We may take the generalized reducing functions

$$Y_1 = \beta_1 X_1,$$
$$Y_2 = \beta_1 X_2 + \beta_2 X_1,$$
$$Y_3 = \gamma_1 X_3,$$
$$Y_4 = \gamma_1 X_4 + \gamma_2 X_3,$$
$$Y_5 = \delta_1 X_5,$$

where $\beta_1, \beta_2, \gamma_1, \gamma_2, \delta_1$ are arbitrary.

Prof. Burnside's two reductions are given by

$$\text{(i.)}\ \beta_1 = -3, \quad \beta_2 = 0, \quad \gamma_1 = -3, \quad \gamma_2 = 0, \quad l_2 = -l_1.$$
$$\text{(ii.)}\ \beta_1 = \tfrac{1}{3}, \quad \beta_2 = \tfrac{5}{9}, \quad \gamma_1 = \tfrac{1}{3}, \quad \gamma_2 = -\tfrac{8}{9}, \quad l_2 = -l_1.$$

[*February* 15*th*.—Since completing the above I have seen Mr. A. C. Dixon's paper (p. 170 of this volume). His method is quite different from mine.]

Notes on the Theory of Automorphic Functions. *By* A. C. Dixon.
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Under the above heading I propose to make some remarks on certain points in the theory of automorphic functions, from the point of view taken by Poincaré in his memoirs (*Acta Mathematica*, Vols. I., III., IV., V.).

In the first place, I show how the theorem given by him that a Fuchsian function exists of the second family and of class 0, and taking assigned values at singular points, may be used to establish the existence theorem on a Riemann surface, so far at least as that theorem relates to uniform functions of position on the surface.

Next I give expressions for Abelian integrals of the first two kinds in terms of series of the type used by Poincaré. Series of the same type are also used to form factorial functions.

It is also shown that a uniform function of the automorphic class exists which will serve as a prime function in the expression of Fuchsian functions as the product of factors. Such have been constructed for automorphic functions existing all over the plane. That which is here given serves for the other class.

The Existence Theorem of Riemann.

1. The existence of a second uniform function of position on a Riemann surface of the ordinary kind, consisting of n spherical sheets connected together, may be proved on the lines of Poincaré's work* as follows.

Let x be the original variable, so that each of the sheets is the inverse of the x-plane, and let $a_1, a_2, a_3, \dots$ be the values of x at the branch-points. Then it is proved by Poincaré (*Acta Mathematica*, Vol. IV., pp. 242–250) that x may be made a Fuchsian function of class (*genre*) 0 of a new variable z, in such a way that the vertices of the fundamental polygon all rest on the circular boundary of the function, the angles at those vertices being accordingly zero, and that only one point in the polygon corresponds to each point on the x-plane or sphere, while the points corresponding to $a_1, a_2, a_3, \dots$ are all vertices of the polygon. The present argument is not affected by the symmetry of the polygon, or the fact that it generally has other vertices as well as those corresponding to the points $a_1, a_2, a_3, \dots$.

This function gives a conformal representation on the Fuchsian polygon of the x-sphere, that is to say, of a *single sheet* of the Riemann surface.

The sides of the polygon, in pairs, represent the parts of a cut in the x-sphere, still a single sheet, the cut passing through, or at least reaching to, the points $a_1, a_2, a_3, \dots$, but not dividing the spherical surface into parts. Let this cut be made right through all the sheets of the Riemann surface. This surface will then be divided into n separate sheets, each consisting of a spherical surface with the one cut, and any one of these sheets is conformally represented on any one of the polygons into which the area enclosed by the circular boundary of the function is divided. Distinguish these polygons as I., II., III., ...

Take the polygon I. to represent the first of these separate sheets, and remove a piece of one of the cuts in such a way as to connect this with another sheet, say the second. As x travels into the second sheet across this cut, z will travel into a new polygon II., adjacent to I. Suppose II. added to I. Now take out another piece of a cut so as to connect the first or second sheet with a third, and suppose III. to be the polygon into which z passes from I. or II. when x crosses this cut into the third sheet. Add III. to I. and II. and

* See especially *Acta Mathematica*, Vol. IV., pp. 301, 302.

carry on this process until there are n polygons joined together, one representing each sheet. Let P_n be the polygon which is made up of these n. Then the whole Riemann surface as bounded by those cuts which are left is conformally represented on P_n by means of the functional relation between x and z. But P_n is a polygon bounded by circular arcs orthogonal to the bounding circle, and all its vertices lie on this bounding circle. Also the correspondence of its sides in pairs is quite definite, being ascertained by noticing the way in which the n sheets must be united again to form the original Riemann surface. Hence P_n is the generating polygon of a Fuchsian group, which is a sub-group of the original one.

Form by Poincaré's method a Fuchsian function of z having this group. Take, for instance, as generating function $\frac{1}{z-a}$, where a is a point within the bounding circle. The theta-Fuchsian function Θ thus formed will not vanish identically, nor will it have the pseudo-automorphic property for any greater group than that of P_n. But x is included among the Fuchsian functions belonging to P_n, and therefore Θ divided by the proper power of $\frac{dx}{dz}$ will be another of them, necessarily distinct from x because its group is not the same as that of x. Thus we have another Fuchsian function of z, say y, which is a uniform function of position on the Riemann surface, and is therefore connected with x by an algebraic equation to which the given Riemann surface will be appropriate; from the way in which y was formed it follows that every uniform function of position can be expressed rationally in terms of x and y, since y becomes infinite at a point in one sheet and not at the corresponding points in the other sheets.

A simple example of the construction may make the argument clearer. Take a surface of three sheets having one branch line AB (Fig. 1), and no branch-points other than A, B. The closed circuit drawn round B shows the order in which the sheets are connected,

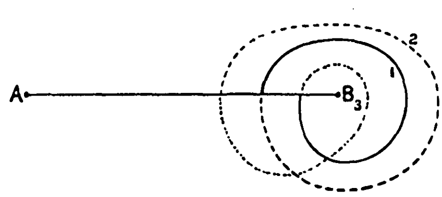

FIG. 1.

the unbroken part being in the first sheet, the broken part in the second, the dotted part in the third.

In Fig. 2, let *acec′bdb′c″a* be the original Fuchsian polygon, the

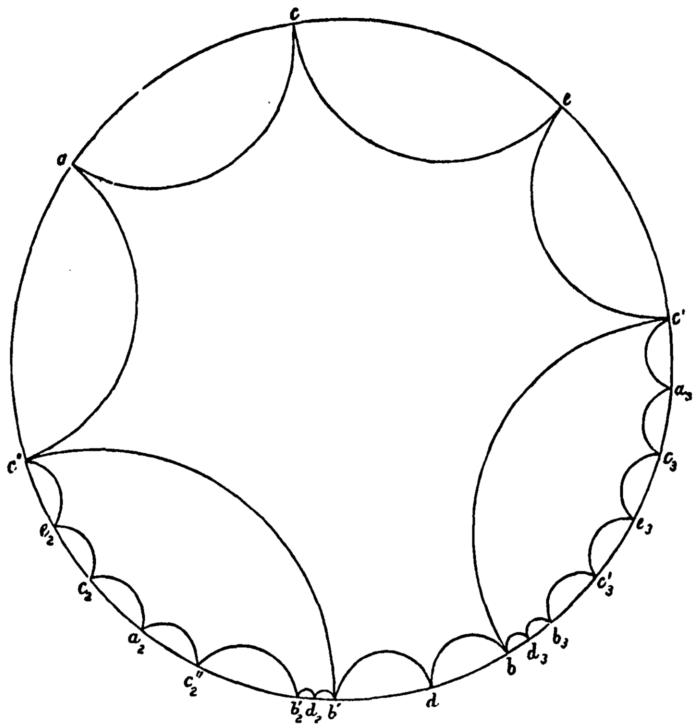

FIG. 2.

pairs of corresponding sides being *ac* and *ac″*, *ce* and *c′e*, *c′b* and *c″b′*, *db* and *db′*; the cycles are then *a*, *bb′*, *cc′c″*, *d*, *e*. Let *A*, *B*, *C*, *D*, *E* be the points of the x-plane corresponding to these cycles respectively (Fig. 3); then what is conformally represented on the Fuchsian

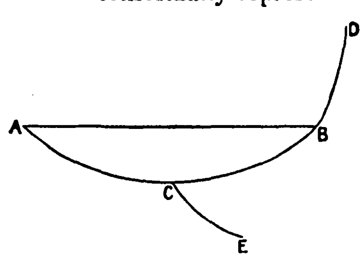

FIG. 3.

polygon is the x-plane, a single sheet only, bounded by a cut consisting of three lines CA, CBD, CE; these are supposed drawn in Fig. 3.

Now distinguish points in the different sheets by suffixes, and suppose the whole surface cut through along the lines CA, CBD, CE; it is thus divided into three sheets whose boundaries are

$$AC_1E_1C_1BD_1BC_2A,$$
$$AC_2E_2C_2BD_2BC_3A,$$
$$AC_3E_3C_3BD_3BC_1A;$$

in fact, the area AC_2B is transferred from the 2nd sheet to the 1st,

„	AC_3B	„	„	3rd	„	„	2nd,
„	AC_1B	„	„	1st	„	„	3rd.

Take the representation of the first sheet $AC_1E_1C_1BD_1BC_2A$ upon the polygon $acec'bdb'c''a$. Since the passage across BD_1 or C_1E_1 does not lead into another sheet, the polygons adjoining $acc \dots c''a$ along bd, $b'd$, ce, $c'e$ will also represent the first sheet. On the other hand, the passage across AC_2 or C_2B leads into the second sheet, which is therefore represented by either of the polygons adjoining $ace \dots c''a$ along ac'' and $c''b'$, and the passage across AC_1 or C_1B leads into the third sheet, which is therefore represented by either of the polygons adjoining $ace \dots c''a$ along ac and bc'.

Remove the cuts BC_1, BC_2; the three sheets are thus joined together into a simply connected whole whose boundary is

$$AC_1E_1C_1AC_3E_3C_3BD_3BD_1BD_2BC_3AC_2E_2C_2A.$$

Let the polygon adjoining $acec'bdb'c''a$ along $c''b'$ be $a_2c_2e_2c''b'd_2b_2'c_2''a_2$, and that adjoining along $c'b$ be $a_3c_3e_3c_3'b_3d_3bc'a_3$. Then the Riemann surface with the boundary

$$AC_1E_1C_1AC_3E_3C_3BD_3BD_1BD_2BC_3AC_2E_2C_2A$$

is conformally represented on the polygon

$$acec'a_3c_3e_3c_3'b_3d_3bdb'd_2b_2'c_2''a_2c_2e_2c''a.$$

The pairs of corresponding sides are ac and a_3c', ce and $c'e$, a_3c_3 and a_2c_2'', c_3e_3 and $c_3'e_3$, $c_3'b_3$ and $c_2''b_2'$, b_3d_3 and bd_3, bd and $b'd$, $b'd_2$ and $b_2'd_2$, a_2c_2 and ac'', c_2e_2 and $c''e_2$; the cycles are aa_3a_2, cc', e, $c_3c_2''c_3'$, e_3, $bb'b_2'b_3$, d_3, d, d_2, c_2c'', e_2.

2. The method here used is the general one (Poincaré, *Acta Mathematica*, Vol. IV., p. 286) for forming sub-groups of a given automorphic group whose generating polygon is, say, R_1. Join together R_1 and any number of the polygons into which it is transformed, say $R_2, R_3, \ldots, R_n$, in such a way as to form a new polygon. Let a_1, b_1 be a pair of conjugate sides* of R_1 and $a_2, b_2, a_3, b_3, \ldots$ the corresponding pairs in $R_2, R_3, \ldots$, any that do not form part of the boundary of the whole being left out. In the new polygon make $a_1, a_2, a_3, \ldots$ conjugate with $b_1, b_2, b_3, \ldots$ in any order; this will be possible since all have the same non-Euclidian length. Then, subject to the condition that the sum of the angles of any cycle in the new polygon shall be a sub-multiple of 2π, the new polygon will generate a new discontinuous group, included in the original one. If all the vertices of R_1 are on the bounding circle in the Fuchsian case, or if R_1 has no vertices, the condition as to the angles disappears; if, on the other hand, the original group includes elliptic substitutions, or, in fact, if R_1 has finite angles, the condition will generally restrict the order in which $b_1, b_2, \ldots$ may be taken as conjugate with $a_1, a_2, \ldots$. For instance, if R_1 has only one cycle, the sum of whose angles is 2π, the new polygon must have at least n cycles.

The consideration of sub-groups of an automorphic group and the associated theory of transformation of automorphic functions is thus closely connected with that of the functions that exist on the surface formed by joining together a number of sheets in Riemann's manner, when each of these sheets is a multiply connected surface, and is, in fact, the original Riemann surface deformed; the new surface is, of course, only equivalent to a spherical Riemann surface of a special class. The order of connexion of the new surface may be the same as that of the old; the theory for such a case will be in close connexion with the transformation theory for Abelian functions, since the Abelian integrals of the first kind on such a multiple surface will be the same as for one of its sheets, but their moduli of periodicity will be multiplied.

As an example, suppose R_1 to have four sides and opposite sides to correspond. Call the vertices a, b, c, d, and let R_2 $(bcef)$ be the region adjoining R_1 along bc, R_3 $(cdgh)$ along cd (Fig. 4). Then R_1, R_2, R_3 may be supposed to form the new polygon; we are to take

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* The argument as stated applies to a Fuchsian group; for the modification in the case of a Kleinian group compare Poincaré, *Acta Mathematica*, Vol. III., p. 72, § 5.

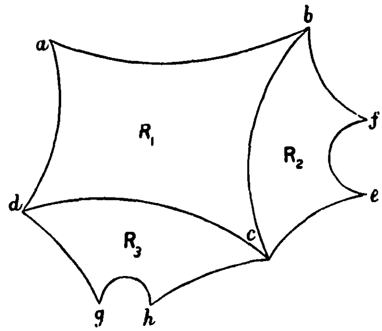

FIG. 4.

as conjugate to *ad*, *dg* either *fe*, *ch* or *ch*, *fe*, and as conjugate to *ab*, *bf* either *gh*, *ce* or *ce*, *gh*. Thus there are four possible arrangements—

I.	*ad*	*dg*	*ab*	*bf*	giving one cycle *afedcbhg*,
	fe	*ch*	*gh*	*ce*	
II.	*ad*	*dg*	*ab*	*bf*	giving one cycle *acbhdfeg*,
	ch	*fe*	*gh*	*ce*	
III.	*ad*	*dg*	*ab*	*bf*	giving one cycle *afhgbedc*,
	fe	*ch*	*ce*	*gh*	
IV.	*ad*	*dg*	*ab*	*bf*	giving three cycles *ac*, *dhf*, *beg*.
	ch	*fe*	*ce*	*gh*	

Now in the original region R_1 there is only one cycle, and the sum of the angles is therefore a sub-multiple of 2π, say $2\pi/q$. Hence, with the arrangements I., II., III., we have the sum of the angles of a cycle equal to $6\pi/q$. These must therefore be rejected unless q is a multiple of 3, but will be admissible if q is a multiple of 3. The arrangement IV. is always admissible, since the sum of the angles in each cycle is $2\pi/q$, as may be readily seen from the figure.

Let S, T be the generating operations of the original group, and suppose, in fact, that R_2 is SR_1, R_3 is TR_1. The relation satisfied by S, T is

$$(STS^{-1}T^{-1})^q = 1.$$

The generating operations for the sub-groups will be*

$$\text{I.}\quad S^3,\ TST^{-1},\ T^3,\ STS^{-1};$$

$$\text{II.}\quad TS,\ S^2T^{-1},\ T^3,\ STS^{-1};$$

$$\text{III.}\quad S^3,\ TST^{-1},\ ST,\ T^2S^{-1}$$

(these three sub-groups will therefore exist if q is divisible by 3);

$$\text{IV.}\quad TS,\ S^2T^{-1},\ ST,\ T^2S^{-1}$$

(this sub-group exists for all values of q).

In this example there were four possible arrangements, and only one of them satisfied the condition as to the angles in general. There is at present nothing to show that in general even one of the possible arrangements will satisfy the condition.

Poincaré's generating polygons do not always absolutely fix the corresponding groups; take, for instance, the case of a Fuchsian group generated by a polygon with no vertices except on the fundamental circle. When this is so, the operations of the new group which is proposed as a sub-group must be chosen from among the operations of the original group; this can always be done.

The Abelian Integrals.

3. I have not seen it observed that some Abelian integrals on the Riemann surface are zeta-Fuchsian or zeta-Kleinian functions of z, the automorphic argument. If u, for instance, is an Abelian integral and is considered as a function of z, say $u(z)$, we have for any generating substitution of the group, say S_1,

$$u(S_1 z) = u(z) + \mu_1,$$

μ_1 being the modulus of periodicity for the cut on the Riemann surface which corresponds to the boundary between R_1, the original region, and $S_1 R_1$. Thus $u(z)$ and 1 are a pair of functions which undergo a homogeneous linear transformation for any substitution of the group, the determinant being, for S_1,

$$\begin{vmatrix} 1 & \mu_1 \\ 0 & 1 \end{vmatrix}.\dagger$$

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* Here ST, for instance, means T followed by S.

† Prof. Burnside remarks that we are thus led to a class of self-conjugate sub-groups of the automorphic group. Such a sub-group will, in fact, be made up of all substitutions which leave the function $u(z)$, or a set of such functions, unaltered. