Transformations of Surfaces of Guichard and Surfaces Applicable to Quadrics.

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INTRODUCTION.

GUICHARD (*) has discovered a class of surfaces which possess the following characteristic property: If S is such a surface, there exists an *associate* surface $\bar{S}$, such that the lines of curvature on the two surfaces have the same spherical representation, and the principal radii of curvature ρ_1, ρ_2; $\bar{\rho}_1, \bar{\rho}_2$ of the respective surfaces are in the relation

$$\rho_1 \bar{\rho}_2 + \rho_2 \bar{\rho}_1 = \text{const.} \neq 0. \qquad (a)$$

CALAPSO (**) made a study of these surfaces and determined their analytical characterization. He found that there are two types which he called *the surfaces of* GUICHARD *of the first and second kinds.* In the present paper the author establishes transformations of these surfaces such that a surface and a transform constitute the envelope of a two parameter family of spheres with lines of curvature corresponding on the two sheets. These transformations of surfaces of GUICHARD are analogous in some respects to the transformations D_m of isothermic surfaces discovered by DARBOUX (***) and developed by BIANCHI (****). In particular, there are certain *special surfaces of*

(*) *Sur les surfaces isothermiques*, Comptes Rendus de l'Académie des Sciences, vol. 130, p. 159.

(**) *Alcune superficie di* GUICHARD *e le relative trasformazioni*, Annali di Matematica, ser. 3, vol. 11, pp. 201 et seq.

(***) *Sur les surfaces isothermiques*, Annales de l'École Normale Supérieure, ser. 3, vol. 16, pp. 491-508.

(****) *Ricerche sulle superficie isoterme e sulla deformazione delle quadriche*, Annali di Matematica, ser. 3, vol. 11, pp. 93 et seq.

GUICHARD which undergo types of transformations carrying with them surfaces applicable to quadrics. Consequently this memoir makes a contribution to the theory of the deformation of quadrics, which recently has been the object of study by BIANCHI, GUICHARD and others.

In the second volume of his *Leçons* (*) DARBOUX has developed the analysis of the general *transformation of* RIBAUCOUR, that is, the determination of a surface S_1 which together with a given surface S, constitues the envelope of a two-parameter family of spheres upon the two sheets of which lines of curvature correspond. These results are recalled in § 1 and in § 2 are given the equations of surfaces of GUICHARD as derived by CALAPSO. In § 3 we apply the preceding results to the establishment of transformations T_n of surfaces of GUICHARD into surfaces of the same kind. The equations of the transformation involve five functions between which these is a homogeneous quadratic relation, and these functions satisfy a homogeneous differential system of the first order. In addition to the significant constant n which appears in the notation T_n there are consequently three arbitrary constants of integration. We show that each of the *two kinds* of surfaces of GUICHARD is thus transformable into a surface of the same *kind*, but, in order to avoid repetition, in the subsequent sections we develop the theory only for surfaces of the *first kind*. However, each theorem has its analogue for surfaces of the *second kind*.

The relation (a) being reciprocal the *associate surface* $\overline{S}$ is a surface of GUICHARD and it is of the same kind as S. Consequently there are transformations $\overline{T}_n$ of $\overline{S}$. Furthermore, when a transformation of S is known, one finds directly a transformation of $\overline{S}$, such that the transforms S_1 and $\overline{S}_1$ are associates of one another. The interrelation of four such surfaces gives a geometrical interpretation of the constant n.

In §§ 7-10 it is shown that the transformations T_n admit of the following *theorem of permutability:*

If S is a surface of GUICHARD *and S_1 and S_2 an two surfaces obtained from S by transformations T_{n_1} and T_{n_2}, where $n_2 \neq n_1$, there exists a unique surface S' which may be obtained from S_1 by a transformation T'_{n_2} and from S_2 by a transformation T'_{n_1}.*

The determination of S' requires only algebraic processes. For certain values of n_1, it is possible to find two different transformations T_{n_1} which

(*) pp. 338-343.

lead to a similar result, but the determination of S' requires differentiation.

The surfaces of constant curvature are surfaces of GUICHARD of the first or second kind according as the curvature is positive or negative. Of the ∞^4 transforms of such a surface ∞^3 are surfaces of the same constant curvature. The transformations T_n in this case are the same as those which are a consequence of the beautiful theorems announced to the French Academy in 1899 by GUICHARD. BIANCHI has shown also that they may be obtained by suitable combinations of BÄCKLUND transformations of surfaces of constant curvature.

The circles normal to two surfaces S and S_1, which are in the relation of a transformation T_n, form a cyclic system; we call the plane of the circle the *circle-plane* of the transformation. The remainder of the memoir is devoted to the study of a particular type of surfaces of the first kind characterized by the property that for each surface one knows in general three transformations T_n whose circle-planes coincide. Moreover, the envelope of this *singular* circle plane is applicable to the general quadric meeting the circle at infinity in four distinct points. These surfaces are characterized analytically by the requirement that the functions ξ, λ, θ satisfy a differential relation of the first order involving four arbitrary constants A, B, C, D. In some respects these surfaces are analogous to the isothermic surfaces discovered by DARBOUX (*), and following the terminology adopted by BIANCHI in the latter case, we refer to one of our surfaces as a *special surface of* GUICHARD *of the first kind of class* (A, B, C, D). When S is of this type the three known transforms are special surfaces of the same class. Furthermore the associate surface is a special surface of class (C, B, A, D). The investigation closes with the establishment of transformations of special surfaces into surfaces of the same class and the proof of a theorem of permutability for these special transformations.

(*) Loc. cit., p. 506.

§ 1. General Transformation of Ribaucour.

Darboux (*) has developed in an elegant form the formulas of the general transformation of Ribaucour. We recall in this section certain of these results without giving any proofs.

Let S be a surface referred to its lines of curvature; x, y, z the cartesian coordinates of a point on S; u, v the curvilinear coordinates; ρ_1 and ρ_2 the principal radii of normal curvature in the respective directions $v =$ const., $u =$ const.; and we write the linear element of S in the form

$$d s^2 = E\, d u^2 + G\, d v^2. \tag{1}$$

Darboux has shown that if λ and μ are two solutions of the equations

$$\frac{\partial \lambda}{\partial u} + \rho_1 \frac{\partial \mu}{\partial u} = 0, \quad \frac{\partial \lambda}{\partial v} + \rho_2 \frac{\partial \mu}{\partial v} = 0, \tag{2}$$

the envelope of the spheres of radius $\frac{\lambda}{\mu}$ and center

$$\xi = x - X \frac{\lambda}{\mu}, \quad \eta = y - Y \frac{\lambda}{\mu}, \quad \zeta = z - Z \frac{\lambda}{\mu}, \tag{3}$$

where X, Y, Z are the direction-cosines of the normal to S, consists of S and of a second surface S_1 upon which also the lines of curvature are parametric. Moreover, the most general envelope of spheres such that the lines of curvature on the two sheets correspond is given by solving (2). In other words the solution of equations (2) carries with it the determination of the most *general transformation of* Ribaucour of the surface S.

Incidentally we observe that equations (2) are of the same form as the Rodrigues equations

$$\frac{\partial x}{\partial u} + \rho_1 \frac{\partial X}{\partial u} = 0, \quad \frac{\partial x}{\partial v} + \rho_2 \frac{\partial Y}{\partial v} = 0 \text{ (**)}.$$

(*) *Leçons sur la théorie générale des surfaces*, t. II, pp. 338-343.

(**) E. p. 122. A reference of this form is to the author's *Differential Geometry* (Ginn & Co., Boston, 1909).

Hence λ is the general solution of the point equation of S and μ of the tangential equation.

If we define two functions α and β by

$$\frac{\partial \lambda}{\partial u} = \sqrt{E}\,\alpha, \quad \frac{\partial \lambda}{\partial v} = \sqrt{G}\,\beta, \tag{4}$$

the coordinates x, y, z, of the corresponding point on S_1 are given by equations of the form

$$x_1 = x - \frac{1}{\sigma n}(\mu X + \alpha X' + \beta X''), \tag{5}$$

where X', Y', Z' and X'', Y'', Z'' denote the direction-cosines of the tangents to the curves $v = \text{const.}$, $u = \text{const.}$ respectively, n denotes a constant and σ is given by the quadratic relation

$$2\lambda n \sigma = \alpha^2 + \beta^2 + \mu^2 \text{ (*)}. \tag{6}$$

The linear element of S_1 is readily found to be

$$d s_1^2 = \frac{\lambda^2}{\alpha^2}\left(\frac{\partial \log \sigma}{\partial u}\right)^2 d u^2 + \frac{\lambda^2}{\beta^2}\left(\frac{\partial \log \sigma}{\partial v}\right)^2 d v^2 ; \tag{7}$$

we obtains this from the following equations which we get from (5):

$$\left.\begin{aligned} \frac{\partial x_1}{\partial u} &= -\frac{\lambda}{\sigma}\frac{\partial \log \sigma}{\partial u}\left[X' - \frac{1}{\sigma n}\frac{\alpha}{\lambda}(\mu X + \alpha X' + \beta X'')\right] \\ \frac{\partial x_1}{\partial v} &= -\frac{\lambda}{\beta}\frac{\partial \log \sigma}{\partial v}\left[X'' - \frac{1}{\sigma n}\frac{\beta}{\lambda}(\mu X + \alpha X' + \beta X'')\right]. \end{aligned}\right\} \tag{8}$$

If X'_1, Y'_1, Z'_1; X''_1, Y''_1, Z''_1; X_1, Y_1, Z_1 denote the direction-cosines for S_1 analogous to those for S without the subscript, and if the mutual orien-

(*) It is assumed that positive directions are taken on the tangents and normal to S so that

$$\begin{vmatrix} X' & Y' & Z' \\ X'' & Y'' & Z'' \\ X & Y & Z \end{vmatrix} = +1.$$

tation is to be the same as for S, we may put

$$\left.\begin{aligned} X'_1 &= X' - \frac{1}{\sigma n}\frac{\alpha}{\lambda}(\mu X + \alpha X' + \beta X''), \\ X''_1 &= -X'' + \frac{1}{\sigma n}\frac{\beta}{\lambda}(\mu X + \alpha X' + \beta X''), \\ X_1 &= X - \frac{1}{\sigma n}\frac{\mu}{\lambda}(\mu X + \alpha X' + \beta X''). \end{aligned}\right\} \tag{9}$$

Hence if E_1 and G_1 denote the first fundamental coefficients of S_1, we must have

$$\frac{\partial \log \sigma}{\partial u} = -\sqrt{E_1}\,\frac{\alpha}{\lambda}, \quad \frac{\partial \log \sigma}{\partial v} = \sqrt{G_1}\,\frac{\beta}{\lambda}. \tag{10}$$

Since λ must satisfy the equation

$$\frac{\partial}{\partial v}\left(\frac{1}{\rho_1}\frac{\partial \lambda}{\partial u}\right) - \frac{\partial}{\partial u}\left(\frac{1}{\rho_2}\frac{\partial \lambda}{\partial v}\right) = 0, \tag{11}$$

in consequence of the GAUSS and CODAZZI equations (*)

$$\left.\begin{aligned} &\frac{\partial}{\partial v}\left(\frac{1}{\sqrt{G}}\frac{\partial\sqrt{E}}{\partial v}\right) + \frac{\partial}{\partial u}\left(\frac{1}{\sqrt{E}}\frac{\partial\sqrt{G}}{\partial u}\right) + \frac{\sqrt{EG}}{\rho_1\rho_2} = 0, \\ &\frac{\partial}{\partial v}\left(\frac{\sqrt{E}}{\rho_1}\right) = \frac{1}{\rho_2}\frac{\partial\sqrt{E}}{\partial v}, \quad \frac{\partial}{\partial u}\left(\frac{\sqrt{G}}{\rho_2}\right) = \frac{1}{\rho_1}\frac{\partial\sqrt{G}}{\partial u}, \end{aligned}\right\} \tag{12}$$

it follows from (2) and (4) that

$$\left.\begin{aligned} &\frac{\partial \mu}{\partial u} = -\sqrt{E}\,\frac{\alpha}{\rho_1}, \quad \frac{\partial \mu}{\partial v} = -\sqrt{G}\,\frac{\beta}{\rho_2}, \\ &\frac{\partial \beta}{\partial u} = \frac{\alpha}{\sqrt{G}}\frac{\partial\sqrt{E}}{\partial v}, \quad \frac{\partial \alpha}{\partial v} = \frac{\beta}{\sqrt{E}}\frac{\partial\sqrt{G}}{\partial u}. \end{aligned}\right\} \tag{13}$$

If the equation (6) be differentiated and in the reduction use be made of (4), (10) and (13), we obtain

$$\left.\begin{aligned} \frac{\partial \alpha}{\partial u} &= -\frac{1}{\sqrt{G}}\frac{\partial\sqrt{E}}{\partial v}\beta + \mu\frac{\sqrt{E}}{\rho_1} + n\sigma(-\sqrt{E_1} + \sqrt{E}), \\ \frac{\partial \beta}{\partial v} &= -\frac{1}{\sqrt{E}}\frac{\partial\sqrt{G}}{\partial u}\alpha + \mu\frac{\sqrt{G}}{\rho_2} + n\sigma(\sqrt{G_1} + \sqrt{G}). \end{aligned}\right\} \tag{14}$$

(*) E. p. 157.

The conditions of integrability of equations (4) and of the first two of equations (13) are satisfied in virtue of (12) and of the last two of equations (13). Expressing the conditions of integrability of the latter and of (14), we obtain

$$\left.\begin{aligned}\frac{\partial\sqrt{E_1}}{\partial v} &= -\sqrt{G_1}\left[\frac{1}{\sqrt{G}}\frac{\partial\sqrt{E}}{\partial v}+\frac{\beta}{\lambda}(\sqrt{E_1}-\sqrt{E})\right],\\ \frac{\partial\sqrt{G_1}}{\partial u} &= -\sqrt{E_1}\left[\frac{1}{\sqrt{E}}\frac{\partial\sqrt{G}}{\partial u}-\frac{\alpha}{\lambda}(\sqrt{G_1}+\sqrt{G})\right].\end{aligned}\right\} \qquad (15)$$

When these equations are satisfied by two functions E_1 and G_1, so also is the condition of integrability of equations (10).

If we form the equations of Rodrigues for S_1 and make use of (9), we find that the principal radii ρ'_1, ρ'_2 of S_1 are given by

$$\frac{\partial\log\sigma}{\partial u}\left(\frac{1}{\rho'_1}+\frac{\mu}{\lambda}\right)-\frac{\partial}{\partial u}\left(\frac{\mu}{\lambda}\right)=0,\quad \frac{\partial\log\sigma}{\partial v}\left(\frac{1}{\rho'_2}+\frac{\mu}{\lambda}\right)-\frac{\partial}{\partial v}\left(\frac{\mu}{\lambda}\right)=0. \qquad (16)$$

§ 2. Surfaces of Guichard. The Associate Surface.

Following Calapso (*) we say that S is a *surface of* Guichard *of the first kind* if its fundamental coefficients satisfy the relation

$$\sqrt{G}\frac{D}{\sqrt{E}}-\sqrt{E}\frac{D''}{\sqrt{G}}=\sqrt{G-E}. \qquad (17)$$

If we define two functions ξ and θ by

$$\sqrt{E}=e^{\xi}\sinh\theta,\quad \sqrt{G}=e^{\xi}\cosh\theta, \qquad (18)$$

the relation (17) may be replaced by

$$\left.\begin{aligned}D &= e^{\xi}\sinh\theta(\cosh\theta+h\sinh\theta),\\ D'' &= e^{\xi}\cosh\theta(\sinh\theta+h\cosh\theta),\end{aligned}\right\} \qquad (19)$$

the function h being thus defined. From (18) and (19) follow (**)

$$\frac{1}{\rho_1}=e^{-\xi}(\coth\theta+h),\quad \frac{1}{\rho_2}=e^{-\xi}(\tanh\theta+h). \qquad (20)$$

(*) Loc. cit.

(**) E. p. 122.

Expressing the condition that the above functions satisfy the Codazzi and Gauss equations (12), we obtain the following equations to be satisfied by h, θ and ξ:

$$\frac{\partial h}{\partial u} = (h + \coth\theta)\frac{\partial \xi}{\partial u}, \quad \frac{\partial h}{\partial v} = (h + \tanh\theta)\frac{\partial \xi}{\partial v}, \tag{21}$$

$$\left.\begin{aligned} &\frac{\partial^2 \theta}{\partial u^2} + \frac{\partial^2 \theta}{\partial v^2} + \coth\theta\frac{\partial^2 \xi}{\partial u^2} + \tanh\theta\frac{\partial^2 \xi}{\partial v^2} - \operatorname{csch}^2\theta\frac{\partial \theta}{\partial u}\frac{\partial \xi}{\partial u} + \\ &+ \operatorname{sech}^2\theta\frac{\partial \theta}{\partial v}\frac{\partial \xi}{\partial v} + (\cosh\theta + h\sinh\theta)(\sinh\theta + h\cosh\theta) = 0. \end{aligned}\right\} \tag{22}$$

Moreover, the condition of integrability of (21) requires that

$$\frac{\partial^2 \xi}{\partial u \partial v} = \frac{\partial \xi}{\partial u}\frac{\partial \xi}{\partial v} + \coth\theta\frac{\partial \xi}{\partial u}\frac{\partial \theta}{\partial v} + \tanh\theta\frac{\partial \xi}{\partial v}\frac{\partial \theta}{\partial u}. \tag{23}$$

From the general theory of surfaces it follows that every set of solutions of equations (21), (22), (23) defines a surface of Guichard of the first kind.

Later it will desirable to know the expressions for the derivatives of the direction-cosines X', X'', X for S. From well-known general formula (*) we find

$$\left.\begin{aligned} &\frac{\partial X'}{\partial u} = -\left(\tanh\theta\frac{\partial \xi}{\partial v} + \frac{\partial \theta}{\partial v}\right)X'' + (\cosh\theta + h\sinh\theta)X, \\ &\frac{\partial X'}{\partial v} = \left(\coth\theta\frac{\partial \xi}{\partial u} + \frac{\partial \theta}{\partial u}\right)X'', \quad \frac{\partial X''}{\partial u} = \left(\tanh\theta\frac{\partial \xi}{\partial v} + \frac{\partial \theta}{\partial v}\right)X', \\ &\frac{\partial X''}{\partial v} = -\left(\coth\theta\frac{\partial \xi}{\partial u} + \frac{\partial \theta}{\partial u}\right)X' + (\sinh\theta + h\cosh\theta)X, \\ &\frac{\partial X}{\partial u} = -(\cosh\theta + h\sinh\theta)X', \quad \frac{\partial X}{\partial v} = (\sinh\theta + h\cosh\theta)X''. \end{aligned}\right\} \tag{24}$$

From the last two equations it follows that the linear element of the spherical representation of S is

$$d s'^2 = (\cosh\theta + h\sinh\theta)^2 d u^2 + (\sinh\theta + h\cosh\theta) d v^2. \tag{25}$$

Calapso has shown (**) that if ξ, θ, h satisfy equations (21), (22), (23),

(*) E. p. 157.

(**) Loc. cit., p. 14.

so also do $\bar{\xi}$, $\bar{\theta}$, h, where the first two are given by

$$\left.\begin{aligned} e^{\bar{\xi}} &= e^{-\xi}(1-h^2), \\ \sinh\bar{\theta} &= \frac{1}{h^2-1}\left[\sinh\theta\,(1+h^2)+2\,h\cosh\theta\right], \\ \cosh\bar{\theta} &= -\frac{1}{h^2-1}\left[\cosh\theta\,(1+h^2)+2\,h\sinh\theta\right]. \end{aligned}\right\} \tag{26}$$

From these equations we have

$$\left.\begin{aligned} \cosh\bar{\theta}+h\sinh\bar{\theta} &= \quad(\cosh\theta+h\sinh\theta), \\ \sinh\bar{\theta}+h\cosh\bar{\theta} &= -(\sinh\theta+h\cosh\theta). \end{aligned}\right\} \tag{27}$$

From (24) and (27) it follows that for the surface $\bar{S}$, determined by $\bar{\xi}$, $\bar{\theta}$, h, the direction cosines are given by

$$\bar{X}=-X,\quad \bar{X}'=-X',\quad \bar{X}''=X'', \tag{28}$$

so that the proper orientation may be obtained. In view of this we take for the principal radii of $\bar{S}$

$$\bar{\rho}_1=\frac{-e^{\bar{\xi}}\sinh\bar{\theta}}{\cosh\bar{\theta}+h\sinh\bar{\theta}},\quad \bar{\rho}_2=\frac{-e^{\bar{\xi}}\cosh\bar{\theta}}{\sinh\bar{\theta}+h\cosh\bar{\theta}}. \tag{29}$$

With the aid of (26) and (27) we show that

$$\rho_1\bar{\rho}_2+\rho_2\bar{\rho}_1=2. \tag{30}$$

Hence $\bar{S}$ is the associate of S.

A *surface of* GUICHARD *of the second kind* is one whose fundamental coefficients satisfy the relation

$$\sqrt{G}\frac{D}{\sqrt{E}}-\sqrt{E}\frac{D''}{\sqrt{G}}=\sqrt{E+G}. \tag{17*}$$

An expression or equation for these surfaces will be denoted by (a*) when the equation (a) is the analogous one for a surface of the first kind. We have

$$\sqrt{E}=e^{\xi}\sin\theta,\quad \sqrt{G}=e^{\xi}\cos\theta, \tag{18*}$$

$$D=e^{\xi}\sin\theta\,(\cos\theta+h\sin\theta),\quad D''=e^{\xi}\cos\theta\,(-\sin\theta+h\cos\theta), \tag{19*}$$

$$\frac{1}{\rho_1} = e^{-\xi}(\cot\theta + h), \quad \frac{1}{\rho_2} = e^{-\xi}(-\tan\theta + h), \tag{20*}$$

$$\frac{\partial h}{\partial u} = (h + \cot\theta)\frac{\partial \xi}{\partial u}, \quad \frac{\partial h}{\partial v} = (h - \tan\theta)\frac{\partial \xi}{\partial v}, \tag{21*}$$

$$\left.\begin{aligned} &\frac{\partial^2 \theta}{\partial u^2} - \frac{\partial^2 \theta}{\partial v^2} + \cot\theta\frac{\partial^2 \xi}{\partial u^2} + \tan\theta\frac{\partial^2 \xi}{\partial v^2} - \frac{1}{\sin^2\theta}\frac{\partial \theta}{\partial u}\frac{\partial \xi}{\partial u} + \frac{1}{\cos^2\theta}\frac{\partial \theta}{\partial v}\frac{\partial \xi}{\partial v} - \\ &\quad - (\cos\theta + h\sin\theta)(\sin\theta - h\cos\theta) = 0, \end{aligned}\right\} \tag{22*}$$

$$\frac{\partial^2 \xi}{\partial u \partial v} = \frac{\partial \xi}{\partial u}\frac{\partial \xi}{\partial v} + \cot\theta\frac{\partial \theta}{\partial v}\frac{\partial \xi}{\partial u} - \tan\theta\frac{\partial \theta}{\partial u}\frac{\partial \xi}{\partial v}. \tag{23*}$$

For the associate surface $\bar{S}$ the function h is the same as for S, and the other functions are given by

$$\left.\begin{aligned} &e^{\bar{\xi}} = e^{-\xi}(1 + h^2), \\ &\sin\bar{\theta} = -\frac{1}{1+h^2}\left[\sin\theta(1 - h^2) - 2h\cos\theta\right], \\ &\cos\bar{\theta} = \frac{1}{1+h^2}\left[\cos\theta(1 - h^2) + 2h\sin\theta\right]. \end{aligned}\right\} \tag{26*}$$

§ 3. Transformations of Surfaces of Guichard.

If S a surface of Guichard of the first kind and S_1 is to be a surface of Guichard of the first kind with its fundamental functions expressed by means of ξ_1, θ_1, h_1, equations (4), (10), (13) and (14) assume the form

$$\left.\begin{aligned} &\frac{\partial \log\sigma}{\partial u} = -e^{\xi_1}\sinh\theta_1\frac{\alpha}{\lambda}, \quad \frac{\partial \log\sigma}{\partial v} = e^{\xi_1}\cosh\theta_1\frac{\beta}{\lambda}, \\ &\frac{\partial \lambda}{\partial u} = \alpha e^{\xi}\sinh\theta, \quad \frac{\partial \lambda}{\partial v} = \beta e^{\xi}\cosh\theta, \\ &\frac{\partial \mu}{\partial u} = -\alpha(\cosh\theta + h\sinh\theta), \quad \frac{\partial \mu}{\partial v} = -\beta(\sinh\theta + h\cosh\theta), \\ &\frac{\partial \alpha}{\partial u} = -\beta\left(\tanh\theta\frac{\partial \xi}{\partial v} + \frac{\partial \theta}{\partial v}\right) + \mu(\cosh\theta + h\sinh\theta) + \\ &\qquad\qquad + n\sigma(-e^{\xi_1}\sinh\theta_1 + e^{\xi}\sinh\theta), \end{aligned}\right\} \tag{31}$$

$$\left.\begin{aligned}
&\frac{\partial \alpha}{\partial v}=\beta\left(\coth\theta\frac{\partial \xi}{\partial u}+\frac{\partial \theta}{\partial u}\right), \quad \frac{\partial \beta}{\partial u}=\alpha\left(\tanh\theta\frac{\partial \xi}{\partial v}+\frac{\partial \theta}{\partial v}\right),\\
&\frac{\partial \beta}{\partial v}=-\alpha\left(\coth\theta\frac{\partial \xi}{\partial u}+\frac{\partial \theta}{\partial u}\right)+\mu(\sinh\theta+h\cosh\theta)+\\
&\qquad\qquad\qquad\qquad+n\sigma(e^{\xi_1}\cosh\theta_1+e^{\xi}\cosh\theta).
\end{aligned}\right\} \tag{31}$$

Moreover, equations (15) may be replaced by

$$\left.\begin{aligned}
&\tanh\theta_1\frac{\partial \xi_1}{\partial v}+\tanh\theta\frac{\partial \xi}{\partial v}+\frac{\partial \theta_1}{\partial v}+\frac{\partial \theta}{\partial v}+\frac{\beta}{\lambda}(e^{\xi_1}\sinh\theta_1-e^{\xi}\sinh\theta)=0,\\
&\coth\theta_1\frac{\partial \xi_1}{\partial u}+\coth\theta\frac{\partial \xi}{\partial u}+\frac{\partial \theta_1}{\partial u}+\frac{\partial \theta}{\partial u}-\frac{\alpha}{\lambda}(e^{\xi_1}\cosh\theta_1+e^{\xi}\cosh\theta)=0.
\end{aligned}\right\} \tag{32}$$

From (31) it follows that

$$\frac{\partial}{\partial u}\left(\frac{\mu}{\lambda}\right)=-\frac{\alpha}{\lambda}\varphi, \quad \frac{\partial}{\partial v}\left(\frac{\mu}{\lambda}\right)=-\frac{\beta}{\lambda}\psi, \tag{33}$$

where

$$\left.\begin{aligned}
\varphi&=\cosh\theta+\left(h+\frac{\mu}{\lambda}e^{\xi}\right)\sinh\theta,\\
\psi&=\sinh\theta+\left(h+\frac{\mu}{\lambda}e^{\xi}\right)\cosh\theta.
\end{aligned}\right\} \tag{34}$$

When we express the condition that the principal radii of S_1 have forms analogous to those for S (20), but in terms of ξ_1, θ_1, h_1, it follows from equations (16) that

$$\left.\begin{aligned}
\varphi&=\cosh\theta_1+\left(h_1+\frac{\mu}{\lambda}e^{\xi_1}\right)\sinh\theta_1,\\
-\psi&=\sinh\theta_1+\left(h_1+\frac{\mu}{\lambda}e^{\xi_1}\right)\cosh\theta_1.
\end{aligned}\right\} \tag{35}$$

From (34) and (35) it follows that

$$\varphi^2-\psi^2=1-\left(h+\frac{\mu}{\lambda}e^{\xi}\right)^2=1-\left(h_1+\frac{\mu_1}{\lambda_1}e^{\xi_1}\right)^2,$$

from which we find, in order that (34) and (35) be consistent, that

$$h_1+\frac{\mu}{\lambda}e^{\xi_1}=h+\frac{\mu}{\lambda}e^{\xi}=t, \tag{36}$$

t being a function thus defined.

One finds readily that the foregoing equations may be replaced by

$$\left.\begin{aligned} t &= -\tanh\frac{1}{2}(\theta_1+\theta), \\ \varphi &= \operatorname{sech}\frac{1}{2}(\theta+\theta_1)\cosh\frac{1}{2}(\theta_1-\theta), \\ \psi &= -\operatorname{sech}\frac{1}{2}(\theta+\theta_1)\sinh\frac{1}{2}(\theta_1-\theta). \end{aligned}\right\} \tag{37}$$

In consequence of equations (21) and analogous equations for S_1 namely,

$$\left.\begin{aligned} \frac{\partial h_1}{\partial u} &= (h_1+\coth\theta_1)\frac{\partial \xi_1}{\partial u}, \\ \frac{\partial h_1}{\partial v} &= (h_1+\tanh\theta_1)\frac{\partial \xi_1}{\partial v}, \end{aligned}\right\} \tag{38}$$

we have from (36)

$$\frac{\partial t}{\partial u} = \left(\frac{\partial \xi}{\partial u}\operatorname{csch}\theta - e^{\xi}\frac{\alpha}{\lambda}\right)\varphi, \qquad \frac{\partial t}{\partial v} = \left(\frac{\partial \xi}{\partial v}\operatorname{sech}\theta - e^{\xi}\frac{\beta}{\lambda}\right)\psi, \tag{39}$$

$$\frac{\partial t}{\partial u} = \left(\frac{\partial \xi_1}{\partial u}\operatorname{csch}\theta_1 - e^{\xi_1}\frac{\alpha}{\lambda}\right)\varphi, \qquad \frac{\partial t}{\partial v} = -\left(\frac{\partial \xi_1}{\partial v}\operatorname{sech}\theta_1 + e^{\xi_1}\frac{\beta}{\lambda}\right)\psi, \tag{40}$$

from which it follows that

$$\left.\begin{aligned} \frac{\partial \xi_1}{\partial u} &= \sinh\theta_1\left[\operatorname{csch}\theta\frac{\partial \xi}{\partial u} + \frac{\alpha}{\lambda}(e^{\xi_1}-e^{\xi})\right], \\ \frac{\partial \xi_1}{\partial v} &= -\cosh\theta_1\left[\operatorname{sech}\theta\frac{\partial \xi}{\partial v} + \frac{\beta}{\lambda}(e^{\xi_1}-e^{\xi})\right] \end{aligned}\right\} \tag{41}$$

When the expression (37) for t is substituted in (39), we obtain

$$\left.\begin{aligned} \frac{\partial \theta_1}{\partial u} + \frac{\partial \theta}{\partial u} &= -\left(\frac{\partial \xi}{\partial u}\operatorname{csch}\theta - e^{\xi}\frac{\alpha}{\lambda}\right)(\cosh\theta_1+\cosh\theta), \\ \frac{\partial \theta_1}{\partial v} + \frac{\partial \theta}{\partial v} &= \left(\frac{\partial \xi}{\partial v}\operatorname{sech}\theta - e^{\xi}\frac{\beta}{\lambda}\right)(\sinh\theta_1-\sinh\theta). \end{aligned}\right\} \tag{42}$$

One finds readily that in consequence of (31), the conditions of integrability of (41) and (42) are satisfied. Furthermore, if we have two functions θ_1 and ξ_1 satisfying the system (31), (41), (42), equations (32) are satisfied identically, the function h_1 given by (36) satisfies (38) and the functions θ_1, ξ_1

and h_1 satisfy an equation analogous to (22), because of the fundamental relation (6).

It is our purpose now to reduce our equations to a simple form and to this end we observe that as a consequence of equations (35) we have

$$\varphi \cosh \theta_1 + \psi \sinh \theta_1 = 1, \quad \psi \cosh \theta_1 + \varphi \sinh \theta_1 = -t, \tag{43}$$

from which follows

$$\cosh \theta_1 (t\varphi + \psi) + \sinh \theta_1 (t\psi + \varphi) = 0.$$

We replace this equation by

$$e^{\xi_1} \cosh \theta_1 = \rho (t\psi + \varphi), \quad e^{\xi_1} \sinh \theta_1 = -\rho (t\varphi + \psi),$$

where ρ is a factor of proportionality to be determined. To this end we differentiate these equations and express the condition that ξ_1 and θ_1 satisfy equations (41) and (42). This leads to the two equations

$$\frac{\partial \log \rho}{\partial u} + \frac{\partial \log \sigma}{\partial u} + \frac{\partial \xi}{\partial u} - \frac{\alpha}{\lambda} e^{\xi} \sinh \theta = 0,$$

$$\frac{\partial \log \rho}{\partial v} + \frac{\partial \log \sigma}{\partial v} + \frac{\partial \xi}{\partial v} - \frac{\beta}{\lambda} e^{\xi} \cosh \theta = 0.$$

Hence to within a constant factor ρ is equal to $\lambda/\sigma\, e^{\xi}$, which factor may be taken equal to unity since σ has thus far appeared only with the constant multiplier n. Hence we have

$$e^{\xi_1} \cosh \theta_1 = \frac{\lambda e^{-\xi}}{\sigma} (t\psi + \varphi), \quad e^{\xi_1} \sinh \theta_1 = -\frac{\lambda e^{-\xi}}{\sigma} (t\varphi + \psi). \tag{44}$$

Moreover, from (43) it follows that

$$e^{\xi_1} = \frac{\lambda}{\sigma} e^{-\xi} (1 - t^2). \tag{45}$$

From (44) we find readily the formulas

$$e^{\xi_1 + \xi} \sinh (\theta_1 + \theta) = -\frac{2\lambda t}{\sigma}, \quad e^{\xi_1 + \xi} \cosh (\theta_1 + \theta) = \frac{\lambda}{\sigma} (1 + t^2). \tag{46}$$

As a result of the preceding investigation the *fundamental system of equations* of a transformation from a surface of Guichard of the first kind into another surface of the same kind is the following:

$$\left.\begin{aligned}
&\frac{\partial \sigma}{\partial u} = e^{-\xi}\alpha(t\varphi+\psi), \quad \frac{\partial \sigma}{\partial v} = e^{-\xi}\beta(t\psi+\varphi),\\
&\frac{\partial \lambda}{\partial u} = e^{\xi}\alpha \sinh\theta, \quad \frac{\partial \lambda}{\partial v} = e^{\xi}\beta\cosh\theta,\\
&\frac{\partial \mu}{\partial u} = -\alpha(\cosh\theta + h\sinh\theta), \quad \frac{\partial \mu}{\partial v} = -\beta(\sinh\theta + h\cosh\theta),\\
&\frac{\partial \alpha}{\partial u} = -\beta\left(\tanh\theta\frac{\partial \xi}{\partial v}+\frac{\partial \theta}{\partial v}\right) + \mu(\cosh\theta + h\sinh\theta) +\\
&\qquad\qquad + n\sigma e^{\xi}\sinh\theta + n\lambda e^{-\xi}(t\varphi+\psi),\\
&\frac{\partial \alpha}{\partial v} = \beta\left(\coth\theta\frac{\partial \xi}{\partial u}+\frac{\partial \theta}{\partial u}\right), \quad \frac{\partial \beta}{\partial u} = \alpha\left(\tanh\theta\frac{\partial \xi}{\partial v}+\frac{\partial \theta}{\partial v}\right),\\
&\frac{\partial \beta}{\partial v} = -\alpha\left(\coth\theta\frac{\partial \xi}{\partial u}+\frac{\partial \theta}{\partial u}\right) + \mu(\sinh\theta + h\cosh\theta) +\\
&\qquad\qquad + n\sigma e^{\xi}\cosh\theta + n\lambda e^{-\xi}(t\psi+\varphi),
\end{aligned}\right\} \tag{I}$$

and the fundamental quadratic relation

$$2\lambda\sigma n = \alpha^2 + \beta^2 + \mu^2. \tag{II}$$

One finds that the system (I) and (II) is consistent. When we have a set of functions α, β, λ, μ, σ satisfying this system for a given value of the constant n, we say that we have a transformation T_n from one surface S into a second surface S_1. The fundamental functions ξ_1, θ_1 and h_1 are given at once from (44), (45) and (36).

Since one of the constants arising from the integration has been put in evidence, namely n, there remain three other arbitrary constants. Hence we have the theorem:

A surface of Guichard *of the first kind admits* ∞^3 *transformations* T_n *into surfaces of the same kind.*

Evidently these constants can be chosen so that a given point of S shall go into a given point of S_1, and then the transformation T_n is determined.

Incidentally we observe that for a surface of Guichard of the first kind the equations (2) assume the form

$$\frac{\partial \lambda}{\partial u}(\coth\theta + h) + e^{\xi}\frac{\partial \mu}{\partial u} = 0, \quad \frac{\partial \lambda}{\partial v}(\tanh\theta + h) + e^{\xi}\frac{\partial \mu}{\partial v} = 0.$$

Comparing these equations with (I), we find that a set of solutions is of the form

$$\lambda = e^{\xi} + l, \quad \mu = -h + m, \tag{47}$$

where l and m are arbitrary constants. It does not follow that S_1 in this case is a surface of GUICHARD. This question will be investigated later (§ 15).

If we proceed with surfaces of GUICHARD of the second kind in a manner analogous to the foregoing, we find

$$\frac{\partial}{\partial u}\left(\frac{\mu}{\lambda}\right) = -\frac{\alpha}{\lambda}\varphi, \quad \frac{\partial}{\partial u}\left(\frac{\mu}{\lambda}\right) = -\frac{\beta}{\lambda}\psi, \tag{33*}$$

where now

$$\varphi = \cos\theta + t\sin\theta, \quad \psi = -\sin\theta + t\cos\theta, \tag{34*}$$

$$\varphi = \cos\theta_1 + t\sin\theta_1, \quad \psi = \sin\theta_1 - t\cos\theta_1, \tag{35*}$$

$$t = h + \frac{\mu}{\lambda}e^{\xi} = h_1 + \frac{\mu}{\lambda}e^{\xi_1}. \tag{36*}$$

The functions ξ_1 and θ_1 of a transform S_1 are given by

$$e^{\xi_1}\cos\theta_1 = \frac{\lambda}{\sigma}(t\psi - \varphi)e^{-\xi}, \quad e^{\xi_1}\sin\theta_1 = -\frac{\lambda}{\sigma}(t\varphi + \psi)e^{-\xi}, \tag{44*}$$

$$e^{\xi_1} = -\frac{\lambda}{\sigma}(1 + t^2)e^{-\xi}; \tag{45*}$$

and the fundamental system analogous to (I) is

$$\left.\begin{aligned}
&\frac{\partial\sigma}{\partial u} = e^{-\xi}\alpha(t\varphi + \psi), \quad \frac{\partial\sigma}{\partial v} = e^{-\xi}\beta(t\psi - \varphi),\\
&\frac{\partial\lambda}{\partial u} = e^{\xi}\sin\theta\,\alpha, \quad \frac{\partial\lambda}{\partial v} = e^{\xi}\cos\theta\,\beta,\\
&\frac{\partial\mu}{\partial u} = -\alpha(\cos\theta + h\sin\theta), \quad \frac{\partial\mu}{\partial v} = \beta(\sin\theta - h\cos\theta),\\
&\frac{\partial\alpha}{\partial u} = -\beta\left(\tan\theta\frac{\partial\xi}{\partial v} + \frac{\partial\theta}{\partial v}\right) + \mu(\cos\theta + h\sin\theta) +\\
&\qquad\qquad + n\sigma e^{\xi}\sin\theta + n\lambda e^{-\xi}(t\varphi + \psi),\\
&\frac{\partial\alpha}{\partial v} = \beta\left(\cot\theta\frac{\partial\xi}{\partial u} - \frac{\partial\theta}{\partial u}\right), \quad \frac{\partial\beta}{\partial u} = \alpha\left(\tan\theta\frac{\partial\xi}{\partial v} + \frac{\partial\theta}{\partial v}\right),\\
&\frac{\partial\beta}{\partial v} = -\alpha\left(\cot\theta\frac{\partial\xi}{\partial u} - \frac{\partial\theta}{\partial u}\right) - \mu(\sin\theta - h\cos\theta) +\\
&\qquad\qquad + n\sigma e^{\xi}\cos\theta + n\lambda e^{-\xi}(t\psi - \varphi).
\end{aligned}\right\} \tag{I*}$$

In the following sections we limit our considerations to surfaces of the first kind, but it is easy to carry out similar processes for surfaces of the second kind, as the preceding results indicate.

§ 4. Envelope of the Circle-Plane of the Transformation.

It is a well known fact that the circles orthogonal to two surfaces which are in the relation of a transformation of Ribaucour form a cyclic system. We call the plane of this circle the ***circle-plane of the transformation*** and in this section we derive certain results for the envelope S_0 of this plane.

From (9) it follows that the direction-cosines of this plane are proportional to

$$\beta X' - \alpha X'', \quad \beta Y' - \alpha Y'', \quad \beta Z' - \alpha Z''. \tag{48}$$

Consequently the coordinates x_0, y_0, z_0 of a point M_0 on S_0 are of the form

$$x_0 = x + p(\alpha X' + \beta X'') + q X, \tag{49}$$

where p and q are to be determined. If we express the condition that for any infinitesimal variation of u and v, the displacement of M_0 is normal to the direction given by (48), and in the reduction make use of (I), we find that

$$\left.\begin{aligned} p &= -\frac{e^{\xi}}{r}, \quad q = -\frac{1}{r}(e^{\xi}\mu + 2n\lambda t), \\ r &= n\left\{ e^{-\xi}\lambda(1+t^2) - 2ht + e^{\xi}\sigma \right\}. \end{aligned}\right\} \tag{50}$$

We have seen that μ satisfies the tangential equation of S; hence there is a surface Σ_0 whose tangential coordinates are X, Y, Z, μ. The point coordinates, ξ_0, η_0, ζ_0, of Σ_0 are given by expressions of the form

$$\xi_0 = \mu X + \alpha X' + \beta X'', \tag{51}$$

as follows from (I) and the general theory of tangential coordinates (*).

(*) E. p. 163.

With the aid of the functions ξ_0, η_0, ζ_0, the differentials of x_0, y_0, z_0 may be given the form

$$d x_0 = X d\omega + \xi_0 d p, \tag{52}$$

where

$$\omega = q - p\mu. \tag{53}$$

If we put further

$$\psi = \mu\omega + \frac{p}{2}(\mu^2 + \alpha^2 + \beta^2) + \lambda, \tag{54}$$

the linear element of S_0 may be written

$$d s_0^2 = d\omega^2 + 2 d p d\psi. \tag{55}$$

These results which hold for any cyclic system (*) will be applied later to the transformations T_n.

§ 5. Transformations $\overline{T}_n$ of the Associate Surface.

The associate surface $\overline{S}$ of a given S admits transformations $\overline{T}_n$ analogous to those for S. The fundamental system of equations we denote by $(\overline{\text{I}})$ and $(\overline{\text{II}})$; they are analogous to (I) and (II) and differ only in that the functions are $\overline{\alpha}$, $\overline{\beta}$, $\overline{\mu}$, $\overline{\lambda}$, $\overline{\sigma}$. However, it is our purpose to show that the knowledge of each transformation T_n of S carries with it that of a transformation $\overline{T}_n$ of $\overline{S}$.

In § 1 we observed that μ satisfies the tangential equation of S and each solution of this equation leads to a transformation of Ribaucour of S. Since S and $\overline{S}$ correspond with parallelism of tangent planes and the lines of curvature on the two surfaces correspond, there is a transformation of Ribaucour of $\overline{S}$ determined by $\overline{\mu} = \mu$. We shall show that in fact this is a transformation $\overline{T}_n$.

Assuming that $\overline{\mu} = \mu$ leads to a $\overline{T}_n$, we observe from $(\overline{\text{I}})$ and (27) that in this case we must have

$$\overline{\alpha} = \alpha, \quad \overline{\beta} = -\beta. \tag{56}$$

(*) Bianchi, *Lezioni*, vol. II, p. 211.

From (26) and ($\bar{\text{I}}$) it follows that

$$\left.\begin{aligned}\frac{\partial \bar{\lambda}}{\partial u} &= -e^{-\xi}\alpha\left[\sinh\theta\,(1+h^2)+2h\cosh\theta\right],\\ \frac{\partial \bar{\lambda}}{\partial v} &= -e^{-\xi}\beta\left[\cosh\theta\,(1+h^2)+2h\sinh\theta\right].\end{aligned}\right\} \tag{57}$$

In consequence of the foregoing results, we have from (II) and ($\bar{\text{II}}$),

$$\bar{\lambda}\,\bar{\sigma} = \lambda\,\sigma. \tag{58}$$

By means of (26) the first two of ($\bar{\text{II}}$), namely

$$\frac{\partial \bar{\sigma}}{\partial u} = e^{-\bar{\xi}}\,\bar{\alpha}\,(\bar{t}\,\bar{\varphi}+\bar{\psi}), \qquad \frac{\partial \bar{\sigma}}{\partial v} = e^{-\bar{\xi}}\,\bar{\beta}\,(\bar{t}\,\bar{\psi}+\bar{\varphi}),$$

are transformable into

$$\frac{\partial \bar{\sigma}}{\partial u} = -\alpha e^{\xi}\left\{\left[h\frac{\mu^2}{\bar{\lambda}^2}e^{-\xi}+\frac{\mu}{\bar{\lambda}}\right]2e^{-\xi}\cosh\theta + \right.$$
$$\left. +\left[1+2h\frac{\mu}{\bar{\lambda}}e^{-\xi}+\frac{\mu^2}{\bar{\lambda}^2}e^{-2\xi}(h^2+1)\right]\sinh\theta\right\},$$

$$\frac{\partial \bar{\sigma}}{\partial v} = -\beta e^{\xi}\left\{\left[h\frac{\mu^2}{\bar{\lambda}^2}e^{-\xi}+\frac{\mu}{\bar{\lambda}}\right]2e^{-\xi}\sinh\theta + \right.$$
$$\left. +\left[1+2h\frac{\mu}{\bar{\lambda}}e^{-\xi}+\frac{\mu^2}{\bar{\lambda}^2}e^{-2\xi}(h^2+1)\right]\cosh\theta\right\}.$$

When $\bar{\sigma}$ in these equations is replaced by the value from (58), we obtain two equations of the form

$$A\cosh\theta + B\sinh\theta = 0, \qquad A\sinh\theta + B\cosh\theta = 0.$$

Hence $A = B = 0$, and thus we obtain two equations

$$C \equiv \frac{\mu^2}{\bar{\lambda}^2}+\frac{\lambda}{\bar{\lambda}}+\frac{\lambda\,\sigma}{\bar{\lambda}^2} = 0, \qquad C\left[(h^2+1)\,e^{-\xi}+e^{\xi}\,\frac{\bar{\lambda}}{\lambda}\right] = 0.$$

This necessitates

$$\bar{\lambda} = -\left(\frac{\mu^2}{\lambda}+\sigma\right),$$

which value satisfies (57) as is readily shown. And from (58) we have

$$\bar{\sigma} = -\frac{\lambda^2 \sigma}{\mu^2 + \sigma\lambda}.$$

Hence have the following theorem:

If α, β, λ, μ, σ determine a transformation T_n of a surface of GUICHARD *of the first kind, the functions*

$$\bar{\alpha} = \alpha, \quad \bar{\beta} = -\beta, \quad \bar{\lambda} = -\left(\frac{\mu^2}{\lambda} + \sigma\right), \quad \bar{\mu} = \mu, \quad \bar{\sigma} = -\frac{\lambda^2 \sigma}{\mu^2 + \sigma\lambda} \tag{59}$$

determine a transformation $\bar{T}_n$ of the associate surface $\bar{S}$. Moreover, the surface $\bar{S}_1$ resulting from $\bar{T}_n$ is the associate of the transform S_1 of S.

In order to prove the latter part of the theorem, we remark that the direction cosines of the normal to $\bar{S}_1$ are of the form

$$\bar{X}_1 = \bar{X} - \frac{1}{\bar{\sigma} n}\frac{\mu}{\bar{\lambda}}(\mu \bar{X} + \bar{\alpha}\bar{X}' + \bar{\beta}\bar{X}''),$$

as follows from (9). In consequence of (28) and (59) we have

$$\bar{X}_1 = -X_1, \quad \bar{Y}_1 = -Y_1, \quad \bar{Z}_1 = -Z_1. \tag{60}$$

From (45), (36) and (26) it follows that the functions e^{ξ} of $\bar{S}_1$ and of the associate of S_1 have the respective forms

$$\frac{\bar{\lambda}}{\bar{\sigma}} e^{-\bar{\xi}}\left[1 - \left(\frac{\mu}{\bar{\lambda}} e^{\bar{\xi}} + \bar{h}\right)^2\right], \qquad e^{-\xi_1}\left[1 - \left(t - \frac{\mu}{\lambda} e^{\xi_1}\right)^2\right].$$

One finds readily that each of these is equal to

$$e^{-\xi}\frac{\mu^2}{\lambda\sigma}(t^2 - 1) + 2t\frac{\mu}{\lambda} + e^{\xi}\frac{\sigma}{\lambda}. \tag{61}$$

Again from (36) and analogous equations for the transformation of $\bar{S}$ we have

$$\frac{\bar{h}_1 - h_1}{\mu} = \frac{1}{\bar{\lambda}}(e^{\bar{\xi}} - e^{\bar{\xi}_1}) + \frac{1}{\lambda}(e^{\xi_1} - e^{\xi}).$$

When the value (61) of $e^{\bar{\xi}_1}$ is substituted in the right-hand member of this equation, and also the expressions for $e^{\bar\xi}$ and e^{ξ_1}, it vanishes identically. Hence for $\bar S_1$ and the associate of S_1 the functions e^{ξ} and h are equal. From equations for these two surfaces analogous to (21) it follows that the functions θ also are equal. Consequently $\bar S_1$ is the associate of S_1.

The coordinates of $\bar S_1$ are given by

$$\bar x_1 - \bar x = -\frac{1}{\bar\sigma n}(\bar\mu \bar X + \bar\alpha \bar X' + \bar\beta \bar X''), \tag{62}$$

as follows from (5). In consequence of (59) and (28), we have

$$\frac{\bar x_1 - \bar x}{x_1 - x} = \frac{\bar y_1 - \bar y}{y_1 - y} = \frac{\bar z_1 - \bar z}{z_1 - z} = -\frac{\sigma}{\bar\sigma}. \tag{63}$$

If $\bar d$ and $\bar d$ denote the lengths of the joins of corresponding points on S and S_1 and on $\bar S$ and $\bar S_1$ respectively, we have from (5), (62) and (63)

$$d^2 = \frac{2\lambda}{\sigma n}, \quad \bar d^2 = \frac{2\bar\lambda}{\bar\sigma n}, \quad \frac{\bar d}{d} = -\frac{\sigma}{\bar\sigma}. \tag{64}$$

Since the radius R of the sphere of the transformation T_n is $\frac{\lambda}{\mu}$, it follows from (59) and (64) that

$$d\bar d + \frac{d^2}{R^2} = -\frac{2}{n}. \tag{65}$$

This gives a geometric interpretation of the constant n. We resume these results in the theorem:

If S and S_1 are in the relation of a transformation T_n, so also are the associates of these surfaces, namely $\bar S$ and $\bar S_1$. The joins of corresponding points on S and S_1 and on $\bar S$ and $\bar S_1$ are parallel and the lengths of these joins satisfy the relation

$$d\bar d + \frac{d^2}{R^2} = -\frac{2}{n}.$$

An interesting particular case is afforded when $n = -\frac{1}{2}$. Later (§ 11) it will be seen that this is a sort of critical value of n. If now ω denotes the angle between the join of corresponding points on S and S_1 and the normal to S, we have $d = 2\,R \cos \omega$.

Hence when $n = -\frac{1}{2}$ equation (65) reduces to

$$d\,\bar{d} = 4 \sin^2 \omega. \tag{66}$$

§ 6. The Inverse Transformation.

It is evident that the relation between a surface S and a trasform S_1 is reciprocal. It is our purpose now to fund the expressions for the functions α^{-1}, β^{-1}, λ^{-1}, μ^{-1}, σ^{-1} by which one obtanis S from S_1 by a transformation of the type discussed in § 3. From (65) it is evident that n has the same value as for the transformation from S to S_1. Hence we refer to the desired transformation as T_n^{-1}.

The analogue of equation (5) is

$$x - x_1 = -\frac{1}{\sigma^{-1}\,n}(\mu^{-1} X_1 + \alpha^{-1} X'_1 + \beta^{-1} X''_1).$$

If we replace X_1, X'_1 and X''_1 by their expressions (9) and equate the expressions for $(x_1 - x)$ given by this equation and (5), we obtain an equation of the form

$$A\,X + B\,X' + C\,X'' = 0,$$

where A, B, C are determinate functions.

Since we obtain also two equations of the same form but with the $X's$ replaced by $Y's$ and $Z's$ respectively, it follows that A, B, C must be zero. This gives the three equations

$$\sigma\frac{\mu^{-1}}{\mu} + \sigma^{-1} = \sigma\frac{\alpha^{-1}}{\alpha} + \sigma^{-1} = -\sigma\frac{\beta^{-1}}{\beta} + \sigma^{-1} = \frac{\mu\,\mu^{-1} + \alpha\,\alpha^{-1} - \beta\,\beta^{-1}}{n\,\lambda}.$$

From these equations follow

$$\alpha^{-1} = \rho\,\alpha, \quad \beta^{-1} = -\rho\,\beta, \quad \mu^{-1} = \rho\,\mu, \quad \sigma^{-1} = \rho\,\sigma,$$

where ρ is a factor of proportionality to be determined. To these may be added also $\lambda^{-1} = \rho\lambda$, since the radius of the sphere is given by

$$R = \frac{\lambda}{\mu} = \frac{\lambda^{-1}}{\mu^{-1}} \cdot$$

When this value of λ^{-1} is substituded in the equations

$$\frac{\partial \lambda^{-1}}{\partial u} = \alpha^{-1} e^{\xi_1} \sinh\theta_1, \quad \frac{\partial \lambda^{-1}}{\partial v} = \beta^{-1} e^{\xi_1} \cosh\theta_1,$$

which are analogous to the equations for λ in the system (I), it is found that $\rho = \frac{1}{\sigma\lambda}\cdot$ Hence the above equations become

$$\alpha^{-1} = \frac{\alpha}{\sigma\lambda}, \quad \beta^{-1} = -\frac{\beta}{\sigma\lambda}, \quad \mu^{-1} = \frac{\mu}{\sigma\lambda}, \quad \lambda^{-1} = \frac{1}{\sigma}, \quad \sigma^{-1} = \frac{1}{\lambda} \cdot \tag{67}$$

It is readily shown that these values satisfy the fundamental system for S_1 analogous to (I).

§ 7. The Theorem of Permutability.

In the succeeding sections we establish the following theorem of permutability for the transformations T_n of surfaces of Guichard of the first kind:

If S is a surface of Guichard *of the first kind and S_1 and S_2 are two surfaces of the same kind obtained from S by transformations T_{n_1} and T_{n_2}, where $n_2 \neq n_1$, these exists a unique surface S' of the same kind which may be obtained from S_1 by a transformation T'_{n_2} and from S_2 by a transformation T'_{n_1}.*

We say that four such surfaces S, S_1, S_2, S' form a *quatern.*

Let α_1, β_1, λ_1, μ_1, σ_1 denote the functions of the transformation T_{n_1} which gives rise to the surface S_1. The functions α'_1, β'_1, λ'_1, μ'_1, σ'_1 of a transformation T'_{n_2} of S_1 must satisfy the equations which are obtained from (I) on replacing ξ, θ, h and n by ξ_1, θ_1, h_1 and n_2. By means of the relations of § 3 these equations may be put in the form

$$
\left.
\begin{aligned}
&\frac{\partial \sigma'_1}{\partial u} = -\frac{\lambda_1}{\sigma_1} e^{-\xi} e^{-2\xi_1} K_1 \alpha'_1, \quad \frac{\partial \sigma'_1}{\partial v} = \frac{\lambda_1}{\sigma_1} e^{-\xi} e^{-2\xi_1} L_1 \beta'_1, \\
&\frac{\partial \lambda'_1}{\partial u} = -\frac{\lambda_1}{\sigma_1} e^{-\xi} (t_1 \varphi_1 + \psi_1) \alpha'_1, \quad \frac{\partial \lambda'_1}{\partial v} = \frac{\lambda_1}{\sigma_1} e^{-\xi} (t_1 \psi_1 + \varphi_1) \beta'_1, \\
&\frac{\partial \mu'_1}{\partial u} = -\alpha'_1 \left[\varphi_1 + \frac{\nu_1}{\sigma_1} e^{-\xi} (t_1 \varphi_1 + \psi_1)\right], \\
&\qquad\qquad \frac{\partial \mu'_1}{\partial v} = \beta'_1 \left[\psi_1 + \frac{\mu_1}{\sigma_1} e^{-\xi} (t_1 \psi_1 + \varphi_1)\right], \\
&\frac{\partial \alpha'_1}{\partial u} = \beta'_1 \left[\tanh\theta . \frac{\partial \xi}{\partial v} + \frac{\partial \theta}{\partial v} - \frac{\beta_1}{\lambda_1} M_1\right] + \mu'_1 \left[\varphi_1 + \frac{\mu_1}{\sigma_1} e^{-\xi} (t_1 \varphi_1 + \psi_1)\right] - \\
&\qquad - n_2 \sigma'_1 \frac{\lambda_1}{\sigma_1} e^{-\xi} (t_1 \varphi_1 + \psi_1) - n_2 \lambda'_1 \frac{\lambda_1}{\sigma_1} e^{-\xi} e^{-2\xi_1} K_1, \\
&\frac{\partial \alpha'_1}{\partial v} = -\beta'_1 \left[\coth\theta \frac{\partial \xi}{\partial u} + \frac{\partial \theta}{\partial u} - \frac{\alpha_1}{\lambda_1} N_1\right], \\
&\qquad\qquad \frac{\partial \beta'_1}{\partial u} = -\alpha'_1 \left[\tanh\theta \frac{\partial \xi}{\partial v} + \frac{\partial \theta}{\partial v} - \frac{\beta_1}{\lambda_1} M_1\right], \\
&\frac{\partial \beta'_1}{\partial v} = \alpha'_1 \left[\coth\theta \frac{\partial \xi}{\partial u} + \frac{\partial \theta}{\partial u} - \frac{\alpha_1}{\lambda_1} N_1\right] - \mu'_1 \left[\psi_1 + \frac{\mu_1}{\sigma_1} e^{-\xi} (t_1 \psi_1 + \varphi_1)\right] + \\
&\qquad + n_2 \sigma'_1 \frac{\lambda_1}{\sigma_1} e^{-\xi} (t_1 \psi_1 + \varphi_1) + n_2 \lambda'_1 \frac{\lambda_1}{\sigma_1} e^{-\xi} e^{-\xi_1} L_1,
\end{aligned}
\right\} \quad \text{(III)}
$$

where

$$
\left.
\begin{aligned}
&t_1 = h + \frac{\mu_1}{\lambda_1} e^{\xi} = h_1 + \frac{\mu_1}{\lambda_1} e^{\xi_1}, \quad t'_1 = h_1 + \frac{\mu'_1}{\lambda'_1} e^{\xi_1}, \\
&\varphi_1 = \cosh\theta + t_1 \sinh\theta, \quad \psi_1 = \sinh\theta + t_1 \cosh\theta, \\
&K_1 = \sinh\theta \left[(t'^2_1 + 1)(t^2_1 + 1) - 4 t_1 t'_1\right] + 2\cosh\theta (t_1 t'_1 - 1)(t'_1 - t_1), \\
&L_1 = \cosh\theta \left[(t'^2_1 + 1)(t^2_1 + 1) - 4 t_1 t'_1\right] + 2\sinh\theta (t_1 t'_1 - 1)(t'_1 - t_1), \\
&M_1 = e^{\xi} \sinh\theta + \frac{\lambda_1}{\sigma_1} e^{-\xi} (t_1 \varphi_1 + \psi_1), \\
&N_1 = e^{\xi} \cosh\theta + \frac{\lambda_1}{\sigma_1} e^{-\xi} (t_1 \psi_1 + \varphi_1).
\end{aligned}
\right\} \quad \text{(III')}
$$

To these equations must be added also the quadratic relation

$$\alpha_1'^2 + \beta_1'^2 + \mu_1'^2 = 2\,\lambda'_1\,\sigma'_1\,n_2\,. \tag{IV}$$

In like manner if α_2, β_2, λ_2, μ_2, σ_2 are the functions of the transformation T_{n_2} by which S is transformed into S_2, the fundamental system of equations to be satisfied by the functions α'_2, β'_2, λ'_2, μ'_2, σ'_2 of a transformation T'_{n_1} of S_2 may be obtained from (III), (III') and (IV) by replacing the subscripts 1 and 2 by 2 and 1 respectively.

It is our purpose to show that there exists a surface S' which may be obtained without quadratures from S_1 by a transformation T'_{n_2} and from S_2 by a transformation T'_{n_1}.

§ 8. Relations for General Transformations of Ribaucour.

In this section we shall derive preparatory to the proof of the theorem of permutability certain relations connecting the transformation functions which hold for any transformation of Ribaucour possessing a theorem of permutability (*).

Equations (9) may be written in the form

$$\left.\begin{aligned} X'_1 &= a_{11}\,X' + a_{12}\,X'' + a_{13}\,X, \\ X''_1 &= a_{21}\,X' + a_{22}\,X'' + a_{23}\,X, \\ X_1 &= a_{31}\,X' + a_{32}\,X'' + a_{33}\,X, \end{aligned}\right\} \tag{68}$$

where

$$\left.\begin{aligned} &a_{11} = 1 - \frac{\alpha_1^2}{n_1\,\sigma_1\,\lambda_1}, \quad a_{22} = -1 + \frac{\beta_1^2}{n_1\,\sigma_1\,\lambda_1}, \quad a_{33} = 1 - \frac{\mu_1^2}{n_1\,\sigma_1\,\lambda_1}, \\ &a_{12} = -a_{21} = -\frac{\alpha_1\,\beta_1}{n_1\,\sigma_1\,\lambda_1}, \quad a_{13} = a_{31} = -\frac{\alpha_1\,\mu_1}{n_1\,\sigma_1\,\lambda_1}, \\ &a_{23} = -a_{32} = \frac{\beta_1\,\mu_1}{n_1\,\sigma_1\,\lambda_1}. \end{aligned}\right\} \tag{69}$$

From (5) it follows that the coordinates x', y', z' of S', a transform of

(*) Cf. Bianchi, *Ricerche sulle superficie isoterme, etc.*, p. 18.

S_1 by a T'_{n_2}, are given by

$$x' = x_1 - \frac{1}{n_2 \sigma'_1} (\alpha'_1 X'_1 + \beta'_1 X''_1 + \mu'_1 X_1). \tag{70}$$

If S' is at the same time a transform of S_2 by a T'_{n_1}, we must have

$$x' = x_2 - \frac{1}{n_1 \sigma'_2} (\alpha'_2 X'_2 + \beta'_2 X''_2 + \mu'_2 X_2), \tag{71}$$

where X'_2, X''_2 and X_2 are of the form

$$\left.\begin{aligned} X'_2 &= b_{11} X' + b_{12} X'' + b_{13} X, \\ X''_2 &= b_{21} X' + b_{22} X'' + b_{23} X, \\ X_2 &= b_{31} X' + b_{32} X'' + b_{33} X, \end{aligned}\right\} \tag{72}$$

a coefficient b_{ij} being given by a_{ij} in (69) when the subscripts 1 are replaced by 2.

If these two values of x' be equated, the resulting equation is reducible by means of equations of the form (5) for S_1 and S_2 and by (68) and (72) to the form

$$P X' + Q X'' + R X = 0.$$

There are two similar equations obtained by replacing the X' s by Y' s and Z' s. Hence we must have $P = Q = R = 0$. If we put for brevity

$$A_i = \frac{\alpha'_1 a_{1i} + \beta'_1 a_{2i} + \mu'_1 a_{3i}}{n_2 \sigma'_1}, \quad B_i = \frac{\alpha'_2 b_{1i} + \beta'_2 b_{2i} + \mu'_2 b_{3i}}{n_1 \sigma'_2},$$

for $i = 1, 2, 3$, this gives the three equations

$$\left.\begin{aligned} A_1 - B_1 = \frac{\alpha_2}{n_2 \sigma_2} - \frac{\alpha_1}{n_1 \sigma_1}, \quad A_2 - B_2 = \frac{\beta_2}{n_2 \sigma_2} - \frac{\beta_1}{n_1 \sigma_1}, \\ A_3 - B_3 = \frac{\mu_2}{n_2 \sigma_2} - \frac{\mu_1}{n_1 \sigma_1}. \end{aligned}\right\} \tag{73}$$

Another set of necessary conditions may be obtained by considering the two sets of expressions for the direction-cosines of the principal directions and normal to S'. These are of the form

$$\begin{aligned} X'_3 &= a'_{11} X'_1 + a'_{12} X''_1 + a'_{13} X_1 = b'_{11} X'_2 + b'_{12} X''_2 + b'_{13} X_2, \\ X''_3 &= a'_{21} X'_1 + a'_{22} X''_1 + a'_{23} X_1 = b'_{21} X'_2 + b'_{22} X''_2 + b'_{23} X_2, \\ X_3 &= a'_{31} X'_1 + a'_{32} X''_1 + a'_{33} X_1 = b'_{31} X'_2 + b'_{32} X''_2 + b'_{33} X_2, \end{aligned}$$

where the functions a'_{ij} and b'_{ij} are of the same form as (69) in the functions of T'_{n_2} and T'_{n_1} respectively. Proceeding as above we find the following nine equations of condition:

$$\left.\begin{aligned}
&\frac{\alpha'_1}{\lambda'_1} A_i - \frac{\alpha'_2}{\lambda'_2} B_i = a_{1i} - b_{1i}, \\
&\frac{\beta'_1}{\lambda'_1} A_i - \frac{\beta'_2}{\lambda'_2} B_i = a_{2i} - b_{2i}, \qquad (i = 1, 2, 3) \\
&\frac{\mu'_1}{\lambda'_1} A_i - \frac{\mu'_2}{\mu'_2} B_i = a_{3i} - b_{3i}.
\end{aligned}\right\} \tag{74}$$

In order that these equations may be consistent with (73), it is necessary that for the following matrix

$$\left\| \begin{array}{cccccc}
\alpha'_1 & \alpha'_2 & a_{11} - b_{11} & a_{12} - b_{12} & a_{13} - b_{13} \\
\beta'_1 & \beta'_2 & a_{21} - b_{21} & a_{22} - b_{22} & a_{23} - b_{23} \\
\mu'_1 & \mu'_2 & a_{31} - b_{31} & a_{32} - b_{32} & a_{33} - b_{33} \\
\lambda'_1 & \lambda'_2 & \dfrac{\alpha_2}{n_2 \sigma_2} - \dfrac{\alpha_1}{n_1 \sigma_1} & \dfrac{\beta_2}{n_2 \sigma_2} - \dfrac{\beta_1}{n_1 \sigma_1} & \dfrac{\mu_2}{n_2 \sigma_2} - \dfrac{\mu_1}{n_1 \sigma_1}
\end{array} \right\|$$

every minor of the third order involving the terms of the first two columns be zero. From the form of the expressions for the functions a_{ij} and b_{ij}, and the fact that the fundamental relation (II) is satisfied by T_{n_1} and T_{n_2} the preceding condition necessitates the proportionality of the corresponding minors of the second order of the matrices

$$\left\| \begin{array}{cccc} \alpha'_1 & \beta'_1 & \mu'_1 & \lambda'_1 \\ \alpha'_2 & \beta'_2 & \mu'_2 & \lambda'_2 \end{array} \right\|, \qquad \left\| \begin{array}{cccc} \alpha_1 & -\beta_1 & \mu_1 & \lambda_1 \\ \alpha_2 & -\beta_2 & \mu_2 & \lambda_2 \end{array} \right\|.$$

From this it follows in turn that the minors of the third order of the following matrices must vanish

$$\left\| \begin{array}{cccc} \alpha'_1 & \beta'_1 & \mu'_1 & \lambda'_1 \\ \alpha_1 & -\beta_1 & \mu_1 & \lambda_1 \\ \alpha_2 & -\beta_2 & \mu_2 & \lambda_2 \end{array} \right\|, \qquad \left\| \begin{array}{cccc} \alpha'_2 & \beta'_2 & \mu'_2 & \lambda'_2 \\ \alpha_1 & -\beta_1 & \mu_1 & \lambda_1 \\ \alpha_2 & -\beta_2 & \mu_2 & \lambda_2 \end{array} \right\|. \tag{75}$$

§ 9. Determination of T'_{n_2}.

We now apply the general results of the preceding section to the case of surfaces of Guichard.

For the sake of brevity we use the notation

$$(\lambda_1\,\mu_2) = \lambda_1\,\mu_2 - \lambda_2\,\mu_1,$$

and we consider the following identitstet which follows from (75):

$$\alpha'_1\,(\mu_1\,\beta_2) + \beta'_1\,(\mu_1\,\alpha_2) + \mu'_1\,(\beta_1\,\alpha_2) = 0. \tag{76}$$

If this equation be differentiated with respect to u and v separately, and the derivatives of the various functions be replaced by their expressions from (I) and (III), we obtain the following equations:

$$\begin{aligned}(\mu_1\,\beta_2)\Big[(\alpha_1\,\alpha'_1 - \beta_1\,\beta'_1 + \mu_1\,\mu'_1)\frac{M_1}{\lambda_1} - n_2\,\sigma'_1\,\frac{\lambda_1}{\sigma_1}\,e^{-\xi}\,(t_1\,\varphi_1 + \psi_1) -\\ - n_2\,\lambda'_1\,\frac{\lambda_1}{\sigma_1}\,e^{-\xi}\,e^{-2\xi_1}\,K_1\Big] +\\ + n_2\,\sigma_2\,(\mu'_1\,\beta_1 + \beta'_1\,\mu_1)\,M_2 - n_1\,\sigma_1\,(\mu'_1\,\beta_2 + \beta'_1\,\mu_2)\,M_1 = 0,\end{aligned}$$

$$\begin{aligned}(\mu_1\,\alpha_2)\Big[(\alpha_1\,\alpha'_1 - \beta_1\,\beta'_1 + \mu_1\,\mu'_1)\frac{N_1}{\lambda_1} - n_2\,\sigma'_1\,\frac{\lambda_1}{\sigma_1}\,e^{-\xi}\,(t_1\,\psi_1 + \varphi_1) -\\ - n_2\,\lambda'_1\,\frac{\lambda_1}{\sigma_1}\,e^{-\xi}\,e^{-\xi_1}\,L_1\Big] +\\ + n_2\,\sigma_2\,(\mu'_1\,\alpha_1 - \alpha'_1\,\mu_1)\,N_2 - n_1\,\sigma_1\,(\mu'_1\,\alpha_2 - \alpha'_1\,\mu_2)\,N_1 = 0,\end{aligned}$$

where M_2 and N_2 have expressions analogous to those of M_1 and N_1.

With the aid of the identities

$$\left.\begin{aligned}\beta'_1\,(\mu_1\,\lambda_2) + \mu'_1\,(\beta_1\,\lambda_2) + \lambda'_1\,(\mu_1\,\beta_2) = 0,\\ \alpha'_1\,(\mu_1\,\lambda_2) + \mu'_1\,(\lambda_1\,\alpha_2) + \lambda'_1\,(\alpha_1\,\mu_2) = 0,\end{aligned}\right\} \tag{77}$$

which arise from (75), the preceding equations are reducible to

$$\left.\begin{aligned}\mu'_1\,\Big\{M_1\left(\frac{\beta_1}{\lambda_1}\,\Phi - n_1\,\sigma_1\,\beta_2\right) - n_2\,\sigma_2\,\beta_1\,M_2 + n_2\,\frac{\lambda_1}{\sigma_1}\,e^{-\xi}\,e^{-2\xi_1}\,K_1\,.\,(\lambda_1\,\beta_2)\Big\} +\\ + \beta'_1\,\Big\{M_1\left(\frac{\mu_1}{\lambda_1}\,\Phi - n_1\,\sigma_1\,\mu_2\right) - n_2\,\sigma_2\,\mu_1\,M_2 + n_2\,\frac{\lambda_1}{\sigma_1}\,e^{-\xi}\,e^{-2\xi_1}\,K_1\,(\lambda_1\,\mu_2)\Big\} +\\ + n_2\,\sigma'_1\,(\mu_1\,\beta_2)\,\frac{\lambda_1}{\sigma_1}\,e^{-\xi}\,(t_1\,\varphi_1 + \psi_1) = 0,\end{aligned}\right\} \tag{78}$$

$$\left.\begin{aligned}
&\mu'_1 \left\{ N_1 \left(\frac{\beta_1}{\lambda_1}\Phi - n_1\sigma_1\beta_2\right) - n_2\sigma_2\beta_1 N_2 + n_2\frac{\lambda_1}{\sigma_1}e^{-\xi}e^{-2\xi_1}L_1(\lambda_1\beta_2)\right\} + \\
&+ \beta'_1 \left\{ N_1 \left(\frac{\mu_1}{\lambda_1}\Phi - n_1\sigma_1\mu_2\right) - n_2\sigma_2\mu_1 N_2 + n_2\frac{\lambda_1}{\sigma_1}e^{-\xi}e^{-2\xi_1}L_1(\lambda_1\mu_2)\right\} + \\
&+ n_2\sigma'_1(\mu_1\beta_2)\frac{\lambda_1}{\sigma_1}e^{-\xi}(t_1\psi_1 + \varphi_1) = 0,
\end{aligned}\right\} \quad (78)$$

where

$$\Phi = \alpha_1\alpha_2 + \beta_1\beta_2 + \mu_1\mu_2. \quad (79)$$

When one operates upon any other equation arising from (75) after the manner in which (76) was treated, the resulting equations are reducible by means of (75) to equations (78).

When one eliminates σ'_1 from the equations (78) and then obtains a linear relation between β'_1 and μ'_1, the latter may be replaced by

$$\left.\begin{aligned}
\mu'_1 &= \rho\left[\mu_1\Omega + \sigma_1(\mu_1\lambda_2)(n_1 t_1 - n_2 t'_1)\right], \\
\beta'_1 &= -\rho\left[\beta_1\Omega + \sigma_1(\beta_1\lambda_2)(n_1 t_1 - n_2 t'_1)\right],
\end{aligned}\right\} \quad (80)$$

where

$$\Omega = t_1(\Phi - n_2\sigma_2\lambda_1 - n_1\sigma_1\lambda_2) - \lambda_1\lambda_2 n_2 e^{-2\xi}(t_1 t_2 - 1)(t_2 - t_1). \quad (81)$$

From (77) and (80) it follows that

$$\left.\begin{aligned}
\alpha'_1 &= \rho\left[\alpha_1\Omega + \sigma_1(\alpha_1\lambda_2)(n_1 t_1 - n_2 t'_1)\right], \\
\lambda'_1 &= \rho\lambda_1\Omega.
\end{aligned}\right\} \quad (82)$$

We are now confronted with the probleme of evaluating t'_1 and ρ. To this end we remark that in consequence of the first two of equations (III') we must have

$$t'_1 - t_1 = \left(\frac{\mu'_1}{\lambda'_1} - \frac{\mu_1}{\lambda_1}\right)e^{\xi_1}. \quad (83)$$

Expressing the condition that μ'_1 and λ'_1 as given by (80) and (81) satisfy this relation, we have

$$\left.\begin{aligned}
(t'_1 - t_1)\Omega &= (\mu_1\lambda_2)(n_1 t_1 - n_2 t'_1)e^{-\xi}(1 - t_1^2) = \\
&= \lambda_1\lambda_2 e^{-2\xi}(1 - t_1^2)(t_1 - t_2)(n_1 t_1 - n_2 t'_1);
\end{aligned}\right\} \quad (84)$$

this second expression is a consequence of (36). With the aid of this result equations (80) and (82) can be given the form

$$\left.\begin{aligned}
\mu'_1 &= \rho'\left[\mu_1(\mu_1\lambda_2)e^{-\xi}(1-t_1^2)+(\mu_1\lambda_2)\sigma_1(t'_1-t_1)\right],\\
\beta'_1 &= -\rho'\left[\beta_1(\mu_1\lambda_2)e^{-\xi}(1-t_1^2)+(\beta_1\lambda_2)\sigma_1(t'_1-t_1)\right],\\
\alpha'_1 &= \rho'\left[\alpha_1(\nu_1\lambda_2)e^{-\xi}(1-t_1^2)+(\alpha_1\lambda_2)\sigma_1(t'_1-t_1)\right],\\
\lambda'_1 &= \rho'\lambda_1(\nu_1\lambda_2)e^{-\xi}(1-t_1^2),
\end{aligned}\right\} \tag{85}$$

where

$$\rho' = \frac{\rho\,\Omega}{(\nu_1\lambda_2)e^{-\xi}(1-t_1^2)}. \tag{86}$$

If we put

$$\Psi_1 = \Phi - n_2(\sigma_2\lambda_1+\sigma_1\lambda_2) - \lambda_1\lambda_2 e^{-2\xi} n_2 (t_1-t_2)^2, \tag{87}$$

equation (84) may be written

$$(t'_1-t_1)\left[\Psi_1+(n_2-n_1)\lambda_2\sigma_1\right] = e^{-\xi}(\nu_1\lambda_2)(1-t_1^2)(n_1-n_2). \tag{88}$$

By means of this expression equations (85) are reducible to the form

$$\left.\begin{aligned}
\mu'_1 &= \bar\rho\left[\mu_1\Psi_1+\mu_2\lambda_1\sigma_1(n_2-n_1)\right],\\
\beta'_1 &= -\bar\rho\left[\beta_1\Psi_1+\beta_2\lambda_1\sigma_1(n_2-n_1)\right],\\
\alpha'_1 &= \bar\rho\left[\alpha_1\Psi_1+\alpha_2\lambda_1\sigma_1(n_2-n_1)\right],\\
\lambda'_1 &= \bar\rho\left[\lambda_1\Psi_1+\lambda_2\lambda_1\sigma_1(n_2-n_1)\right],
\end{aligned}\right\} \tag{89}$$

where

$$\bar\rho = \frac{\rho\,\Omega}{\Psi_1+(n_2-n_1)\lambda_2\sigma_1}.$$

Before determining $\bar\rho$ we derive from equations (78) the expressions for σ'_1. To this end we multiply these equations by *cosh* θ and *sinh* θ and add the resulting equations. In consequence of (46) and (III') this gives on the elimination of μ'_1 and β'_1 by means of (89)

$$\sigma'_1 = \bar{\rho}\left[\sigma_1 \Psi_1 + \sigma_2 \lambda_1 \sigma_1 (n_2 - n_1) + \frac{\lambda_1 \lambda_2 e^{-2\xi} (n_2 - n_1)(t_1 - t_2)^2 \sigma_1 \Psi_1}{\Psi_1 + (n_2 - n_1) \lambda_2 \sigma_1}\right]. \quad (90)$$

In order to determine $\bar{\rho}$ we calculate the first derivatives of Ψ_1 and find

$$\left.\begin{aligned} \frac{\partial \Psi_1}{\partial u} &= (n_1 - n_2)\, \alpha_2 \left[e^{\xi} \sinh \theta\, \sigma_1 + e^{-\xi} \lambda_1 (t_1 \varphi_1 + \psi_1)\right], \\ \frac{\partial \Psi_1}{\partial v} &= (n_1 - n_2)\, \beta_2 \left[e^{\xi} \cosh \theta\, \sigma_1 + e^{-\xi} \lambda_1 (t_1 \psi_1 + \varphi_1)\right]. \end{aligned}\right\} \quad (91)$$

If we express the conditions that the functions λ'_1, α'_1, β'_1 satisfy the third and fourth of equations (III) we obtain

$$\frac{\partial \log \bar{\rho}\, \lambda_1 \sigma_1}{\partial u} = 0, \qquad \frac{\partial \log \bar{\rho}\, \lambda_1 \sigma_1}{\partial v} = 0.$$

To within a constant multiplier $\bar{\rho}$ is $\frac{1}{\lambda_1 \sigma_1}\cdot$ Without loss of generality we take this constant equal to unity and obtain the following fundamental set of values:

$$\left.\begin{aligned} \mu'_1 &= \frac{1}{\lambda_1 \sigma_1}\left[\mu_1 \Psi_1 + \mu_2 \lambda_1 \sigma_1 (n_2 - n_1)\right], \\ \beta'_1 &= -\frac{1}{\lambda_1 \sigma_1}\left[\beta_1 \Psi_1 + \beta_2 \lambda_1 \sigma_1 (n_2 - n_1)\right], \\ \alpha'_1 &= \frac{1}{\lambda_1 \sigma_1}\left[\alpha_1 \Psi_1 + \alpha_2 \lambda_1 \sigma_1 (n_2 - n_1)\right], \\ \lambda'_1 &= \frac{1}{\lambda_1 \sigma_1}\left[\lambda_1 \Psi_1 + \lambda_2 \lambda_1 \sigma_1 (n_2 - n_1)\right], \\ \sigma'_1 &= \frac{1}{\lambda_1 \sigma_1}\Big[\sigma_1 \Psi_1 + \sigma_2 \lambda_1 \sigma_1 (n_2 - n_1) + \\ &\qquad + \frac{\lambda_1 \lambda_2 e^{-2\xi} (n_2 - n_1)(t_1 - t_2)^2 \sigma_1 \Psi_1}{\Psi_1 + (n_2 - n_1) \lambda_2 \sigma_1}\Big]. \end{aligned}\right\} \quad (V)$$

One shows readily that these expressions satisfy the systems (III) and (IV).

§ 10. Determination of S'.

From (V) we have

$$\alpha_1 \alpha'_1 - \beta_1 \beta'_1 + \mu_1 \mu'_1 = 2 \Psi_1 n_1 + \Phi (n_2 - n_1). \tag{92}$$

We put

$$\left.\begin{aligned} \chi_1 &= \Psi_1 + (n_2 - n_1) \lambda_2 \sigma_1 = \Phi - n_2 \sigma_2 \lambda_1 - n_1 \sigma_1 \lambda_2 - \lambda_1 \lambda_2 e^{-2\xi} n_2 (t_1 - t_2)^2, \\ \chi_2 &= \Psi_2 + (n_1 - n_2) \lambda_1 \sigma_2 = \Phi - n_1 \sigma_1 \lambda_2 - n_2 \sigma_2 \lambda_1 - \lambda_1 \lambda_2 e^{-2\xi} n_1 (t_1 - t_2)^2. \end{aligned}\right\} \tag{93}$$

When the expressions (V) are substituted in (70), the resulting equation is reducible by (5), (68) and (69) to

$$\left.\begin{aligned} x' - x = \frac{n_1 - n_2}{n_1 n_2 \sigma'_1 \lambda_1 \chi_1} \Big\{ & X' (n_1 \lambda_1 \alpha_2 \chi_1 - n_2 \lambda_2 \alpha_1 \chi_2) + \\ & + X'' (n_1 \lambda_1 \beta_2 \chi_1 - n_2 \lambda_2 \beta_1 \chi_2) + \\ & + X (n_1 \lambda_1 \mu_2 \chi_1 - n_2 \lambda_2 \mu_1 \chi_2) \Big\}, \end{aligned}\right\} \tag{VI}$$

where

$$\lambda_1 \sigma'_1 \chi_1 = \lambda_1 \lambda_2 \sigma_1 \sigma_2 (n_1 - n_2)^2 + \Psi_1 \Psi_2 . \tag{94}$$

It is evident that the espressions (VI) and (94) are symmetric in the functions of the transformations T_{n_1} and T_{n_2}. Hence if we had applied the preceding methods to S_2 we should have been brought to the same result. Accordingly we have established the theorem of permutability stated at the beginning of § 7, provided we show that S' is different from S.

According to the theorem the case $n_2 = n_1$ is excluded. It follows from (VI) that if S' and S coincide, either $\chi_1 = \chi_2 = 0$, or

$$\frac{n_1 \lambda_1 \alpha_2}{n_2 \lambda_2 \alpha_1} = \frac{n_1 \lambda_1 \beta_2}{n_2 \lambda_2 \alpha_1} = \frac{n_1 \lambda_1 \mu_2}{n_2 \lambda_2 \mu_1} = \frac{\chi_2}{\chi_1} . \tag{95}$$

The first of these conditions necessitates

$$\Phi - n_2 \sigma_2 \lambda_1 - n_1 \sigma_1 \lambda_2 = 0, \quad t_1 - t_2 = 0, \tag{96}$$

as follows from (93). But the second of (96) gives $\frac{\mu_2}{\lambda_2} - \frac{\mu_1}{\lambda_1} = 0$, and consequently, the radii of the two spheres are equal, which means that S_1 and S_2 coincide.

If (95) is satisfied, then

$$\frac{\alpha_2}{\alpha_1} = \frac{\beta_2}{\beta_1} = \frac{\mu_2}{\mu_1} = \rho,$$

where ρ is a factor of proportionality, which is found to be constant when this value of μ_2 is substituted in equations (I) for $\frac{\partial \mu}{\partial u}$ and $\frac{\partial \mu}{\partial v}$.

Since the equations of the system (I) and (II) are homogeneous, the foregoing proportionalities may be replaced by

$$\alpha_2 = \alpha_1, \quad \beta_2 = \beta_1, \quad \mu_2 = \mu_1. \tag{97}$$

From the equations for $\frac{\partial \lambda}{\partial u}$ and $\frac{\partial \lambda}{\partial v}$, it follows that

$$\lambda_2 - \lambda_1 = c,$$

where c is constant. In like manner from the equations for $\frac{\partial \alpha}{\partial u}$ and $\frac{\partial \beta}{\partial v}$, we obtain the condition

$$n_2 \lambda_2 t_2 = n_1 \lambda_1 t_1.$$

If this equation be differentiated with respect to u and v separately, the resulting equations are reducible to

$$\alpha_1 e^{\xi} \sinh \theta = \left(\lambda_1 + \frac{n_2 c}{n_2 - n_1}\right) \frac{\partial \xi}{\partial u}, \quad \beta_1 e^{\xi} \cosh \theta = \left(\lambda_1 + \frac{n_2 c}{n_2 - n_1}\right) \frac{\partial \xi}{\partial v}.$$

In consequence of the expressions for $\frac{\partial \lambda}{\partial u}$ and $\frac{\partial \lambda}{\partial v}$ in (I), the foregoing equations lead by integration to

$$\lambda_1 = e^{\xi} + l_1,$$

by a suitable choice of the constant integration. From (I) and the foregoing, it follows that μ_1 is equal to $-h$ to within an additive constant. Hence we write the above results thus:

$$\lambda_1 = e^{\xi} + l_1, \quad \mu_1 = m_1 - h, \quad \alpha_1 = \cosh \theta \frac{\partial \xi}{\partial u}, \quad \beta_1 = \operatorname{senh} \theta \frac{\partial \xi}{\partial v}. \tag{98}$$

Later (§ 15) we find that this transformation is not possible for the general surface of Guichard, and furthermore that it is impossible to have (98) and (97) hold at the same time; as $n_2 \neq n_1$. Accordingly we have the theorem:

If the constants n_1 and n_2 are unequal, the surface S' is distinct from S.

§ 11. Case where $n_2 = n_1$.

Thus far we have excluded from the discussion the case where $n_2 = n_1$. In taking it up for discussion, we observe that as a consequence of (VI), (93) and (94), S' coincides with S unless $\Psi_1 = 0$, since $\Psi_1 = \Psi_2$ for $n_2 = n_1$. But it follows from (91) that now $\Psi_1 = \text{const.}$ Hence we assume that the arbitrary constants in the functions of $\lambda_2, \mu_2, \alpha_2, \beta_2, \sigma_2$ satisfying (I) and (II) are chosen so that $\Psi_1 = 0$. Consequently

$$\Phi = \alpha_1 \alpha_2 + \beta_1 \beta_2 + \mu_1 \mu_2 = n_1 (\sigma_2 \lambda_1 + \sigma_1 \lambda_2) + \lambda_1 \lambda_2 e^{-2\xi} n_1 (t_1 - t_2)^2. \qquad (99)$$

From this and (II) we have

$$(\alpha_1 \lambda_2 - \alpha_2 \lambda_1)^2 + (\beta_1 \lambda_2 - \beta_2 \lambda_1)^2 + (\mu_1 \lambda_2 - \mu_2 \lambda_1)^2 (1 + 2 n_1) = 0. \qquad (100)$$

Hence when $1 + 2 n_1 \geqq 0$, the only real solutions of the problem are such that

$$\frac{\alpha_2}{\alpha_1} = \frac{\beta_2}{\beta_1} = \frac{\mu_2}{\mu_1} = \frac{\lambda_2}{\lambda_1}.$$

As shown at the close of the last section, the factor of proportionality is constant and may be taken equal to unity. Thus S_2 coincides with S_1. However, when $2 n_1 + 1 < 0$, this condition is not imposed by (100). Accordingly we investigate this case.

From (81) we obtain as the expression for Ω

$$\Omega = \lambda_1 \lambda_2 e^{-2\xi} n_1 (t_1 - t_2)(t_1^2 - 1) = n_1 (\lambda_1 \mu_2 - \lambda_2 \mu_1) e^{\xi_1} \frac{\sigma_1}{\lambda_1}. \qquad (101)$$

When this value is substituted in (84), the latter equation is satisfied identically, and hence does not, as in the general case, give an expression for $t'_1 - t_1$. In order to obtain the latter we proceed as follows.

As suggested by (88), we seek a function r such that

$$t'_1 - t_1 = r\,\lambda_1\,\lambda_2\,e^{-2\xi}\,(t_1 - t_2)\,(1 - t_1^2) = r\,\lambda_2\,\sigma_1\,e^{-\xi+\xi_1}\,(t_1 - t_2). \qquad (102)$$

Evidently t'_1 must satisfy equations analogous to (39), namely

$$\frac{\partial t'_1}{\partial u} = \left(\frac{\partial \xi_1}{\partial u}\,\operatorname{csch}\theta_1 - e^{\xi_1}\,\frac{\alpha'_1}{\lambda'_1}\right)(\cosh\theta_1 + t'_1\sinh\theta_1),$$

$$\frac{\partial t'_1}{\partial v} = \left(\frac{\partial \xi_1}{\partial v}\,\operatorname{sech}\theta_1 - e^{\xi_1}\,\frac{\beta'_1}{\lambda'_1}\right)(\sinh\theta_1 + t'_1\cosh\theta_1),$$

in which α'_1, β'_1, λ'_1 have the values (80) and (82). If we express this condition, the resulting equations for the determination of r may be put in the form

$$\left.\begin{aligned} \frac{\partial}{\partial u}\left(\frac{1}{r} + \lambda_2\,\sigma_1\right) &= \alpha_2\left[\lambda_1\,e^{-\xi}\,(t_1\,\varphi_1 + \psi_1) + \sigma_1\,e^{\xi}\sinh\theta\right], \\ \frac{\partial}{\partial v}\left(\frac{1}{r} + \lambda_2\,\sigma_1\right) &= \beta_2\left[\lambda_1\,e^{-\xi}\,(t_1\,\psi_1 + \varphi_1) + \sigma_1\,e^{\xi}\cosh\theta\right]. \end{aligned}\right\} \qquad (103)$$

These equations satisfy the condition of integrability (*) and hence r can be found by quadratures.

In order to find the coordinates of S' we must derive from equations (78) the expression for σ'_1 when μ'_1 and β'_1 have the values (80). This is

$$\left.\begin{aligned} \frac{\sigma'_1\,t_1}{\rho} = \Omega\left[t_1\,\sigma_1 - \lambda_1\,e^{-2\xi_1}\,(t_1\,t'_1 - 1)\,(t'_1 - t_1)\right] + \\ + \sigma_1\,n_1\,(t_1 - t'_1)\left[(\sigma_1\,t_2\,\lambda_2 - \sigma_2\,t_1\,\lambda_1) - t_1\,\lambda_1\,\lambda_2\,e^{-2\xi}\,(t_1 - t_2)^2\right]. \end{aligned}\right\} \qquad (104)$$

From equations (5), (70), (68), (69), (80), and (82) we obtain

$$\begin{aligned} x' - x = \frac{(t'_1 - t_1)\,\rho}{n_1\,\sigma_1\,\sigma'_1\,t_1}\Big\{ & X'\left[n_1\,\sigma_1^2\,t_1\,(\alpha_1\,\lambda_2) + \alpha_1\,\lambda_1\,\Omega\,e^{-2\xi_1}\,t_1\,(t'_1 - t_1)\right] + \\ & + X''\left[n_1\,\sigma_1^2\,t_1\,(\beta_1\,\lambda_2) + \beta_1\,\lambda_1\,\Omega\,e^{-2\xi_1}\,t_1\,(t'_1 - t_1)\right] + \\ & + X\left[n_1\,\sigma_1^2\,t_1\,(\mu_1\,\lambda_2) + \mu_1\,\lambda_1\,\Omega\,e^{-2\xi_1}\,t_1\,(t'_1 - t_1)\right]\Big\}. \end{aligned}$$

(*) This can be seen directly from equations (91).

When the expression (102) of $t'_1 - t_1$ is substituted in this equation, the result is reducible to

$$\left.\begin{aligned} x' - x = \frac{r}{n_1 P} \Big\{ & X' (\alpha_2 \lambda_1 - \alpha_1 \lambda_2 R) + X'' (\beta_2 \lambda_1 - \beta_1 \lambda_2 R) + \\ & + X (\mu_2 \lambda_1 - \mu_1 \lambda_2 R) \Big\}, \end{aligned}\right\} \tag{105}$$

where

$$R = 1 - r e^{-2\xi} \lambda_1 \lambda_2 (t_1 - t_2)^2, \quad P = (1 + r \lambda_2 \sigma_1) R - r \lambda_1 \sigma_2 .$$

We have seen that r may be found by quadratures, but the latter may be avoided in the following manner. Let $\bar{\alpha}_2, \bar{\beta}_2, \bar{\lambda}_2, \bar{\mu}_2, \bar{\sigma}_2$ be a set of integrals of (I) and (II) for $n = \bar{n}$, such that as $\bar{n}$ approaches n_1, we have

$$\left.\begin{aligned} \lim (\bar{\alpha}_2) = \alpha_2 , \quad \lim (\bar{\beta}_2) = \beta_2 , \quad \lim (\bar{\lambda}_2) = \lambda_2 , \\ \lim (\bar{\mu}_2) = \mu_2 , \quad \lim (\bar{\sigma}_2) = \sigma_2 . \end{aligned}\right\} \tag{106}$$

The function $\dfrac{\bar{\Psi}}{n_1 - \bar{n}}$, where

$$\bar{\Psi} = \alpha_1 \bar{\alpha}_2 + \beta_1 \bar{\beta}_2 + \mu_1 \bar{\mu}_2 - \bar{n} (\bar{\sigma}_2 \lambda_1 + \sigma_1 \bar{\lambda}_2) - \lambda_1 \bar{\lambda}_2 e^{-2\xi} \bar{n} (t_1 - \bar{t}_2)^2, \tag{107}$$

satisfies the equations

$$\frac{\partial \bar{\Psi}}{\partial u} = (n_1 - \bar{n}) \bar{\alpha}_2 \left[\lambda_1 e^{-\xi} (t_1 \varphi_1 + \psi_1) + \sigma_1 e^{\xi} \sinh \theta \right],$$

$$\frac{\partial \bar{\Psi}}{\partial v} = (n_1 - \bar{n}) \bar{\beta}_2 \left[\lambda_1 e^{-\xi} (t_1 \psi_1 + \varphi_1) + \sigma_1 e^{\xi} \cosh \theta \right].$$

As $\bar{n}$ approaches n_1, we have accordingly that the solution of (103) is given by

$$\frac{1}{r} + \lambda_2 \sigma_1 = - \lim_{\bar{n} = n_1} \frac{d \bar{\Psi}}{d \bar{n}} \cdot$$

Hence if we have a set of solutions of (I) and (II), looked upon for the time being as functions of n as well as of u and v, satisfying the conditions (106), the function r can be found by differentiation.

§ 12. Transformations of a Transform S_1.

As a consequence of the theorem of permutability we have the following theorem:

If one knows all the surfaces arising from a surface of Guichard ***of the first type by transformations T_n, the transformations of these surfaces can be effected without integration.***

If S_1 is a surface arising from S by means of a transformation T_{n_1} whose functions are α_1, β_1, λ_1, μ_1, σ_1, the preceding theorem follows at once for the case of all transformations T_n of S_1, so long as $n \neq n_1$. In § 11 we saw also that by means of differentiation the transformations T_{n_1} of S_1 can be found where the functions α_2, β_2, μ_2, λ_2, σ_2 satisfy the conditions (I), (II), (99).

We consider now the remaining case where we take for α_2, β_2, λ_2, μ_2, σ_2 functions α, β, λ, μ, σ looked upon as functions of n as well as of u and v, such that

$$\lim_{n=n_1} (\alpha) = \alpha_1, \quad \lim (\beta) = \beta_1, \quad \lim (\lambda) = \lambda_1,$$

$$\lim (\mu) = \mu_1, \quad \lim (\sigma) = \sigma_1 .$$

Equation (VI) may be replaced by

$$\left.\begin{aligned} x' - x = \frac{n - n_1}{n\, n_1 A} \Big\{ & X' \left[(n \lambda \alpha_1 - n_1 \lambda_1 \alpha) \chi + n_1 \lambda_1^2 \alpha \lambda e^{-2\xi} (n - n_1)(t - t_1)^2\right] + \\ & + X'' \left[(n \lambda \beta_1 - n_1 \lambda_1 \beta) \chi + n_1 \lambda_1^2 \beta \lambda e^{-2\xi} (n - n_1)(t - t_1)^2\right] + \\ & + X \left[(n \lambda \mu_1 - n_1 \lambda_1 \mu) \chi + n_1 \lambda_1^2 \mu \lambda e^{-2\xi} (n - n_1)(t - t_1)^2\right] \Big\}, \end{aligned}\right\} \qquad (108)$$

where

$$\left.\begin{aligned} \chi &= \alpha \alpha_1 + \beta \beta_1 + \mu \mu_1 - n \sigma \lambda_1 - n_1 \sigma_1 \lambda - \lambda \lambda_1 e^{-2\xi} n_1 (t - t_1)^2, \\ A &= \chi \left[\chi + (n - n_1)(\lambda_1 \sigma - \lambda \sigma_1)\right] + \\ & \qquad + \lambda \lambda_1 e^{-2\xi} (n_1 - n)(t - t_1)^2 \left[\chi + (n - n_1) \lambda_1 \sigma\right]. \end{aligned}\right\} \qquad (109)$$

From this expression for χ it follows that

$$\lim_{n=n_1} (\chi) = 0, \quad \lim_{n=n_1} \left(\frac{\partial \chi}{\partial n}\right) = 0.$$

We introduce a function B by the equation

$$\lim_{n=n_1} \left(\frac{\partial^2 \chi}{\partial n^2}\right) = B(u, v),$$

and for the sake of brevity we write

$$\Pi = \lambda \lambda_1 e^{-\xi} (t - t_1)(n - n_1),$$

so that the expression for A assumes the form

$$A = \chi \left[\chi + (n - n_1)(\lambda_1 \sigma - \lambda \sigma_1)\right] - \frac{\Pi^2}{n n_1 \lambda \lambda_1} \left(\lambda_1 \sigma + \frac{\chi}{n - n_1}\right). \tag{110}$$

One sees that $\lim_{n=n_1} (\Pi) = 0$, $\lim_{n=n_1} \left(\frac{\partial \Pi}{\partial n}\right) = 0$. We put also

$$\lim_{n=n_1} \left(\frac{\partial^2 \Pi}{\partial n^2}\right) = C(u, v).$$

Referring to (108) and (110), we note that in order to evaluate the indeterminate form in the right-hand member of (108), we must differentiate the numerators and denominator four times with respect to n. If we adopt the notation

$$\bar{\lambda} = \lim_{n=n_1} \left(\frac{\partial \lambda}{\partial n}\right), \quad \bar{\mu} = \lim_{n=n_1} \left(\frac{\partial \mu}{\partial n}\right), \quad \text{etc.,}$$

we find the following expression for $(x' - x)$ when $n = n_1$:

$$\left.\begin{aligned} x' - x = \frac{1}{E} \Big\{ & X'(D \alpha_1 - n_1 \lambda_1^2 B \bar{\alpha}) + X''(D \beta_1 - n_1 \lambda_1^2 B \bar{\beta}) + \\ & + X(D \mu_1 - n_1 \lambda_1^2 B \bar{\mu}) \Big\}, \end{aligned}\right\} \tag{111}$$

where

$$\left.\begin{aligned} D &= (\lambda_1 + n_1 \bar{\lambda}) \cdot {}_1 B + 6 n_1 C^2, \\ E &= n_1^2 \lambda_1 \left[6 B^2 + B(\lambda_1 \bar{\sigma} - \bar{\lambda} \sigma_1)\right] - 6 C^2 \sigma_1 . \end{aligned}\right\} \tag{112}$$

Since only differentiations have been performed in arriving at this result, the theorem is established.

It is interesting to see that the functions B and C may be obtained by quadratures, which in some cases may be a simpler process than the foregoing.

In fact from (109), (91) and (I) it follows that

$$\left.\begin{aligned}\frac{\partial \chi}{\partial u} &= (n - n_1)\, e^{-\xi}(\lambda\, \alpha_1 - \lambda_1\, \alpha)(t\,\varphi + \psi),\\ \frac{\partial \chi}{\partial v} &= (n - n_1)\, e^{-\xi}(\lambda\, \beta_1 - \lambda_1\, \beta)(t\,\psi + \varphi),\end{aligned}\right\} \tag{113}$$

and

$$\left.\begin{aligned}\frac{\partial \Pi}{\partial u} &= (n - n_1)\left\{(\mu\, \alpha_1 - \mu_1\, \alpha)\, e^{\xi} \sinh\theta + (\alpha_1\, \lambda - \alpha\, \lambda_1)(\cosh\theta + h \sinh\theta)\right\},\\ \frac{\partial \Pi}{\partial v} &= (n - n_1)\left\{(\mu\, \beta_1 - \mu_1\, \beta)\, e^{\xi} \cosh\theta + (\beta_1\, \lambda - \beta\, \lambda_1)(\sinh\theta + h \cosh\theta)\right\}.\end{aligned}\right\} \tag{114}$$

Hence B and C satisfy the equations

$$\left.\begin{aligned}\frac{\partial B}{\partial u} &= e^{-\xi}(\bar\lambda\, \alpha_1 - \lambda_1\, \bar\alpha)(t_1\, \varphi_1 + \psi_1),\\ \frac{\partial B}{\partial v} &= e^{-\xi}(\bar\lambda\, \beta_1 - \lambda_1\, \bar\beta)(t_1\, \psi_1 + \varphi_1),\\ \frac{\partial C}{\partial u} &= (\bar\mu\, \alpha_1 - \mu_1\, \bar\alpha)\, e^{\xi} \sinh\theta + (\alpha_1\, \bar\lambda - \bar\alpha\, \lambda_1)(\cosh\theta + h \sinh\theta),\\ \frac{\partial C}{\partial v} &= (\bar\mu\, \beta_1 - \mu_1\, \bar\beta)\, e^{\xi} \cosh\theta + (\beta_1\, \bar\lambda - \bar\beta\, \lambda_1)(\sinh\theta + h \cosh\theta).\end{aligned}\right\} \tag{115}$$

§ 13. Spherical Surfaces and Their Transformations.

If we put

$$e^{\xi} = a, \quad h = 0, \tag{116}$$

equations (21) are satisfied identically and equation (22) reduces to

$$\frac{\partial^2 \theta}{\partial u^2} + \frac{\partial^2 \theta}{\partial v^2} + \sinh\theta \cosh\theta = 0. \tag{117}$$

Now

$$\left.\begin{aligned}\sqrt{E} &= a \sinh \theta, \quad \sqrt{G} = a \cosh \theta, \\ D &= D'' = a \sinh \theta \cosh \theta, \\ \rho_1 &= a \tanh \theta, \quad \rho_2 = a \coth \theta.\end{aligned}\right\} \tag{118}$$

From the latter it follows that $\rho_1 \rho_2 = a^2$. Hence S in this case is a *spherical* surface. Moreover, it can be shown that the fundamental coefficients of any spherical surface can be put in the form (118) (*).

Conversely, suppose that ξ is constant. It follows from (21) that h is constant. If $h \neq 0$, and we put

$$\cosh k = \frac{h}{\sqrt{h^2 - 1}}, \quad \sinh k = \frac{1}{\sqrt{h^2 - 1}},$$

$$u' = \sqrt{h^2 - 1}\, u, \quad v' = \sqrt{h^2 - 1}\, v, \quad \theta' = \theta + k,$$

the function θ' satisfies an equation of the form (117). Hence it is perfectly general to take $h = 0$. From these results we have the theorem:

Spherical surfaces are surfaces of Guichard *of the first kind; they are characterized analytically by the condition $\xi = const.$*

In the present case as follows from (26) the functions of the associate surface $\bar{S}$ have the values

$$e^{\bar{\xi}} = \frac{1}{a}, \quad \sinh \bar{\theta} = -\sinh \theta, \quad \cosh \bar{\theta} = \cosh \theta,$$

and

$$\bar{\rho}_1 = \frac{1}{a} \tanh \theta, \quad \bar{\rho}_2 = \frac{1}{a} \coth \theta.$$

Hence *the associate surface of a spherical surface is a spherical surface, homothetic to the original surface.*

We consider the transformations T_n of a spherical surface S. If the new surface is to be spherical, it follows from (41) that $e^{\xi_1} = e^{\xi} = a$, and from (36) it follows that $h_1 = 0$. Hence S and S_1 have the same constant Gaussian curvature. From (45) we see that if e^{ξ_1} is to be equal to a, we must have

$$\frac{\sigma}{\lambda} + \frac{\mu^2}{\lambda^2} - \frac{1}{a^2} = 0. \tag{119}$$

(*) E. p. 278.

If we put $e^{\xi} = a$, $h = 0$, in the system (I), we obtain a system which is entirely consistent, and ordinarily the condition (119) is not satisfied. Hence we have the theorem:

Of the ∞^4 transforms of a spherical surface ∞^3 are spherical surfaces. We consider these transformations further.

If σ be eliminated from (119) and (II), we obtain

$$\alpha^2 + \beta^2 - 2n\frac{\lambda^2}{\alpha^2} + (1 + 2n)\mu^2 = 0. \tag{120}$$

One sees that if $0 > n > -1/2$, the surface S_1 is imaginary.

If σ be eliminated from (I) by means of (119), we obtain the system

$$\left.\begin{aligned}
&\frac{\partial\lambda}{\partial u} = a\,\alpha\sinh\theta, \qquad \frac{\partial\lambda}{\partial v} = a\,\beta\cosh\theta,\\
&\frac{\partial\mu}{\partial u} = -\alpha\cosh\theta, \qquad \frac{\partial\mu}{\partial v} = -\beta\sinh\theta,\\
&\frac{\partial\alpha}{\partial u} = -\beta\frac{\partial\theta}{\partial v} + \mu(1+2n)\cosh\theta + \frac{2\lambda n}{a}\sinh\theta, \quad \frac{\partial\alpha}{\partial v} = \beta\frac{\partial\theta}{\partial u},\\
&\frac{\partial\beta}{\partial u} = \alpha\frac{\partial\theta}{\partial v}, \quad \frac{\partial\beta}{\partial v} = -\alpha\frac{\partial\theta}{\partial u} + \mu(1+2n)\sinh\theta + \frac{2\lambda n}{a}\cosh\theta.
\end{aligned}\right\} \tag{121}$$

Since the functions α, β, λ, μ must satisfy (120), we verify that there are ∞^3 transformations of spherical surfaces into spherical surfaces.

Suppose now that S_1 and S_2 are two spherical surfaces which are obtained from S by transformations T_{n_1} and T_{n_2}. In order to make use of the general formulas, we retain σ_1 and σ_2, given by

$$\frac{\sigma_1}{\lambda_1} + \frac{\mu_1^2}{\lambda_1^2} - \frac{1}{a^2} = 0, \quad \frac{\sigma_2}{\lambda_2} + \frac{\mu_2^2}{\lambda_2^2} - \frac{1}{a^2} = 0. \tag{122}$$

If the fourth surface S' of the quatern is to be spherical also, we must have

$$\frac{\sigma_1'}{\lambda_1'} + \frac{\mu_1'^2}{\lambda_1'^2} - \frac{1}{a^2} = 0. \tag{123}$$

If the values of the functions as given by (V) be substituted in this equation, the result is reducible to

$$\left.\begin{aligned}
&\Psi_1\left(\frac{\sigma_1}{\lambda_1} + \frac{\mu_1^2}{\lambda_1^2} - \frac{\sigma_2}{\lambda_2} - \frac{\mu_2^2}{\lambda_2^2}\right) + (n_2 - n_1)\left[\lambda_2\,\sigma_1\left(\frac{\sigma_1}{\lambda_1} - \frac{\sigma_2}{\lambda_2}\right) + \right.\\
&\left. + 2\lambda_2\,\sigma_1\frac{\mu_1}{\lambda_1}\left(\frac{\mu_1}{\lambda_1} - \frac{\mu_2}{\lambda_2}\right) + \lambda_1\lambda_2\left(\frac{\mu_1}{\lambda_1} - \frac{\mu_2}{\lambda_2}\right)^2\left(\frac{\mu_1^2}{\lambda_1^2} - \frac{1}{a^2}\right)\right] = 0.
\end{aligned}\right\} \tag{124}$$

One finds readily that this equation is satisfied identically by (122). Hence

If S_1 and S_2 are spherical surfaces obtained from a spherical surface S by transformations T_{n_1} and T_{n_2}, the fourth surface S' of the quatern is spherical and all four surfaces have the same constant Gaussian curvature.

Equations (123) and (124) follow on expressing the condition that $e^{\xi'} = a$. We raise the question if this is possible, when the surfaces S_1 and S_2 are not spherical. Now equation (124) must hold. We write it in the form

$$\chi_1 + (n_2 - n_1) \frac{\left(\frac{\sigma_1}{\lambda_1} + \frac{\mu_1^2}{\lambda_1^2} - \frac{1}{a^2}\right) \lambda_1 \lambda_2 \left(\frac{\mu_1}{\lambda_1} - \frac{\mu_2}{\lambda_2}\right)^2}{\frac{\sigma_1}{\lambda_1} + \frac{\mu_1^2}{\lambda_1^2} - \frac{\sigma_2}{\lambda_2} - \frac{\mu_2^2}{\lambda_2^2}} = 0, \tag{125}$$

where χ_1 is given by (93). Moreover, in the general case we have

$$\left.\begin{aligned} \frac{\partial \chi_1}{\partial u} &= (n_1 - n_2) \frac{1}{a} (\lambda_1 \alpha_2 - \lambda_2 \alpha_1)(t_1 \varphi_1 + \psi_1), \\ \frac{\partial \chi_1}{\partial v} &= (n_1 - n_2) \frac{1}{a} (\lambda_1 \beta_2 - \lambda_2 \beta_1)(t_1 \psi_1 + \varphi_1). \end{aligned}\right\} \tag{126}$$

If equation (125) be differentiated separately with respect to u and v, and in the reduction use be made of (126) and (I), the resulting equations are of the form

$$A \cosh \theta + B \sinh \theta = 0, \quad A \sinh \theta + B \cosh \theta = 0,$$

where A and B are determinate functions. It is evident that A and B must be zero. Since $n_1 \neq n_2$, this gives, if we denote by P_1 and P_2 the left-hand members of equations (122),

$$(P_1 - P_2)\left(\frac{\mu_2}{\lambda_2} P_1 - \frac{\mu_1}{\lambda_1} P_2\right) = 0,$$

$$\frac{1}{a^2}(P_1 - P_2)^2 + \left(\frac{\mu_2}{\lambda_2} P_1 - \frac{\mu_1}{\lambda_1} P_2\right) = 0.$$

Hence P_1 and P_2 must be zero, and consequently S_1 and S_2 must be spherical.

We apply also to the case of spherical surfaces the results of § 11. One finds readily from (80), (102) and (103) that

$$\frac{\mu_1'}{\lambda_1'} = \frac{\mu_1}{\lambda_1} + \frac{1}{a}(t'_1 - t_1) = \frac{\mu_1}{\lambda_1} + r \frac{\sigma_1}{\lambda_1} (\mu_1 \lambda_2 - \mu_2 \lambda_1),$$

$$\frac{\sigma'_1}{\lambda'_1} = \frac{\sigma_1}{\lambda_1}\left[1 - 2r\frac{\mu_1}{\lambda_1}(\mu_1\lambda_2 - \mu_2\lambda_1) - r^2\frac{\sigma_1}{\lambda_1}(\mu_1\lambda_2 - \mu_2\lambda_1)^2\right].$$

Since these values satisfy (123), the surface S' is spherical.

When one adopts the limiting process in finding the transforms of S_1 as set forth in the preceding section, if both S_1 and $\overline{S}_2$ are spherical, the condition (123) is satisfied durinig the limiting process and consequently S' is spherical. In view of these results we have the theorem:

When one knows the transformations of a spherical surface S into spherical surfaces, the similar transformations of the latter require differentiation at most.

§ 14. Envelope of the Circle-Plane for Transformations of Surfaces of Constant Curvature. The Surface of Centres of the Spheres.

When S is a spherical surface, the functions p, q, ω, ψ which enter in the equations of the envelope S_0 of the circle-plane of a transformations (cf. § 4) take on the values

$$p = -\frac{a^2}{2n\lambda}, \quad q = -\frac{a^2}{2n\lambda}\mu(1 + 2n),$$

$$\omega = -a^2\frac{\mu}{\lambda}, \quad \psi = \frac{\lambda}{2}\left(1 - a^2\frac{\mu^2}{\lambda^2}\right).$$

Now the linear element of S_0 (55) is

$$ds^2 = d\omega^2 + 2\,dp\,d\psi$$

$$= a^4\left(d\frac{\mu}{\lambda}\right)^2 - a^4\frac{\mu}{\lambda}\frac{1}{n}d\frac{\mu}{\lambda}\cdot\frac{d\lambda}{\lambda} + \frac{a^2}{2n}\left(1 - a^2\frac{\mu^2}{\lambda^2}\right)\frac{d\lambda^2}{\lambda^2}.$$

If we put

$$\overline{x} = -a^2\frac{\mu}{\lambda}, \quad \overline{y} - i\overline{z} = -\frac{a^2}{2n\lambda}, \quad \overline{y} + i\overline{z} = \lambda\left(1 - a^2\frac{\mu^2}{\lambda^2}\right),$$

the surface of coordinates $\overline{x}$, $\overline{y}$, $\overline{z}$ is applicable to S_0. Moreover, on eliminating μ and λ from the above equations, we see that

$$\bar{y}^2 + \bar{z}^2 - \frac{\bar{x}^2}{2n} = -\frac{a^2}{2n}. \tag{127}$$

Hence *when n is negative and different from $-1/2$, the envelope of the circle-plane is applicable to an ellipsoid of revolution; when n is positive, to a hyperboloid of revolution of two sheets; when $n = -1/2$, to a sphere.*

We shall show that the surface Σ of centres of the spheres is the surface complementary to S_0, when the geodesics on the latter are the deforms of the meridians on (127). In order to do this we must show that the joins of corresponding points on Σ and S_0 are tangent to the orthogonal trajectories of the curve $\frac{\mu}{\lambda} = \text{const.}$ on S_0, and that they are the intersection of the tangent planes to Σ and S_0.

The orthogonal trajectories are defined by $\lambda^2 - a^2\mu^2 = \text{const.}$ and from (121) it follows that along one of these geodesics

$$\alpha\psi\, du + \beta\varphi\, dv = 0.$$

Hence the direction-cosines of the tangent to one of these curves are proportional to

$$\beta\varphi\frac{\partial x_0}{\partial u} - \alpha\psi\frac{\partial x_0}{\partial v}, \quad \beta\varphi\frac{\partial y_0}{\partial u} - \alpha\psi\frac{\partial y_0}{\partial v}, \quad \beta\varphi\frac{\partial z_0}{\partial u} - \alpha\psi\frac{\partial z_0}{\partial v}.$$

With the aid of (52) we find that

$$\beta\varphi\frac{\partial x_0}{\partial u} - \alpha\psi\frac{\partial x_0}{\partial v} = \alpha^2\frac{\alpha\beta\mu}{\lambda^2}\left[p(\alpha X' + \beta X'') + \left(q + \frac{\lambda}{\mu}\right)X\right].$$

But from (3) and (49) it follows that the quantity in parenthesis is equal to $x_0 - \xi$. Hence the first condition is satisfied.

The direction-cosines of the tangent plane to S_0 are proportional to

$$\beta X' - \alpha X'', \quad \beta Y' - \alpha Y'', \quad \beta Z' - \alpha Z''.$$

One shows readily that to within a factor the direction-cosines of the normal to the surface Σ are of the form

$$\alpha X' + \beta X'' + \mu X.$$

Since these two normals are perpendicular to the joins of corresponding points on S_0 and Σ, we have proved that Σ is complementary to S_0.

BIANCHI has shown (*) that every deform of a quadric of revolution is applicable to the complementary surface determined by the tangents to the deforms of the meridians. Hence we have the following theorem:

The envelope of the circle-plane of a transformation T_n of spherical surfaces and the surface of centres of the spheres of T_n are the focal surfaces of a normal congruence, and are applicable to one another and to an hyperboloid or ellipsoid of revolution according as n is positive or negative ($\neq -1/2$).

We have seen that S_0 is imaginary, when $0 < n < -1/2$. Hence the real transformations are those for which the surface of centres is applicable to a prolate ellipsoid of revolution or an hyperboloid of two sheets of revolution. Consequentty they are the transformations growing out of the beautiful theorem announced by GUICHARD (**), which have been the starting point of the recent theory of surfaces applicable to quadrics. BIANCHI (***) showed that transformations of this kind can be obtained by combining certain conjugate-imaginary transformations of BÄCKLUND for spherical surfaces.

Pseudospherical surfaces are surfaces of GUICHARD of the second kind, being characterized by $\xi = \text{const.}$ We shall not repeat the preceding investigations for this kind of surfaces, but it is worth while to call attention to a few results.

For pseudospherical surfaces we have the relation

$$\sigma = -\lambda\left(\frac{1}{a^2} + \frac{\mu^2}{\lambda^2}\right). \tag{119*}$$

From this equation and (II) it follows that if the transformation is to be real n must be negative.

The functions determining S_0, the envelope of the circle-plane have the values

$$p = \frac{a^2}{2n\lambda}, \quad q = \frac{a^2\mu}{2n\lambda}(1+2n), \quad \omega = a^2\frac{\mu}{\lambda}, \quad \psi = \frac{\lambda}{a}\left(1 + a^2\frac{\mu^2}{\lambda^2}\right).$$

If we put

$$\bar{x} = a^2\frac{\mu}{\lambda}, \quad \bar{y} - i\bar{z} = \frac{a^2}{2n\lambda}, \quad \bar{y} + i\bar{z} = \lambda\left(1 + n^2\frac{\mu^2}{\lambda^2}\right),$$

(*) *Lezioni*, vol. III, p. 281.

(**) *Sur la déformation des quadriques de révolution.* Comptes Rendus, 23 janvier, 1899.

(***) *Lezioni*, vol. II, pp. 464-465.

the locus of the point $\bar{x}$, $\bar{y}$, $\bar{z}$ is the quadric

$$\bar{y}^2 + \bar{z}^2 = \frac{a^2}{2n}\left(1 + \frac{\bar{x}^2}{a^2}\right). \tag{127*}$$

When $n = -\frac{1}{2}$, S_0 is a pseudospherical surface. Since

$$\alpha^2 + \beta^2 = \frac{\lambda^2}{a^2}, \quad p = -\frac{a^2}{\lambda}, \quad q = 0,$$

we find that $\sum (x - x_0)^2 = a^2$. Hence S and S_1 are two BIANCHI transforms of S_0.

For the other values of n, S_0 and the surface Σ of centres of the spheres are the focal surfaces of a normal congruence of tangents to the geodesics on S_0 corresponding to the meridians on (127*). BIANCHI (*) has shown that the ellipsoid (127*) is applicable to a hyperbolic sinusoid or to the abridged catenoid according as $2n$ is less or greater than -1. Consequently these real transformations can be obtained by combining BÄCKLUND transformations B_σ and $B_{-\sigma}$, as BIANCHI (**) has shown.

§ 15. SPECIAL SURFACES OF GUICHARD OF THE FIRST KIND.

The remainder of this memoir will be devoted to the study of a class of surfaces of GUICHARD of the first kind, each of which admits several transformations T_n with a common circle-plane.

From (48) and (I) it follows that if S_1 and S_2 are two transforms determined by α_1, β_1, λ_1, μ_1, σ_1, n_1 and α_2, β_2, λ_2, μ_2, σ_2, n_2 respectively such that the circle-plane is the same for both transformations, we must have

$$\frac{\partial(\lambda_1, \lambda_2)}{\partial(u, v)} = 0, \quad \frac{\partial(\mu_1, \mu_2)}{\partial(u, v)} = 0.$$

Since λ satisfies the point equation of S there can at most be a linear relation between λ_1 and λ_2; the same is true of μ_1 and μ_2. Since equations (I)

(*) *Lezioni*, vol. III, p. 283.
(**) *Lezioni*, vol. II, p. 432.

and (II) are homogeneous, the most general case to be considered is

$$\lambda_2 = \lambda_1 + \lambda', \quad \mu_2 = \mu_1 + \mu', \quad n_2 = n_1 + n', \tag{128}$$

where λ', μ' and n' are constants. We must have also

$$\alpha_2 = \alpha_1, \quad \beta_2 = \beta_1. \tag{129}$$

In order that the expressions in (I) for $\frac{\partial \alpha}{\partial u}$ and $\frac{\partial \beta}{\partial v}$ shall be the same for the two transformations, the following equations must be satisfied:

$$\left.\begin{aligned} &\mu' + 2 n_2 \lambda_2 e^{-\xi} t_2 - 2 n_1 \lambda_1 e^{-\xi} t_1 = 0, \\ &\mu' h + (n_2 \sigma_2 - n_1 \sigma_1) e^{\xi} + n_2 \lambda_2 e^{-\xi}(1 + t_2^2) - n_1 \lambda_1 e^{-\xi}(1 + t_1^2) = 0, \end{aligned}\right\} \tag{130}$$

If these equations be differentiated with respect to u and v, the results are reducible in consequence of (I) to

$$\frac{n_2 \lambda_2 - n_1 \lambda_1}{n_2 - n_1} e^{-\xi} \operatorname{csch} \theta \frac{\partial \xi}{\partial u} = \alpha, \quad \frac{n_2 \lambda_2 - n_1 \lambda_1}{n_2 - n_1} e^{-\xi} \operatorname{sech} \theta \frac{\partial \xi}{\partial v} = \beta.$$

When these expressions for α and β are substituted in equations (I) for $\frac{\partial \lambda}{\partial u}$ and $\frac{\partial \lambda}{\partial v}$, one finds by integration that

$$\xi = c \cdot \log (n_2 \lambda_2 - n_1 \lambda_1),$$

where c is the constant of integration. Without loss of generality we choose c so that

$$\frac{n_2 \lambda_2 - n_1 \lambda_1}{n_2 - n_1} = e^{\xi}, \quad \alpha = \operatorname{csch} \theta \frac{\partial \xi}{\partial u}, \quad \beta = \operatorname{sech} \theta \frac{\partial \xi}{\partial v}. \tag{131}$$

With this choice the equations (I) for $\frac{\partial \lambda}{\partial u}$, $\frac{\partial \lambda}{\partial v}$, $\frac{\partial \mu}{\partial u}$ and $\frac{\partial \mu}{\partial v}$ assume such a form that we have

$$\left.\begin{aligned} \lambda_1 &= e^{\xi} + l_1, & \lambda_2 &= e^{\xi} + l_2, \\ \mu_1 &= -h + m_1, & \mu_2 &= -h + m_2, \end{aligned}\right\} \tag{132}$$

where the l's and m's are constants. From (130), (131), (132) we obtain

$$n_2 l_2 - n_1 l_1 = 0, \quad (m_2 - m_1) + 2(n_2 m_2 - n_1 m_1) = 0. \tag{133}$$

When the above values for α and β are substituted in equations (41), the latter become

$$\frac{\partial \xi_1}{\partial u} = \sinh \theta_1 \frac{\alpha_1}{\lambda_1} (l_1 + e^{\xi_1}), \quad \frac{\partial \xi_1}{\partial v} = - \cosh \theta_1 \frac{\beta_1}{\gamma_1} (l_1 + e^{\xi_1}). \tag{134}$$

Comparing these equations with the first two of (31), we have from the latter by integration

$$\sigma_1 = \frac{c_1}{l_1 + e^{\xi_1}}, \tag{135}$$

where c_1 is the constant of integration.

If we replace the first of equations (133) by

$$n_2 l_2 = n_1 l_1 = -\frac{1}{2} A, \tag{136}$$

the second is equivalent to

$$A = \frac{l_1 l_2 (m_2 - m_1)}{l_1 m_2 - l_2 m_1}. \tag{137}$$

From (135) and (45) we get

$$\sigma_1 = \frac{1}{l_1} \left[c_1 - (l_1 + e^{\xi}) e^{-\xi} (1 - t_1^2) \right], \tag{138}$$

where now in consequence of (132) we have from (36)

$$t_1 = \frac{l_1 h + m_1 e^{\xi}}{l_1 + e^{\xi}}. \tag{139}$$

When the above values for α, β, λ, γ, σ are substituted in equation (II), we obtain

$$h^2 + \operatorname{csch}^2 \theta \left(\frac{\partial \xi}{\partial u}\right)^2 + \operatorname{sech}^2 \theta \left(\frac{\partial \xi}{\partial v}\right)^2 + A e^{-\xi} (h^2 - 1) + 2 B h + C e^{\xi} + D = 0, \tag{VII}$$

A, B, C, D being constants, given by (137) and by

$$\left. \begin{aligned} B = -(1 + 2 n_1) m_1, \quad C = \frac{2 n_1}{l_1} (1 - m_1^2 - c_1), \\ D = m_1^2 + 4 n_1 - 2 c_1 n_1. \end{aligned} \right\} \tag{140}$$

From these follow

$$\left.\begin{aligned} B &= m_1\left(\frac{A}{l_1}-1\right), \quad C = -\frac{2n_1c_1}{l_1}+\frac{m_1^2-1}{l_1^2}A,\\ D &= -2n_1c_1+m_1^2-2\frac{A}{l_1}, \quad l_1C-D=\frac{A}{l_1}+m_1^2\left(\frac{A}{l_1}-1\right).\end{aligned}\right\} \quad (141)$$

If l_1 and m_1 be eliminated from the first and last of these equations and (137), we obtain the following cubic for n_1:

$$(D-2n_1)\,2n_1(1+2n_1)+AC(1+2n_1)-B^2\,2n_1=0. \quad (142)$$

In consequence of (137) and (140) the expressions for λ_1, μ_1, σ_1 may be given the form

$$\left.\begin{aligned} \lambda_1 &= e^{\xi}-\frac{A}{2n_1}, \quad \mu_1=-\left(\frac{B}{1+2n_1}+h\right),\\ n_1\sigma_1 &= -\frac{C}{2}+n_1e^{-\xi}(h^2-1)-n_1\frac{\mu_1^2}{\lambda_1}.\end{aligned}\right\} \quad (143)$$

Conversely, suppose we have a surface S of Guichard of the first kind whose functions satisfy equation (VII). For convenience we refer to S as a *special surface of class* $(A\,B\,C\,D)$, thus putting the constants in evidence. Each of the roots of (142) when substituted in (143) gives a set of functions which together with the values (131) of α and β satisfy equations (I) and (II). Hence if the roots if (142) are distinct, three circles lie in the same plane, and this plane is unique for the surface.

In § 3 we observed that a transformation of Ribaucour of a surface of Guichard is always given by

$$\lambda = e^{\xi}+l, \quad \mu=-h+m.$$

But if this is to be a transformation T_n, all the foregoing steps follow at once and so we have the theorem:

A necessary and sufficient condition that a surface be a special surface of Guichard *of the first kind is that equations* (I) *admit the solutions*

$$\left.\begin{aligned} \lambda_1 &= e^{\xi}+l_1, \quad \mu_1=-h+m_1, \quad \alpha_1=\operatorname{csch}\theta\,\frac{\partial\xi}{\partial u}.\\ \beta_1 &= \operatorname{sech}\theta\,.\,\frac{\partial\xi}{\partial v}, \quad \sigma_1=\frac{1}{l_1}\left[c_1-(l_1+e^{\xi})\,e^{-\xi}(1-l_1^2)\right].\end{aligned}\right\} \quad (144)$$

It should be observed that the functions of a spherical surface satisfies equation (VII). However, in this case λ_1 and μ_1, as given by (144), are constants, and α and β are zero. Accordingly we exclude spherical surfaces when considering special surfaces.

§ 16. Complementary Transformations of Special Surfaces. Envelope of the Singular Circle-Plane.

We have just seen that in general a special surface admits of three transformations for each of which the functions are of the form (144). We say that these functions determine *complementary* transformations of S and that a resulting surface S_1 is *complementary* to S.

From (67) and (36) it follows that the functions of the inverse transformation are, to within the constant factor $1/l.\ c.$,

$$\left.\begin{aligned} \alpha^{-1} = \operatorname{csch}\theta_1 \frac{\partial \xi_1}{\partial u}, \quad \beta^{-1} = \operatorname{sech}\theta_1 \frac{\partial \xi_1}{\partial v}, \quad \lambda^{-1} = l_1 + e^{\xi_1}, \\ \mu^{-1} = m_1 - h_1, \quad \sigma^{-1} = \frac{1}{l_1 + e^{\xi}} \cdot \end{aligned}\right\} \quad (145)$$

Since all of the equations are homogeneous this constant factor is unessential. Furthermore, one notices that the same constants l_1, m_1, n_1 appear in (145) as in (144), and in the same way. Hence

If S is a special surface of Guichard *of class* $(A\,B\,C\,D)$, *each complementary surface is a special surface of the same class.*

From (26) we find that

$$\operatorname{csch}\overline{\theta}\frac{\partial \overline{\xi}}{\partial u} = \operatorname{csch}\theta\frac{\partial \xi}{\partial u}, \quad \operatorname{sech}\overline{\theta}\frac{\partial \overline{\xi}}{\partial v} = \operatorname{sech}\theta\frac{\partial \xi}{\partial v}\,.$$

Combining this result with the first of (26), we have from (VII) that

The surface $\overline{S}$ associate to a special surface of Guichard *of class* $(A\ B\ C\ D)$ *is a special surface of class* (C, B, A, D).

We have seen that the circle-plane of a complementary transformation of a special surface of Guichard possesses the property that in general it contains three circles associated with complementary transformations. It will be found that the envelope of this ***singular circle-plane*** is applicable to a quadric.

If we apply the results of § 4 to the particular case under discussion, the functions ω, r, ψ have the form

$$\left.\begin{aligned} \omega &= \frac{A}{l_1 r}(l_1 h + m_1 e^{\xi}), \quad r = \frac{A}{2}(h^2 - 1) e^{-\xi} - e^{\xi}\frac{C}{2}, \\ \psi &= \frac{A}{2r}\left[2 m_1 h - 2 + (m_1^2 + 1)\frac{e^{\xi}}{l_1} + l_1 e^{-\xi}(h^2 - 1)\right]. \end{aligned}\right\} \tag{146}$$

Putting

$$\left.\begin{aligned} \bar{x} &= \frac{A}{l_1 r}(h l_1 + e^{\xi} m_1), \quad \bar{y} - i\bar{z} = -\frac{e^{\xi}}{r}, \\ \bar{y} + i\bar{z} &= \frac{A}{r}\left[2 m_1 h - 2 + (m_1^2 + 1)\frac{e^{\xi}}{l_1} + l_1 e^{-\xi}(h^2 - 1)\right], \end{aligned}\right\} \tag{147}$$

we have in consequence of (55)

$$d s_0^2 = d\bar{x}^2 + d\bar{y}^2 + d\bar{z}^2.$$

Hence the surface whose coordinates are given by (147) is applicable to S_0.

From (147) it follows that

$$\frac{A}{r} = m_1 \bar{x} - \frac{1}{2}(\bar{y} + i\bar{z}) + \frac{1}{2}(\bar{y} - i\bar{z})\left[(m_1^2 - 1)\frac{A}{l_1} - C l_1\right] + l_1. \tag{148}$$

Eliminating e^{ξ} and h from the expressions (146) for r and (147) for $\bar{x}$ and $\bar{y} - i\bar{z}$, we obtain

$$A\left\{\frac{1}{r^2} - \frac{1}{A^2}\left[\bar{x} + \frac{m_1 A}{l_1}(\bar{y} - i\bar{z})\right]^2\right\} - 2(\bar{y} - i\bar{z}) + C(\bar{y} - i\bar{z})^2 = 0.$$

When the expression (148) for $\frac{1}{r}$ is substituted in this equation, we see that the surface of coordinates $\bar{x}$, $\bar{y}$, $\bar{z}$ is a general quadric meeting the circle at infinity in four distinct points.

The equation of the quadric as thus found involves the four constants A, C, l_1 and m_1. But by means of equations (141) l_1 and m_1 are expressible in terms of A, B, C and D. Recalling these results, we have the theorem:

The envelope of the singular circle-plane of a special surface of Guichard *of the first kind is applicable to a general quadric which meets the circle at infinity in four distinct points; this quadric is the same for all special surfaces of the same class.*

§ 17. Transformations of Special Surfaces into Surfaces of the Same Class.

We have seen that the surfaces complementary to a special surface are surfaces of the same class. With the aid of this result and the theorem of permutability for general transformations T_n, we shall show that, if S_1 is a surface arising from a special surface S by a complementary transformation T_{n_1}, then ∞^2 of the ∞^3 surfaces arising from S by transformations T_{n_2}, where $n_2 \neq n_1$, are special surfaces of the same class as S.

Let S_2 be one of the latter surfaces. From § 9 it follows that the transformation fonctions of S_2 are of the form

$$\left.\begin{aligned}
\alpha'_2 &= \frac{1}{\lambda_2 \sigma_2}\left[\alpha_2 \Psi_2 + \alpha_1 \lambda_2 \sigma_2 (n_1 - n_2)\right], \\
\beta'_2 &= -\frac{1}{\lambda_2 \sigma_2}\left[\beta_2 \Psi_2 + \beta_1 \lambda_2 \sigma_2 (n_1 - n_2)\right], \\
\lambda'_2 &= \frac{1}{\lambda_2 \sigma_2}\left[\lambda_2 \Psi_2 + \lambda_1 \lambda_2 \sigma_2 (n_1 - n_2)\right], \\
\mu'_2 &= \frac{1}{\lambda_2 \sigma_2}\left[\mu_2 \Psi_2 + \mu_1 \lambda_2 \sigma_2 (n_1 - n_2)\right], \\
\sigma'_2 &= \frac{1}{\lambda_2 \sigma_2}\left[\sigma_2 \Psi_2 + \sigma_1 \lambda_2 \sigma_2 (n_1 - n_2) + \frac{\lambda_1 \lambda_2 e^{-2\xi}(n_1 - n_2)(t_1 - t_2)^2 \sigma_2 \Psi_2}{\Psi_2 + (n_1 - n_2)\lambda_1 \sigma_2}\right],
\end{aligned}\right\} \tag{149}$$

where

$$\Psi_2 = \alpha_1 \alpha_2 + \beta_1 \beta_2 + \mu_1 \mu_2 - n_1 (\sigma_1 \lambda_2 + \sigma_2 \lambda_1) - \lambda_1 \lambda_2 e^{-2\xi} n_1 (t_1 - t_2)^2. \tag{150}$$

We inquire whether it is possible to determine $\alpha_2, \beta_2, \lambda_2, \mu_2, \sigma_2$ so that S' shall be a complementary surface to S_2, when $\alpha_1, \beta_1, \lambda_1, \mu_1, \sigma_1$ have the values (144). If this is possible, the functions $\alpha'_2, \ldots, \sigma'_2$ given by (149) must be equal, to within a constant factor, to

$$\left.\begin{aligned}
&\operatorname{csch}\theta_2 \frac{\partial \xi_2}{\partial u}, \quad \operatorname{sech}\theta_2 \frac{\partial \xi_2}{\partial v}, \quad e^{\xi_2} + l_1, \quad m_1 - h_2, \\
&\frac{1}{l_1}\left[c_1 - (e^{\xi_2} + l_1) e^{-\xi_2} (1 - t_2'^2)\right].
\end{aligned}\right\} \tag{151}$$

One finds readily that if the constant multiplier in (151) be taken as $(n_1 - n_2)$, the equivalence of the two values of the first four quantities in (149) and (151) requires that

$$\Omega \equiv \Psi_2 - \sigma_2 (n_1 - n_2)(e^{\xi_2} - e^{\xi}) = 0.$$

In consequence of (45) this may be written

$$\Omega = \Psi_2 - (n_1 - n_2)\left[\lambda_2 e^{-\xi}(1 - t_2^2) - e^{\xi}\sigma_2\right] = 0. \tag{152}$$

Without difficulty one shows that

$$t'_2 = \frac{l_1 e^{-\xi}(e^{\xi} - e^{\xi_2})\, t_2}{l_1 + e^{\xi_2}} + e^{\xi_2 - \xi}\, t_1 \frac{l_1 + e^{\xi}}{l_1 + e^{\xi_2}}\,.$$

With the aid of this result it is readily shown that when $\Omega = 0$, the expressions for σ'_2 from (149) and (151) differ only by the factor $(n_1 - n_2)$.

We return to the consideration of the condition $\Omega = 0$. From equations analogous to (91) we have in the present case

$$\frac{\partial \Psi_2}{\partial u} = (n_2 - n_1)\operatorname{csch}\theta \frac{\partial \xi}{\partial u}\left[e^{\xi}\sinh\theta\,\sigma_2 + e^{-\xi}\lambda_2(t_2\varphi_2 + \psi_2)\right],$$

$$\frac{\partial \Psi_2}{\partial v} = (n_2 - n_1)\operatorname{sech}\theta \frac{\partial \xi}{\partial v}\left[e^{\xi}\cosh\theta\,\sigma_2 + e^{-\xi}\lambda_2(t_2\psi_2 + \varphi_2)\right].$$

Making use of these expressions, we find that

$$\frac{\partial \Omega}{\partial u} = 0, \quad \frac{\partial \Omega}{\partial v} = 0.$$

Hence if the initial values of $\alpha_2, \ldots, \sigma_2$ be chosen so that $\Omega = 0$, this equation will be true for all values of u and v. Accordingly we have the theorem:

If S is a special surface of Guichard *and S_1 is a complementary surface by a transformation T_{n_1}, of the ∞^3 transformations T_{n_2}, where $n_2 \neq n_1$, ∞^2 give rise to special surfaces S_2 of the same class as S; since the fourth surface S' of the quatern is complementary to S_2, it also is a special surface of the same class.*

§ 18. First Integral of the System (I) for Special Surfaces, and Resulting Transformations.

If in (150) and (152) we replace $\alpha_1, \beta_1, \ldots, \sigma_1$ by the values (144) and in the reduction we make use of (138) and (140), the function Ω can be given the form

$$\left.\begin{aligned} \Omega = \operatorname{csch}\theta \frac{\partial \xi}{\partial u}\alpha_2 + \operatorname{sech}\theta \frac{\partial \xi}{\partial v}\beta_2 + \left(\frac{A}{2} - n_2 e^{\xi}\right)\left(\sigma_2 + \frac{\mu_2^2}{\lambda_2}\right) - \\ - \mu_2 (B + h + 2 n_2 h) + \lambda_2 \left[n_2 (1 - h^2) e^{-\xi} + \frac{1}{2} C\right]. \end{aligned}\right\} \quad (153)$$

We have just seen that Ω is constant, provided that $n_2 \neq n_1$.

However, this restriction is not necessary. For if A, B, C are constants entering in the class of a special surface and n_2 is any constant, the first derivatives of Ω given by (153) are reducible to zero in consequence of equation (VII). Hence we have the theorem:

If S is a special surface of class $(A\,B\,C\,D)$, the fundamental system (I) *admits the first integral*

$$\left.\begin{aligned} \Omega \equiv \operatorname{csch}\theta \frac{\partial \xi}{\partial u}\alpha + \operatorname{sech}\theta \frac{\partial \xi}{\partial v}\beta + \left(\frac{A}{2} - n e^{\xi}\right)\left(\sigma + \frac{\mu^2}{\lambda}\right) - \\ - \quad (B + h + 2 n h) + \lambda \left[n (1 - h^2) e^{-\xi} + \frac{1}{2} C\right] = \text{const.} \end{aligned}\right\} \quad (\text{VIII})$$

Suppose now that we have a set of functions $\alpha, \ldots, \sigma$ satisfying (I) and whose initial values are such that $\Omega = 0$; then for all values of u and v $\Omega = 0$. We shall show that the surface arising from the transformation determined by these functions gives rise to a special surface S_1 of the same class as S. In fact, if we substitute the expressions for h_1, $\frac{\partial \xi_1}{\partial u}$, $\frac{\partial \xi_1}{\partial v}$, e^{ξ_1}, given by (36), (41) and (45), in the expression

$$h_1^2 + \operatorname{csch}^2\theta_1 \left(\frac{\partial \xi_1}{\partial u}\right)^2 + \operatorname{sech}^2\theta_1 \left(\frac{\partial \xi_1}{\partial v}\right)^2 + A e^{-\xi_1}(h_1^2 - 1) + \\ + 2 B h_1 + C e^{\xi_1} + D,$$

the latter vanishis identically, and consequently the desired result is established.

One shows without difficulty that certain of these transformations are real. For, if the equation $\Omega = 0$ be written in the form

$$\operatorname{csch}\theta \,.\, \frac{\partial \xi}{\partial u}\, \alpha + \operatorname{sech}\theta \frac{\partial \xi}{\partial v}\, \beta - \left(\frac{B}{1+2n} + h\right) \mu\, (1+2n) + \\ + \left(\frac{A}{2} - n e^{\xi}\right)\left(\sigma + \frac{\mu^2}{\lambda}\right) + \lambda \left[n(1-h^2)\, e^{-\xi} + \frac{C}{2}\right] = 0,$$

the coefficients of α, β, $\mu\,(1+2n)$ are the values of α, β, μ for a complementary transformation. If σ be eliminated from this equation and (II), the result may be written

$$\left.\begin{aligned} &\frac{\alpha^2}{\lambda n} + \frac{2 \operatorname{csch}\theta \frac{\partial \xi}{\partial u}\, \alpha}{\frac{A}{2} - n e^{\xi}} + \frac{\beta^2}{\lambda n} + \frac{2 \operatorname{sech}\theta \frac{\partial \xi}{\partial v}\, \beta}{\frac{A}{2} - n e^{\xi}} + \\ &+ \left[\frac{\mu^2}{\lambda n} - \frac{2\left(\frac{B}{1+2n} + h\right)\mu}{\frac{A}{2} - n e^{\xi}}\right](1+2n) + \frac{\lambda}{\frac{A}{2} - n e^{\xi}} \left[2n(1-h^2)\, e^{-\xi} + C\right] = 0. \end{aligned}\right\} \quad (154)$$

We consider now the function

$$\left.\begin{aligned} \Pi \equiv &\left(\frac{\alpha}{n\lambda} - \frac{\operatorname{csch}\theta \frac{\partial \xi}{\partial u}}{n e^{\xi} - \frac{A}{2}}\right)^2 + \left(\frac{\beta}{n\lambda} - \frac{\operatorname{sech}\theta \frac{\partial \xi}{\partial v}}{n e^{\xi} - \frac{A}{2}}\right)^2 + \\ &+ \left(\frac{\mu}{n\lambda} + \frac{\frac{B}{1+2n} + h}{n e^{\xi} - \frac{A}{2}}\right)^2 (1+2n). \end{aligned}\right\} \quad (155)$$

When n is equal to one of the roots of the cubic (142), and α, β, μ are given the values (144), the function Π is equal to zero. We shall show that these are the only real expressions of α, β, μ for these values of such that $2n + 1 > 0$. In fact, in consequence of (154), we find that

$$\left.\begin{aligned} \Pi = -\frac{1}{2n(1+2n)\left(n e^{\xi} - \frac{A}{2}\right)^2} &\Big[(D - 2n)\, 2n\, (1+2n) + \\ &+ A\, C\, (1+2n) - 2n B^2\Big]. \end{aligned}\right\} \quad (156)$$

Since the expression on the right differs only by a factor from the left-hand member of (142), the function Π vanishes when n is a root of equation (142), thus establishing the above mentioned result.

Equation (156) in which Π has the meaning (155) is equivalent to $\Omega = 0$ for special surfaces. Evidently Π changes sign as n passes through the roots of equation (142). Hence for certain values of n the function Π is positive, in which case the transformation is real.

These results may be stated as follows:

If S is a special surface of class $(A\,B\,C\,D)$ and α, β, λ, μ, σ are solutions of the system (I) *satisfying the condition*

$$\operatorname{csch}\theta \frac{\partial \xi}{\partial u}\alpha + \operatorname{sech}\theta \frac{\partial \xi}{\partial v}\beta + \left(\frac{A}{2} - n e^{\xi}\right)\left(\sigma + \frac{\mu^2}{\lambda}\right) - \\ - \mu\left[B + h(1+2n)\right] + \lambda\left[n(1-h^2)e^{-\xi} + \frac{1}{2}C\right] = 0,$$

the surface S_1 determined by this transformation is of the same class as S; for certain values of n these surfaces S_1 are real; the complementary surfaces are the only real ones when n is a root of the fundamental cubic equation such that $2n+1>0$.

§ 19. Theorem of Permutability for Transformations of Special Surfaces and Surfaces Applicable to the General Quadric.

We close the discussion of special surfaces with the proof of the following theorem:

If a special surface S of class $(A\,B\,C\,D)$ is transformed by a T_{n_1} and T_{n_2} respectively into surfaces S_1 and S_2 of the same class, the fourth surface of the quatern is a special surface of the same class.

By hypothesis the functions α_1, β_1, λ_1, μ_1, σ_1; $\alpha_2, \ldots, \sigma_2$ of the transformations T_{n_1} and T_{n_2} satisfy the conditions

$$\left.\begin{aligned} &\operatorname{csch}\theta \frac{\partial \xi}{\partial u}\alpha_1 + \operatorname{sech}\theta \frac{\partial \xi}{\partial v}\beta_1 + \left(\frac{A}{2} - n_1 e^{\xi}\right)\left(\sigma_1 + \frac{\mu_1^2}{\lambda_1}\right) - \\ &\quad - \mu_1\left[B + h(1+2n_1)\right] + \lambda_1\left[n_1(1-h^2)e^{-\xi} + \frac{1}{2}C\right] = 0, \end{aligned}\right\} \quad (157)$$

$$\left.\begin{aligned} &\operatorname{csch}\theta\,\frac{\partial\xi}{\partial u}\,\alpha_2+\operatorname{sech}\theta\,\frac{\partial\xi}{\partial v}\,\beta_2+\left(\frac{A}{2}-n_2\,e^{\xi}\right)\left(\sigma_2+\frac{\mu_2^2}{\lambda_2}\right)-\\ &\quad-\mu_2\left[B+h\,(1+2\,n_2)\right]+\lambda_2\left[n_2\,(1-h^2)\,e^{-\xi}+\frac{1}{2}\,C\right]=0. \end{aligned}\right\}\qquad(158)$$

The theorem will be proved, if we show that the functions α'_1, β'_1, λ'_1, μ'_1, σ'_1 given by (V) satisfy the condition

$$\operatorname{csch}\theta_1\,\frac{\partial\xi_1}{\partial u}\,\alpha'_1+\operatorname{sech}\theta_1\,\frac{\partial\xi_1}{\partial v}\,\beta'_1+\left(\frac{A}{2}-n_2\,e^{\xi_1}\right)\left(\sigma'_1+\frac{\mu'^2_1}{\lambda'_1}\right)-$$

$$-\mu'_1\left[B+h_1\,(1+2\,n_2)\right]+\lambda'_1\left[n_2\,(1-h_1^2)\,e^{-\xi_1}+\frac{1}{2}\,C\right]=0.$$

We replace $\operatorname{csch}\theta_1\,\frac{\partial\xi_1}{\partial u}$ and $\operatorname{sech}\theta_1\,\frac{\partial\xi_1}{\partial v}$ by their expressions from (41), and α'_1 and β'_1 by their values from (V); we subtract from this result equation (157) multiplied by $\frac{\Psi_1}{\lambda_1\,\sigma_1}$ and equation (158) multiplied by (n_2-n_1). The resulting equation assumes the form

$$a\,\frac{A}{2}+b\,B+c\,\frac{C}{2}+d=0,$$

where a, b, c, d are determinate functions. The expressions for b and c vanish in consequence of (V).

The function a is

$$\sigma'_1+\frac{\mu'^2_1}{\lambda'_1}-\frac{\Psi_1}{\lambda_1\,\sigma_1}\left(\sigma_1+\frac{\mu_1^2}{\lambda_1}\right)+(n_1-n_2)\left(\sigma_2+\frac{\mu_2^2}{\lambda_2}\right).$$

When the values of σ'_1, μ'_1, λ'_1 from (V) are substituted, we find that $a=0$.

With the aid of this identity and (45) we obtain for d the following expression multiplied by (n_1-n_2):

$$\frac{\Psi_1}{\lambda_1}\,(e^{\xi_1}-e^{\xi})+n_2\,(e^{\xi_1}-e^{\xi})\left(\sigma_2+\frac{\mu_2^2}{\lambda_2}\right)-\mu_2\,(1+2\,n_2)\,h+\lambda_2\,n_2\,(1-h^2)\,e^{-\xi}$$

$$-(e^{\xi_1}-e^{\xi})\,\frac{\alpha_1\,\alpha_2+\beta_1\,\beta_2}{\lambda_1}-\lambda_2\,n_2\,e^{-\xi_1}\,(1-h_1^2)+h_1\,(1+2\,n_2)\,\mu_2.$$

When the value (87) of Ψ_1 is substituted in the above, and use is made of (45) and (36), we see that $d=0$. Hence the theorem is established.

The singular circle-planes of each of the surfaces S, S_1, S_2, S'_1 envelope surfaces applicable to the same quadric. Hence the preceding results lead to a transformation of surfaces S_0 applicable to the general quadric, and the last theorem shows that the transformations of these surfaces S_0 possess a theorem of permutability.

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