Mit Hülfe der beiden Stetigkeitsaxiome V. 1 und V. 2 ist es noch nicht möglich, die ebene Geometrie als identisch mit der gewöhnlichen analytischen "Cartesischen" Geometrie nachzuweisen, in der *jedem* Zahlenpaar ein Punkt der Ebene entspricht. Zu diesem Nachweise ist aber weder das Axiom vom Dedekindschen Schnitte, noch das Axiom von der Existenz der Grenzpunkte notwendig; vielmehr genügt dasjenige Axiom allgemein philosophischen Charakters, welches ich bereits in der französischen Uebersetzung meiner *Grundlagen der Geometrie** aufgestellt habe, und welches wie folgt lautet:

V. 3. *Axiom der Vollständigkeit.*

Zu dem System der Punkte und Geraden ist es nicht möglich ein anderes System von Dingen hinzuzufügen, so dass in dem durch Zusammensetzung entstehenden System sämtliche aufgeführten Axiome I.–IV., V. 1, 2 erfüllt sind; oder kurz:

Die Elemente der Geometrie bilden ein System von Dingen, welches bei Aufrechterhaltung sämtlicher Axiome keiner Erweiterung mehr fähig ist.

On the Groups defined for an Arbitrary Field by the Multiplication Tables of certain Finite Groups. By L. E. DICKSON.

Received and read April 10th, 1902. Received, in modified form,† October 20th, 1902.

In his paper‡ "On the Continuous Group that is defined by any given Group of Finite Order," Burnside obtained by a new method the most fundamental ones of the theorems of Frobenius§ on group-matrices and group-determinants, and gave a new aspect to the theory. A more general theory for an arbitrary field (domain of rationality) was later developed by the writer.|| An exceptional case

* *Annales de l'Ecole Normale*, 1900. Vgl. auch meinen Vortrag über den Zahlbegriff: *Berichte der deutschen Mathematiker-Vereinigung*, 1900.

† The original manuscript was lost in transit. From rough notes on a part of it, the present paper has been prepared.

‡ *Proc. Lond. Math. Soc.*, Vol. XXIX., pp. 207–224, 546–565.

§ *Berliner Sitzungsberichte*, 1896, pp. 985–1021, 1343–1382; 1897, pp. 994–1015; 1898, pp. 501–515; 1899, pp. 330–339, 482–500; 1900, pp. 516–534; 1901, pp. 303–315. An elementary exposition of Frobenius's theory is given by the writer in the *Annals of Mathematics*, October, 1902.

|| *Trans. Amer. Math. Soc.*, Vol. III. (1902), pp. 285–301. Additional developments are given in the *University of Chicago Decennial Publications*, Vol. IX., pp. 35–51 (separate preprints, pp. 1–17, Oct. 1, 1902).

not treated in the paper just cited is that of a field having a modulus which divides the order of the given finite group. As this exceptional case appears to offer special difficulties, it seems desirable to have a detailed treatment of a number of examples both for the general case and for the various exceptional cases. Such are the contents of §§ 1–7; the methods employed are very elementary and entirely independent of the general theory. The next step in the extension of the general theory is very naturally a detailed study of the explicit developments in the earlier investigations which cease to hold true for the above exceptional cases. The contents of §§ 8–12 are of this nature.

Canonical Forms of the Group-Matrix for the Symmetric Group g_6 on Three Letters.

1. The body of a left-hand multiplication table for g_6 is*

$$(1)\quad \begin{Bmatrix} I & \alpha & \beta & \gamma & \delta & \epsilon \\ \beta & I & \alpha & \delta & \epsilon & \gamma \\ \alpha & \beta & I & \epsilon & \gamma & \delta \\ \gamma & \delta & \epsilon & I & \alpha & \beta \\ \delta & \epsilon & \gamma & \beta & I & \alpha \\ \epsilon & \gamma & \delta & \alpha & \beta & I \end{Bmatrix}.$$

Let T_1 be a linear transformation on $\xi_1, \ldots, \xi_6$, the matrix of whose coefficients is of the form (1) with $I, \alpha, \ldots, \epsilon$ in a given field F.

Employing the standard notation, let $B_{i,j,\lambda}$ denote the linear transformation which alters only the variable ξ_i, replacing it by $\xi_i + \lambda\xi_j$.

The inverse of $B_{i,j,\lambda}$ is evidently $B_{i,j,-\lambda}$. Transforming T_1, of matrix (1), by

$$(t_1)\quad B_{2,1,-1}B_{3,1,-1}B_{4,1,-1}B_{5,1,-1}B_{6,1,-1},$$

we obtain a transformation T_2 with the matrix of coefficients

$$(2)\quad \begin{Bmatrix} \rho & \alpha & \beta & \gamma & \delta & \epsilon \\ 0 & I-\alpha & \alpha-\beta & \delta-\gamma & \epsilon-\delta & \gamma-\epsilon \\ 0 & \beta-\alpha & I-\beta & \epsilon-\gamma & \gamma-\delta & \delta-\epsilon \\ 0 & \delta-\alpha & \epsilon-\beta & I-\gamma & \alpha-\delta & \beta-\epsilon \\ 0 & \epsilon-\alpha & \gamma-\beta & \beta-\gamma & I-\delta & \alpha-\epsilon \\ 0 & \gamma-\alpha & \delta-\beta & \alpha-\gamma & \beta-\delta & I-\epsilon \end{Bmatrix},$$

* Weber, *Algebra*, second edition, Vol. II., p. 124.

where ρ and ρ_1, used below, have the values

$$\rho = I+a+\beta+\gamma+\delta+\epsilon, \quad \rho_1 = I+a+\beta-\gamma-\delta-\epsilon.$$

Transforming T_2, of matrix (2), by the product

$$(t_2) \quad B_{2,3,1}\, B_{2,4,-1}\, B_{2,5,-1}\, B_{2,6,-1},$$

we obtain a transformation T_3 with the matrix

$$(3) \quad \begin{Bmatrix} \rho & a & \beta-a & \gamma+a & \delta+a & \epsilon+a \\ 0 & \rho_1 & 0 & 0 & 0 & 0 \\ 0 & \beta-a & I-2\beta+a & \beta+\epsilon-a-\gamma & \beta+\gamma-a-\delta & \beta+\delta-a-\epsilon \\ 0 & \delta-a & a-\beta-\delta+\epsilon & I-a-\gamma+\delta & 0 & \beta+\delta-a-\epsilon \\ 0 & \epsilon-a & a-\beta+\gamma-\epsilon & \beta+\epsilon-a-\gamma & I-a-\delta+\epsilon & 0 \\ 0 & \gamma-a & a-\beta-\gamma+\delta & 0 & \beta+\gamma-a-\delta & I-a+\gamma-\epsilon \end{Bmatrix}.$$

Transforming T_3, of matrix (3), by the product

$$(t_3) \quad B_{3,4,-1}\, B_{6,3,-1},$$

we obtain a transformation T_4 with the matrix

$$(4) \quad \begin{Bmatrix} \rho & a & \beta+\epsilon & \beta+\gamma & \delta+a & \epsilon+a \\ 0 & \rho_1 & 0 & 0 & 0 & 0 \\ 0 & \beta-\delta & I-\beta+\delta-\epsilon & 0 & \beta+\gamma-a-\delta & 0 \\ 0 & \delta-a & 0 & I+\epsilon-\beta-\gamma & 0 & \beta+\delta-a-\epsilon \\ 0 & \epsilon-a & a+\gamma-\beta-\epsilon & 0 & I+\epsilon-a-\delta & 0 \\ 0 & \gamma+\delta-a-\beta & 0 & a+\delta-\beta-\gamma & 0 & I+\gamma-a-\epsilon \end{Bmatrix}.$$

If F does not have modulus 2 or 3, we transform T_4 by

$$(t_4) \quad \xi_1' = \xi_1+\tfrac{1}{6}\xi_2+\tfrac{1}{3}(\xi_3+\xi_4+\xi_5+\xi_6), \quad \xi_i' = \xi_i \ (i>1),$$

and obtain a transformation T_5 whose matrix differs from (4) only in the first row, that being now $\rho\ 0\ 0\ 0\ 0\ 0$. Transforming T_5 by $t_5 \equiv B_{4,6,-1}$, we obtain a transformation T_6 whose matrix differs from that of T_5 only in the fourth and sixth rows, those being now

$$\begin{matrix} 0 & \beta-\gamma & 0 & I-a-\delta-\epsilon & 0 & \beta+\epsilon-a-\gamma, \\ 0 & \gamma+\delta-a-\beta & 0 & a+\delta-\beta-\gamma & 0 & I+\delta-\beta-\epsilon. \end{matrix}$$

Transforming T_6 by

$$(t_6) \quad \xi_3' = \xi_3-\tfrac{1}{3}\xi_2, \quad \xi_4' = \xi_4-\tfrac{1}{3}\xi_2, \quad \xi_5' = \xi_5+\tfrac{1}{3}\xi_2, \quad \xi_6' = \xi_6+\tfrac{2}{3}\xi_2,$$

we obtain a transformation T_7 whose matrix differs from that of T_6 only in the second column, that now having all zeros except ρ_1 in the second row. Transforming T_7 by

$$(t_7) \quad \xi_4' = -\xi_5, \quad \xi_5' = \xi_6, \quad \xi_6' = \xi_4,$$

we obtain a transformation with the matrix M:

$$M \equiv \begin{Bmatrix} \rho & 0 & 0 & 0 & 0 & 0 \\ 0 & \rho_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & a & b & 0 & 0 \\ 0 & 0 & c & d & 0 & 0 \\ 0 & 0 & 0 & 0 & a & b \\ 0 & 0 & 0 & 0 & c & d \end{Bmatrix} \qquad \begin{pmatrix} a = I-\beta+\delta-\epsilon, \\ b = \alpha-\beta+\delta-\gamma, \\ c = -\alpha+\beta-\gamma+\epsilon, \\ d = I-\alpha-\delta+\epsilon \end{pmatrix}.$$

The product $t = t_1 t_2 t_3 t_4 t_5 t_6 t_7$ is found to be

$$\xi_1' = \tfrac{1}{6}(\xi_1+\xi_2+\xi_3+\xi_4+\xi_5+\xi_6), \qquad \xi_2' = \xi_1+\xi_2+\xi_3-\xi_4-\xi_5-\xi_6,$$
$$\xi_3' = -\tfrac{1}{3}(\xi_1+\xi_2-2\xi_3+2\xi_4-\xi_5-\xi_6), \quad \xi_4' = \tfrac{1}{3}(2\xi_1-\xi_2-\xi_3+\xi_4-2\xi_5+\xi_6),$$
$$\xi_5' = -\tfrac{1}{3}(\xi_1-2\xi_2+\xi_3-\xi_4+2\xi_5-\xi_6), \quad \xi_6' = -\tfrac{1}{3}(\xi_1+\xi_2-2\xi_3-\xi_4-\xi_5+2\xi_6).$$

This transformation* t therefore transforms T_1, of matrix (1), into a transformation of the canonical form M, the cases of moduli 2 and 3 being excluded. The six functions ρ, ρ_1, a, b, c, d are readily seen to be linearly independent. The group of transformations of matrix (1) is therefore simply isomorphic with the product of two general unary linear groups and the general binary linear group, all with coefficients in the given field F, and affecting different variables.

