expressed as the sum of seven fifth powers, for if no six of seven points lie on a conic, it is easy to show that no quintic can have nodes at all of them. [*Richmond*, § 10.]

(5) The quaternary cubic can be expressed as the sum of five cubes, for no cubic surface can have five double points of which four do not lie on a plane. [Consider twisted cubics through the five points; *Richmond*, § 11.]

(6) The quinary cubic cannot be expressed as the sum of seven cubes, for through seven points in four dimensions passes a quartic curve, the chords of which generate a cubic hyper-surface containing the curve as a double curve. [*Richmond*, § 12.]

4. *The binary 2n-ic,*

$$F \equiv X_1^{2n} + X_2^{2n} + \ldots + X_n^{2n} + \frac{2n!}{n!} m X_1^2 X_2^2 \ldots X_n^2.$$

In this case Φ, if it exists, has

$$\frac{1}{2n}\frac{\partial F}{\partial X_r} \equiv X_1^{2m-1} + \frac{(2n-1)!}{2^{n-1}} m X_1^2 \dots X_{r-1}^2 X_r X_{r+1}^2 \dots X_n^2,$$

and

$$\frac{\partial F}{\partial m} \equiv \frac{2n!}{2^n} X_1^2 X_2^2 \dots X_n^2,$$

apolar to it. Put $m = 0$. Then Φ is a $2n$-ic with $X_1, X_2, \dots, X_n$ as double points, that is Φ is $X_1^2 X_2^2 \dots X_n^2$. But this form is not always apolar to itself, unless $n = 1$; hence the proposed form F is canonical for all binary $2n$-ics except the quadratic.

The number of ways in which this reduction can be performed is interesting. For $n = 2, 3, 4$, the numbers are 3, 8, 5 respectively. [*Quantics*, § 211; Wakeford, "A Canonical Form of the Binary Sextic," *Messenger of Mathematics*, Vol. 43 (1914), pp. 25–28; and *Quantics*, § 227.]

It is clear that instead of $X_1^2 X_2^2 \dots X_n^2$, any function of $X_1, X_2, \dots, X_n$ which is not always apolar to $X_1^2 X_2^2 \dots X_n^2$ may be taken. [*Quantics*, § 225.]

5. Any quantic of order $p > 1$ may have its terms of the form $x_1^{p-1} x_2$ removed. Let F be the general p-ic in q variables $X_1, X_2, \dots, X_q$, without such terms as $X_1^{p-1} X_2$.

Consider any term of F, *e.g.* $k X_1^{p-2} X_2 X_3$. Φ if it exists will be apolar to $\partial F/\partial k$, *i.e.* $X_1^{p-2} X_2 X_3$. Now Φ may be written in terms of $U_1, U_2, \dots, U_q$, where U_1 is the common point of the linear forms $X_2, X_3, \dots, X_q$. Written thus, Φ evidently does not contain the term $U_1^{p-2} U_2 U_3$. Similarly for all the other terms of F. Hence Φ consists entirely of such terms as $U_1^{p-1} U_2$.

Consider $\partial F/\partial X_1$, which is apolar to Φ. Put the coefficients of all the terms of F zero, except the terms X_r^p. Then $\partial F/\partial X_1$ is $k X_1^{p-1}$ So Φ can contain no coefficient of U_1^{p-1}, or similarly of U_r^{p-1} Hence Φ does not exist, since it has been shown to contain no terms except those of the form $U_1^{p-1} U_2$. Hence the proposed form is canonical. Particular cases are canonical forms of the binary cubic, quartic, and quintic, and of the ternary cubic.

A similar proof shows that all terms of the form $x_1^{p-r} x_2^r$ can be removed, where r is any fixed number. The following question now arises:—*Is* *there any ready test by which, given $q(q-2)$ terms of a quantic, it can be decided whether they are removable or not?*

It is easy to prove by the method above that the following test is *necessary* :—*It must be possible to associate with each of the terms to be removed a different one of the ratios*

$$x_r/x_s \quad (r = 1, 2, \ldots, q\,;\ s = 1, 2, \ldots, q\,;\ r \neq s),$$

so that the product in each case is a term which is not to be removed.

I cannot prove this condition sufficient, though it seems to be so.

The following particular form is certainly sufficient :—*Choose any set of terms which are prime to one another, that is to say such that the same x_r cannot occur in two of them. These terms may be "isolated" (except in the case of linear forms), i.e. all terms which can be found by multiplying them by x_r/x_s may be removed.*

For instance, $x_1^p, x_2^p, \ldots, x_n^p$ are isolated if the terms $x_r^{p-1}x_s$ are removed.

6. *The lines on a cubic surface.*—It may be inadvisable to make explicit use of the apolar form Φ. It order to dispense with it, note that since the general q-ary p-ic contains m terms, and the forms $\partial F/\partial l_r$ are all apolar to Φ, a syzygy must exist between any m of the n forms $\partial F/\partial l_r$ Conversely, if that is true, Φ must exist. The simplest case is where $n = m$, then the form is canonical or not according as there does or does not exist a syzygy between the forms $\partial F/\partial l_r$. In dealing with a linear form X, the term in the syzygy corresponding to $\partial F/\partial X$ is $X' . \partial F/\partial X$, where X' is an arbitrary linear form. The following example is written out at length.

The study of the lines on a cubic surface may be started by taking

$$F \equiv X_1X_2X_3 - X_4X_5X_6$$

as the canonical form of a cubic surface. In order to justify this, write

$$X_r \equiv l_rx_1 + m_rx_2 + n_rx_3 + p_rx_4 \quad (r = 1, 2, \ldots, 6\,;\ p_r \neq 0).$$

Consider the twenty cubic surfaces

$$\begin{array}{lllllll} X_1X_2X_3, & X_2X_3x_1, & X_2X_3x_2, & X_2X_3x_3, & X_3X_1x_1, & X_3X_1x_2, & X_3X_1x_3, \\ X_1X_2x_1, & X_1X_2x_2, & X_1X_2x_3, & X_4X_5X_6, & X_5X_6x_4, & X_5X_6x_5, & X_5X_6x_6, \\ X_6X_4x_1, & X_6X_4x_2, & X_6X_4x_3, & X_4X_5x_1, & X_4X_5x_2, & X_4X_5x_3. & \end{array}$$

These surfaces are all apolar to Φ, if it exists. Hence, since there are only twenty terms in the equation of a cubic surface, a syzygy must connect the surfaces, viz.

$$X_2X_3X_1'+X_3X_1X_2'+X_1X_2X_3' \equiv X_5X_6X_4'+X_6X_4X_5'+X_4X_4X_6',$$

where neither side can vanish, since $p_r \neq 0$.

Consider the cubic surface represented by both sides of this syzygy. It contains the six lines in which X_2X_3, X_3X_1, X_1X_2, X_5X_6, X_6X_4, X_4X_5 respectively intersect. The plane X_1 meets the surface in two lines, viz. X_1X_2, X_1X_3. The three lines X_5X_6, X_6X_4, X_4X_5 accordingly meet X_1 in collinear points (or else on one of these two lines), and this is not the case for all sets of six planes X_r. Hence the syzygy is impossible and the form canonical.