2. Let next F have modulus 2. Transforming T_2, of matrix (2), by

$$\xi_1' = \xi_1+\xi_4+\xi_5+\xi_6, \quad \xi_2' = \xi_2+\xi_3+\xi_4+\xi_5+\xi_6, \quad \xi_i' = \xi_i \quad (i>2),$$

we obtain a transformation of matrix

$$\begin{Bmatrix} \rho & \gamma+\delta+\epsilon & 0 & 0 & 0 & 0 \\ 0 & \rho & 0 & 0 & 0 & 0 \\ 0 & \alpha+\beta & I+\alpha & \alpha+\beta+\gamma+\epsilon & \alpha+\beta+\gamma+\delta & \alpha+\beta+\delta+\epsilon \\ 0 & \alpha+\delta & \alpha+\beta+\delta+\epsilon & I+\alpha+\gamma+\delta & 0 & \alpha+\beta+\delta+\epsilon \\ 0 & \alpha+\epsilon & \alpha+\beta+\gamma+\epsilon & \alpha+\beta+\gamma+\epsilon & I+\alpha+\delta+\epsilon & 0 \\ 0 & \alpha+\gamma & \alpha+\beta+\gamma+\delta & 0 & \alpha+\beta+\gamma+\delta & I+\alpha+\gamma+\epsilon \end{Bmatrix}.$$

Transforming it by the product $B_{3,4,1} B_{6,3,1} B_{4,6,1}$, we obtain a trans-

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* The product of t on the right by

$$\xi_1' = 6\xi_1, \quad \xi_2' = \xi_2, \quad \xi_3' = -3\xi_3, \quad \xi_4' = -3\xi_4, \quad \xi_5' = -3\xi_5, \quad \xi_6' = -3\xi_6,$$

which transforms M into itself, gives a transformation with integral coefficients of determinant $-2 . 3^5$ which effects the reduction of T_1 to M. It is given, without proof, in the *Trans. Amer. Math. Soc.*, Vol. III., 1902, p. 296 (viz., matrix T).

formation with the matrix

$$(5)\quad \begin{Bmatrix} \rho & \gamma+\delta+\epsilon & 0 & 0 & 0 & 0 \\ 0 & \rho & 0 & 0 & 0 & 0 \\ 0 & \beta+\delta & a & 0 & b & 0 \\ 0 & \beta+\gamma & 0 & d & 0 & c \\ 0 & a+\epsilon & c & 0 & d & 0 \\ 0 & a+\beta+\gamma+\delta & 0 & b & 0 & a \end{Bmatrix}.$$

Transforming it by t_6, which for modulus 2 becomes

$$\xi_3' = \xi_3+\xi_2,\quad \xi_4' = \xi_4+\xi_2,\quad \xi_5' = \xi_5+\xi_2,$$

we obtain a transformation whose matrix differs from (5) only in the second column, that being now $\gamma+\delta+\epsilon$, ρ, 0, 0, 0, 0. Finally, transforming by $(\xi_1\xi_2)(\xi_4\xi_5\xi_6)$, we obtain a transformation with the canonical matrix

$$M_2 \equiv \begin{Bmatrix} \rho & 0 & 0 & 0 & 0 & 0 \\ \gamma+\delta+\epsilon & \rho & 0 & 0 & 0 & 0 \\ 0 & 0 & a & b & 0 & 0 \\ 0 & 0 & c & d & 0 & 0 \\ 0 & 0 & 0 & 0 & a & b \\ 0 & 0 & 0 & 0 & c & d \end{Bmatrix}.$$

The six functions ρ, $\gamma+\delta+\epsilon$, a, b, c, d of I, a, β, γ, δ, ϵ are seen to be linearly independent modulo 2. The group is therefore simply isomorphic with the product of the binary group of transformations $\begin{pmatrix} \rho & 0 \\ \lambda & \rho \end{pmatrix}$ and the general binary linear group, the two affecting different variables.

3. Let, finally, F have modulus 3. For any modulus, the transformed of T_1, with matrix (1), by

$$\xi_1' = \xi_1+\xi_2+\xi_3+\xi_4+\xi_5+\xi_6,\quad \xi_2' = \xi_4+\xi_5+\xi_6,\quad \xi_3' = \xi_3,\quad \xi_4' = \xi_2,$$
$$\xi_5' = \xi_5,\quad \xi_6' = \xi_6,$$

has the matrix of coefficients

$$\begin{Bmatrix} \rho & 0 & 0 & 0 & 0 & 0 \\ \gamma+\delta+\epsilon & \rho_1 & 0 & 0 & 0 & 0 \\ a & \epsilon-a & I-a & \beta-a & \gamma-\epsilon & \delta-\epsilon \\ \beta & \delta-\beta & a-\beta & I-\beta & \epsilon-\delta & \gamma-\delta \\ \delta & \beta-\delta & \gamma-\delta & \epsilon-\delta & I-\beta & a-\beta \\ \epsilon & a-\epsilon & \delta-\epsilon & \gamma-\epsilon & \beta-a & I-a \end{Bmatrix}.$$

Transforming it in succession by the three transformations (mod 3)

$$\xi_3' = \xi_3 - \xi_4 + \xi_5 - \xi_6, \quad \xi_4' = \xi_4 + \xi_5 + \xi_6, \quad \xi_5' = \xi_5 - \xi_6,$$

further simplifications arise, the final matrix being

$$\begin{pmatrix} \rho & 0 & 0 & 0 & 0 & 0 \\ \gamma+\delta+\epsilon & \rho_1 & 0 & 0 & 0 & 0 \\ \tau & \tau & \rho & 0 & 0 & 0 \\ \beta+\delta+\epsilon & \alpha-\epsilon & \alpha-\beta+\gamma-\epsilon & \rho_1 & 0 & 0 \\ \delta-\epsilon & -\tau & \gamma+\delta+\epsilon & 0 & \rho_1 & 0 \\ \epsilon & \alpha-\epsilon & \delta-\epsilon & \gamma+\delta+\epsilon & \beta-\alpha-\gamma+\delta & \rho \end{pmatrix},$$

where $\tau \equiv \alpha-\beta+\delta-\epsilon$. Subsequent simplifications arise by transformation by $B_{2,1,1}$, $B_{5,3,1}$, $B_{6,4,-1}$, $B_{4,6,1}$, $(\xi_2\xi_5\xi_3\xi_6\xi_4)$, $B_{3,2,1}$, $B_{6,5,1}$, in succession, the final matrix having the canonical form*

$$M_3 \equiv \begin{Bmatrix} \rho & 0 & 0 & 0 & 0 & 0 \\ \sigma & \rho_1 & 0 & 0 & 0 & 0 \\ \kappa & -\tau & \rho & 0 & 0 & 0 \\ 0 & 0 & 0 & \rho_1 & 0 & 0 \\ 0 & 0 & 0 & \tau & \rho & 0 \\ 0 & 0 & 0 & \nu & \sigma & \rho_1 \end{Bmatrix},$$

where $\sigma \equiv \alpha-\beta-\delta+\epsilon$, $\kappa \equiv \alpha+\beta+\delta+\epsilon$, $\nu = -\alpha-\beta+\delta+\epsilon$. The six functions ρ, ρ_1, σ, τ, κ, ν are readily seen to be linearly independent modulo 3. For the Galois field of order 3^n, the group is evidently of order $3^{4n}(3^n-1)^2$.

The transformations $[\rho, \rho_1, \sigma, \tau, \kappa, \nu]$, of matrix M_3, form a solvable group. Indeed, the transformations $[1, 1, \sigma, \tau, \kappa, \nu]$ form an invariant sub-group, the quotient-group being a commutative group $\left[\text{of order } (3^n-1)^2 \text{ for the } GF[p^n]\right]$. The sub-group itself has an invariant sub-group formed by the commutative transformations $[1, 1, 0, \tau, \kappa, \nu]$. In fact,

$$[1, 1, 0, \tau, \kappa, \nu]\,[1, 1, 0, \tau', \kappa', \nu'] = [1, 1, 0, \tau+\tau', \kappa+\kappa', \nu+\nu'],$$

while $\begin{pmatrix} 1 & 0 & 0 \\ \sigma & 1 & 0 \\ \kappa & -\tau & 1 \end{pmatrix}$, whose inverse is $\begin{pmatrix} 1 & 0 & 0 \\ -\sigma & 1 & 0 \\ -\kappa-\tau\sigma & \tau & 1 \end{pmatrix}$, transforms a ternary transformation, leaving fixed the first two variables, into a transformation leaving them fixed; likewise $\begin{pmatrix} 1 & 0 & 0 \\ \tau & 1 & 0 \\ \nu & \sigma & 1 \end{pmatrix}$, whose inverse is $\begin{pmatrix} 1 & 0 & 0 \\ -\tau & 1 & 0 \\ -\nu+\sigma\tau & -\sigma & 1 \end{pmatrix}$, transforms $(\xi_4' = \xi_4,\ \xi_5' = \xi_5+\lambda\xi_4,\ \xi_6' = \xi_6+\mu\xi_4)$ into a similar transformation.

* As a check, note that the transformations of matrix M_3 form a group.

Canonical Matrices for the Cyclic G_3.

4. The body of a left-hand multiplication table for G_3 is

$$(6)\quad \begin{pmatrix} I & \alpha & \beta \\ \beta & I & \alpha \\ \alpha & \beta & I \end{pmatrix}.$$

Let ω be an imaginary cube root of unity and set

$$\delta_1 = I+\alpha+\beta,\quad \delta_2 = I+\omega\alpha+\omega^2\beta,\quad \delta_3 = I+\omega^2\alpha+\omega\beta,$$

$$\eta_1 = \xi_1+\xi_2+\xi_3,\quad \eta_2 = \xi_1+\omega^2\xi_2+\omega\xi_3,\quad \eta_3 = \xi_1+\omega\xi_2+\omega^2\xi_3.$$

If F does not have modulus 3, δ_1, δ_2, δ_3 are linearly independent with respect to F, and likewise η_1, η_2, η_3. Expressed in terms of the latter, the transformation on ξ_1, ξ_2, ξ_3, with matrix (6), takes the form

$$(7)\quad \eta_1' = \delta_1\eta_1,\quad \eta_2' = \delta_2\eta_2,\quad \eta_3' = \delta_3\eta_3.$$

Hence, if ω belongs to F, the group of transformations of matrix (6) is simply isomorphic with the direct product of three groups each a general unary linear group. If ω is not in F, it serves to extend* F to a field F_2. Then the conjugate of ω with respect to F is $\bar{\omega} = \omega^2$; so that $\eta_3 = \bar{\eta}_2$, $\delta_3 = \bar{\delta}_2$. The group is therefore simply isomorphic with the direct product of the general unary group in F and the general unary group in F_2. Thus, if F is the $GF[p^n]$, so that F_2 is the $GF[p^{2n}]$, the orders of the two factor groups are p^n-1 and $p^{2n}-1$ respectively.

If $p = 3$, a transformation of matrix (6) is transformed by

$$\xi_1' = -\xi_1-\xi_2-\xi_3,\quad \xi_2' = \xi_2-\xi_3,\quad \xi_3' = -\xi_3,$$

of period 2, into a transformation of canonical matrix

$$(8)\quad \begin{pmatrix} \delta & 0 & 0 \\ \alpha-\beta & \delta & 0 \\ \alpha & \alpha-\beta & \delta \end{pmatrix}.$$

The transformations of matrix (8) form a *commutative* group. Since α, $\alpha-\beta$, $\delta \equiv I+\alpha+\beta$ are linearly independent modulo 3, the order of the group for the $GF[3^n]$ is $3^{2n}(3^n-1)$.

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* To avoid introducing ω, note that a transformation of matrix (6) is transformed into one of matrix $\begin{pmatrix} \delta & 0 & 0 \\ 0 & I-\alpha & \alpha-\beta \\ 0 & \beta-\alpha & I-\beta \end{pmatrix}$ by $\begin{pmatrix} 1 & \frac{1}{3} & \frac{1}{3} \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$.

Canonical Matrices for the Cyclic G_6.

5. The body of a left-hand multiplication table for G_6 is

$$(9)\quad \begin{Bmatrix} a_0 & a_1 & a_2 & a_3 & a_4 & a_5 \\ a_5 & a_0 & a_1 & a_2 & a_3 & a_4 \\ a_4 & a_5 & a_0 & a_1 & a_2 & a_3 \\ a_3 & a_4 & a_5 & a_0 & a_1 & a_2 \\ a_2 & a_3 & a_4 & a_5 & a_0 & a_1 \\ a_1 & a_2 & a_3 & a_4 & a_5 & a_0 \end{Bmatrix}.$$

Let ρ be a primitive sixth root of unity and set

$$\left.\begin{aligned} \delta_i &\equiv a_0+\rho^i a_1+\rho^{2i}a_2+\rho^{3i}a_3+\rho^{4i}a_4+\rho^{5i}a_5 \\ \eta_i &\equiv \xi_0+\rho^{-i}\xi_1+\rho^{-2i}\xi_2+\rho^{-3i}\xi_3+\rho^{-4i}\xi_4+\rho^{-5i}\xi_5 \end{aligned}\right\}\quad (i=0, 1, \ldots, 5).$$

The determinant of the coefficients of $a_0, \ldots, a_5$ in $\delta_0, \ldots, \delta_5$ equals the product of the differences $\rho^r-\rho^s$, and is zero if, and only if, $x^6=1$ has a double root in the field F, *i.e.*, if F has as modulus either 2 or 3.* Excluding these cases, $\delta_0, \ldots, \delta_5$ are linearly independent functions of $a_0, \ldots, a_5$; likewise, $\eta_0, \ldots, \eta_5$ are linearly independent functions of $\xi_0, \ldots, \xi_5$. A transformation on $\xi_0, \ldots, \xi_5$ with coefficients of matrix (9) gives rise to the canonical transformation on $\eta_0, \ldots, \eta_5$:

$$(10)\quad \eta_i' = \delta_i \eta_i \quad (i = 0, 1, \ldots, 5).$$

The structure of the group is now quite evident.† When F is the $GF[p^n]$, the order of the group is $(p^n-1)^6$ or $(p^n-1)^2(p^{2n}-1)^2$ according as ρ does or does not belong to the $GF[p^n]$, viz., according as $p^n = 6k+1$ or $p^n = 6k+5$.

6. Next, let F have modulus 2. Introduce the variables‡

$$Y_0 = \xi_0+\xi_1+\xi_2+\xi_3+\xi_4+\xi_5, \qquad Y_1 = \xi_0+\xi_2+\xi_4,$$

$$Y_2 = \xi_0+\omega\xi_1+\omega^2\xi_2+\xi_3+\omega\xi_4+\omega^2\xi_5, \quad Y_3 = \xi_0+\omega^2\xi_2+\omega\xi_4,$$

$$Y_4 = \xi_0+\omega^2\xi_1+\omega\xi_2+\xi_3+\omega^2\xi_4+\omega\xi_5, \quad Y_5 = \xi_0+\omega\xi_2+\omega^2\xi_4,$$

ω being a root of $\omega^2+\omega+1=0$, and therefore a cube root of unity. Then the transformation on $\xi_0, \ldots, \xi_5$ with matrix (9) becomes

$$(11)\quad \begin{cases} Y_0' = \delta_0 Y_0, & Y_2' = \delta_2 Y_2, \qquad Y_4' = \delta_4 Y_4, \\ Y_1' = \lambda Y_0+\delta_0 Y_1, & Y_3' = \mu Y_2+\delta_2 Y_3, \quad Y_5' = \nu Y_4+\delta_4 Y_5, \end{cases}$$

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* Compare Dickson, "Linear Groups," Corollary of § 74, p. 54.

† *University of Chicago Decennial Publications*, Vol. IX., p. 38 (preprints, p. 4).

‡ The determinant of their coefficients is congruent to unity modulo 2.

where $\lambda \equiv a_1+a_3+a_5$, $\mu \equiv \omega^2 a_1+a_3+\omega a_5$, $\nu \equiv \omega a_1+a_3+\omega^2 a_5$.

Here λ, μ, ν are linearly independent functions of a_1, a_3, a_5, since the determinant of their coefficients is $\equiv 1 \pmod 2$. Then δ_0, δ_3, δ_4, λ, μ, ν are seen to be linearly independent modulo 2. If ω belongs to F, the group of transformations (11) is the direct product of three binary linear groups in F, each a commutative group of transformations $C \equiv \begin{pmatrix} \delta & 0 \\ \sigma & \delta \end{pmatrix}$; for the $GF[2^n]$ its order is $2^{3n}(2^n-1)^3$. If ω extends F to a larger field F_2, the group is the direct product* of a binary group in F and a binary group in F_2, each formed of transformations of type C; for the $GF[2^n]$ its order is

$$2^n(2^n-1)\,2^{2n}(2^{2n}-1).$$

7. Finally, let F have modulus 3. Introduce the variables

$$Z_0 = \xi_0+\xi_1+\xi_2+\xi_3+\xi_4+\xi_5,\quad Z_1 = \xi_1-\xi_2+\xi_4-\xi_5,\quad Z_2 = \xi_0+\xi_3,$$
$$Z_3 = \xi_0-\xi_1+\xi_2-\xi_3+\xi_4-\xi_5,\quad Z_4 = -\xi_1-\xi_2+\xi_4+\xi_5,\quad Z_5 = \xi_0-\xi_3.$$

The transformation on $\xi_0, \ldots, \xi_5$ with matrix (9) becomes

$$(12)\quad \begin{cases} Z_0' = \delta_0 Z_0, & Z_3' = \delta_3 Z_3, \\ Z_1' = qZ_0+\delta_0 Z_1, & Z_4' = sZ_3+\delta_3 Z_4, \\ Z_2' = rZ_0+qZ_1+\delta_0 Z_2, & Z_5' = tZ_3+sZ_4+\delta_3 Z_5, \end{cases}$$

where
$$q \equiv -a_1+a_2-a_4+a_5,\quad r \equiv -a_1-a_2-a_4-a_5,$$
$$s \equiv a_1+a_2-a_4-a_5,\quad t \equiv a_1-a_2-a_4+a_5.$$

To show that q, r, s, t are linearly independent functions modulo 2, we note that

$$q+t \equiv a_4-a_5,\quad q-t \equiv a_1-a_2,\quad r+s \equiv a_4+a_5,\quad r-s \equiv a_1+a_2 \pmod 3;$$

so that a_1, a_2, a_4, a_5 may be expressed in terms of q, r, s, t. It follows that

$$\delta_0 \equiv a_0+a_1+a_2+a_3+a_4+a_5,\quad \delta_3 \equiv a_0-a_1+a_2-a_3+a_4-a_5,$$

and q, r, s, t are six linearly independent functions of $a_0, \ldots, a_5$, modulo 3. Hence the group of transformations (12) is the direct product of two groups, affecting different variables, each a *commutative* group of the type

$$\begin{pmatrix} \delta & 0 & 0 \\ \lambda & \delta & 0 \\ \kappa & \lambda & \delta \end{pmatrix} \quad [\delta, \lambda, \kappa \text{ arbitrary in } F;\ \delta \neq 0].$$

* Given δ_0 and λ arbitrarily in F, δ_2 and μ arbitrarily in F_2, the values of $a_0, \ldots, a_5$ follow uniquely.

Remarks on the General Theory.

8. In the writer's treatment* of a group matrix of order n for an arbitrary field F, the case in which F has a modulus which divides n was excluded. The question of the first invariant factor of the characteristic determinant† of the special group-determinant D depends upon the nature of the greatest common divisor of all its first minors. The adjoint minor of an element ϵ_k in the determinant D equals $\frac{1}{nn_k}\frac{\partial D}{\partial \epsilon_k}$, where n_k is the number of operators in the k-th set of conjugates in the given group of order n. The necessity of a variation in the treatment will be evident from the following examples.

9. For the symmetric group g_6 on three letters, the special group-determinant D is the determinant of matrix (1) when $\alpha = \beta$, $\gamma = \delta = \epsilon$. From the form of matrix M of § 1, we conclude that

$$D \equiv (I+2\alpha+3\gamma)(I+2\alpha-3\gamma)(I-\alpha)^4.$$

Let I_1, α_1, γ_1 denote the adjoint minors of I, α, γ, respectively (by a theorem the same wherever I, α, γ occur in D). Then‡

$$I_1 = \tfrac{1}{6}\frac{\partial D}{\partial I} = (I^2+3I\alpha+2\alpha^2-6\gamma^2)(I-\alpha)^3,$$

$$\alpha_1 = \tfrac{1}{12}\frac{\partial D}{\partial \alpha} = (3\gamma^2-I\alpha-2\alpha^2)(I-\alpha)^3,$$

$$\gamma_1 = \tfrac{1}{18}\frac{\partial D}{\partial \gamma} = -\gamma\,(I-\alpha)^4.$$

For I, α, γ arbitrary in F, the greatest common divisor of I_1, α_1, γ_1 is therefore $(I-\alpha)^3$, if F does not have modulus 3; but is $(I-\alpha)^4$, if F has modulus 3.

10. For the quaternion group (Weber, *Algebra*, 2nd ed., pp. 216-218),

$$D = \sigma_1\sigma_2\sigma_3\sigma_4\,(x_1-x_2)^4,$$

$$\sigma_1 = x_1+x_2+2x_3+2x_5+2x_7,\quad \sigma_2 = x_1+x_2+2x_3-2x_5-2x_7,$$

$$\sigma_3 = x_1+x_2-2x_3+2x_5-2x_7,\quad \sigma_4 = x_1+x_2-2x_3-2x_5+2x_7.$$

* *Trans. Amer. Math. Soc.*, Vol. III., 1902, pp. 285-301. See p. 293.
† It is derived from D by replacing the identity element I by $I-\rho$.
‡ The values of I_1 and γ_1 were also computed direct as determinants of order 5.

Denote the adjoint minor of x_i by X_i. Then

$$X_1 = (x_1-x_2)^3 \{x_1(x_1+x_2)^3 - 2(x_1+x_2)(3x_1+x_2)(x_3^2+x_5^2+x_7^2)$$
$$-32x_3^2x_5^2 + 24x_2x_3x_5x_7 + 40x_1x_3x_5x_7 + 8(x_3^2+x_5^2-x_7^2)^2\},$$

$$X_3 = (x_1-x_2)^4 \{4x_3^3 - 4x_3x_5^2 - 4x_3x_7^2 + 4x_5x_7(x_1+x_2) - x_3(x_1+x_2)^2\}.$$

Since $X_5 \equiv \frac{1}{16}\frac{\partial D}{\partial x_5}$ is derived from $X_3 \equiv \frac{1}{16}\frac{\partial D}{\partial x_3}$ by interchanging x_3 with x_5, D being thereby unaltered, the expression for X_5 may be written down by inspection. Similarly for X_7 and X_2. It follows readily that the greatest common divisor of X_1, X_2, X_3, X_5, X_7 is $(x_1-x_2)^3$, if F does not have modulus 2; but is $(x_1-x_2)^6$ if F has modulus 2. The special character of modulus 2 is also shown by the fact that it is the only case in which the factors σ_1, σ_2, σ_3, σ_4, x_1-x_2 are all equal.

11. For the alternating group G_{12} on four letters, we have*

$$D = (I+3\epsilon+4\alpha+4\beta)(I+3\epsilon+4\rho\alpha+4\rho^2\beta)(I+3\epsilon+4\rho^2\alpha+4\rho\beta)(I-\epsilon)^9,$$

where $\rho = \frac{1}{2}(-1+\sqrt{-3})$ is a cube root of unity. Hence

$$D = \{(I+3\epsilon)^3 + 64\alpha^3 + 64\beta^3 - 48\alpha\beta(I+3\epsilon)\}(I-\epsilon)^9.$$

For the adjoint minors I_1, α_1, β_1, ϵ_1, we get

$$I_1 = \tfrac{1}{12}\frac{\partial D}{\partial I} = (I-\epsilon)^8\{(I+3\epsilon)^2(I+2\epsilon) + 48\alpha^3 + 48\beta^3 - 40\alpha\beta I - 104\alpha\beta\epsilon\},$$

$$\alpha_1 = \tfrac{1}{48}\frac{\partial D}{\partial \alpha} = (I-\epsilon)^9(4\alpha^2 - I\beta - 3\beta\epsilon),$$

$$\beta_1 = \tfrac{1}{48}\frac{\partial D}{\partial \beta} = (I-\epsilon)^9(4\beta^2 - I\alpha - 3\alpha\epsilon),$$

$$\epsilon_1 = \tfrac{1}{36}\frac{\partial D}{\partial \epsilon} = (I-\epsilon)^8\{-\epsilon(I+3\epsilon)^2 - 16\alpha^3 - 16\beta^3 + 8\alpha\beta I + 40\alpha\beta\epsilon\}.$$

The greatest common divisor of I_1, α_1, β_1, ϵ_1 is therefore $(I-\epsilon)^8$ if F does not have modulus 2; but is $(I-\epsilon)^{10}$ if F has modulus 2, since then

$$I_1 = I(I-\epsilon)^{10}, \quad \alpha_1 = \beta(I-\epsilon)^{10}, \quad \beta_1 = \alpha(I-\epsilon)^{10}, \quad \epsilon_1 = \epsilon(I-\epsilon)^{10}.$$

The exceptional character of the case of modulus 2 is also shown by

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* Using table of characters, Weber, *Algebra*, 2nd ed., II., p. 206.

two further facts. The linear factors of D are then all equal. Again, the general group-determinant then decomposes entirely into three linear factors each to the fourth power. For any field, the theory of Frobenius gives* for the general group-determinant of G_{12},

$$\Delta = ff_1f_2\Phi^3,$$

$$f \equiv I+\epsilon+\epsilon_1+\epsilon_2+\alpha+\alpha_1+\alpha_2+\alpha_3+\beta+\beta_1+\beta_2+\beta_3,$$

$$f_1 \equiv I+\epsilon+\epsilon_1+\epsilon_2+\rho\,(\alpha+\alpha_1+\alpha_2+\alpha_3)+\rho^2\,(\beta+\beta_1+\beta_2+\beta_3),$$

$$f_2 \equiv I+\epsilon+\epsilon_1+\epsilon_2+\rho^2\,(\alpha+\alpha_1+\alpha_2+\alpha_3)+\rho\,(\beta+\beta_1+\beta_2+\beta_3),$$

$$\begin{aligned}
\Phi \equiv\ & I^3+\epsilon^3+\epsilon_1^3+\epsilon_2^3+\alpha^3+\alpha_1^3+\alpha_2^3+\alpha_3^3+\beta^3+\beta_1^3+\beta_2^3+\beta_3^3 \\
& -I^2\epsilon-I^2\epsilon_1-I^2\epsilon_2-I\epsilon^2-I\epsilon_1^2-I\epsilon_2^2-\epsilon^2\epsilon_1-\epsilon^2\epsilon_2-\epsilon_1^2\epsilon-\epsilon_1^2\epsilon_2 \\
& -\epsilon_2^2\epsilon-\epsilon_2^2\epsilon_1-\alpha^2\alpha_1-\alpha^2\alpha_2-\alpha^2\alpha_3-\alpha_1^2\alpha-\alpha_1^2\alpha_2-\alpha_1^2\alpha_3-\alpha_2^2\alpha-\alpha_2^2\alpha_1 \\
& -\alpha_2^2\alpha_3-\alpha_3^2\alpha-\alpha_3^2\alpha_1-\alpha_3^2\alpha_2-\beta^2\beta_1-\beta^2\beta_2-\beta^2\beta_3-\beta_1^2\beta-\beta_1^2\beta_2-\beta_1^2\beta_3 \\
& -\beta_2^2\beta-\beta_2^2\beta_1-\beta_2^2\beta_3-\beta_3^2\beta-\beta_3^2\beta_1-\beta_3^2\beta_2 \\
& +2\epsilon\epsilon_1\epsilon_2+2I\epsilon\epsilon_1+2I\epsilon\epsilon_2+2I\epsilon_1\epsilon_2+2\alpha\alpha_1\alpha_2+2\alpha\alpha_1\alpha_3+2\alpha\alpha_2\alpha_3 \\
& +2\alpha_1\alpha_2\alpha_3+2\beta\beta_1\beta_2+2\beta\beta_2\beta_3+2\beta\beta_1\beta_3+2\beta_1\beta_2\beta_3 \\
& +I(\alpha\beta_1+\alpha\beta_2+\alpha\beta_3+\alpha_1\beta+\alpha_1\beta_2+\alpha_1\beta_3+\alpha_2\beta+\alpha_2\beta_1+\alpha_2\beta_3 \\
& \qquad\qquad +\alpha_3\beta+\alpha_3\beta_1+\alpha_3\beta_2) \\
& +(\epsilon+\epsilon_1+\epsilon_2-3I)(\alpha\beta+\alpha_1\beta_1+\alpha_2\beta_2+\alpha_3\beta_3) \\
& -3\epsilon\,(\alpha\beta_1+\alpha_1\beta+\alpha_2\beta_3+\alpha_3\beta_2) \\
& -3\epsilon_1\,(\alpha\beta_2+\alpha_1\beta_3+\alpha_2\beta+\alpha_3\beta_1) \\
& -3\epsilon_2\,(\alpha\beta_3+\alpha_1\beta_2+\alpha_2\beta_1+\alpha_3\beta) \\
& +\epsilon\,(\alpha\beta_2+\alpha\beta_3+\alpha_1\beta_2+\alpha_1\beta_3+\alpha_2\beta+\alpha_2\beta_1+\alpha_3\beta+\alpha_3\beta_1) \\
& +\epsilon_1\,(\alpha\beta_1+\alpha\beta_3+\alpha_1\beta+\alpha_1\beta_2+\alpha_2\beta_1+\alpha_2\beta_3+\alpha_3\beta+\alpha_3\beta_2) \\
& +\epsilon_2\,(\alpha\beta_1+\alpha\beta_2+\alpha_1\beta+\alpha_1\beta_3+\alpha_2\beta+\alpha_2\beta_3+\alpha_3\beta_1+\alpha_3\beta_2).
\end{aligned}$$

For the case of modulus 2, we find that

$$\Phi \equiv ff_1f_2 \pmod 2, \qquad \Delta \equiv f^4f_1^4f_2^4 \pmod 4.$$

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* The computation of Φ was checked as follows. For $\epsilon = \epsilon_1 = \epsilon_2$, $\alpha = \alpha_1 = \alpha_2 = \alpha_3$, $\beta = \beta_1 = \beta_2 = \beta_3$, $\Phi = (I-\epsilon)^3$, thereby agreeing with the above result for the special group-determinant D. Again, the coefficients of I^3, $I^2\epsilon_i$, $I^2\alpha_i$, $I^2\beta_i$ in Φ agree with those given by using the characters for the factor Φ.

12. For the symmetric group G_{24} on four letters,*

$$D = \alpha\epsilon\beta^9\gamma^9\delta^4,$$

$$\alpha = a+3b+6c+8d+6e, \quad \epsilon = a+3b-6c+8d-6e,$$

$$\beta = a-b+2c-2e, \quad \gamma = a-b-2c+2e, \quad \delta = a+3b-4d.$$

By direct computation (see § 8), we find for the adjoint minors

$$d_1 = \beta^9\gamma^9\delta^3\left\{3(c+e)^2-d(a+3b+8d)\right\},$$

$$c_1 = \beta^8\gamma^8\delta^4\left\{4eb^2+4eab-c(a^2+2ab+5b^2)+(c^2-e^2)(20c+16e)\right.$$
$$\left.-(c-e)(8ad+24bd+32d^2)\right\},$$

$$b_1 = \beta^8\gamma^8\delta^3\left\{(a-b)(a+3b+8d)(8d^2-ab-3b^2-4bd)+24(c^2-e^2)^2\right.$$
$$+(a-b)(c+e)^2(3a+33b-36d)$$
$$\left.-(c-e)^2(a+3b+8d)(a+3b+4d)\right\},$$

$$a_1 = \beta^8\gamma^8\delta^3\left\{(a-b)(a+3b+8d)(a^2+5ab+4ad+6b^2+8bd-24d^2)\right.$$
$$-(c-e)^2(a+3b+8d)(a+3b+4d)+24(c^2-e^2)^2$$
$$\left.+(a-b)(c+e)^2(-33a-75b+108d)\right\}.$$

Since D is unaltered by the interchange of c with e, we derive $e_1 \equiv \frac{1}{144}\frac{\partial D}{\partial e}$ from $c_1 \equiv \frac{1}{144}\frac{\partial D}{\partial c}$ by interchanging c with e.

If F does not have modulus 2 or 3, the greatest common divisor g of a_1, b_1, c_1, d_1, e_1 is $\beta^8\gamma^8\delta^3$, in accord with the general theory. If F has modulus 2, so that $\alpha\equiv\beta\equiv\gamma\equiv\delta\equiv\epsilon\equiv a+b \pmod 2$, then

$$d_1 = \alpha^{21}\left\{(c+e)^2+d(a+b)\right\}, \qquad c_1 = \alpha^{20}(ca^2+cb^2),$$

$$b_1 = \alpha^{19}\left\{b(a+b)^3\right\}, \qquad a_1 = \alpha^{19}\left\{a(a+b)^3\right\}.$$

Hence the greatest common divisor g is $(a+b)^{21}$. For modulus 3,

$$d_1 = \beta^9\gamma^9\delta^3\left\{-d(a-d)\right\},$$

$$c_1 = \beta^8\gamma^8\delta^4\left\{eb(a+b)+c(b^2+ab-a^2)+(e-c)(c^2-e^2-ad-d^2)\right\},$$

$$b_1 = \beta^8\gamma^8\delta^3(a-d)\left\{(a-b)(-ab-bd-d^2)-(c-e)^2(a+d)\right\},$$

$$a_1 = \beta^8\gamma^8\delta^3(a-d)(a+d)\left\{(a-b)^2-(c-e)^2\right\},$$

and $\alpha = \epsilon = \delta = a-d$. Since $a-d$ is not a factor of the expression in brackets in c_1, the greatest common divisor g is $\beta^8\gamma^8\delta^4$. Hence the cases of moduli 2 and 3 are both special.

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* Frobenius, *Berliner Sitzungsberichte*, 1896, p. 1012.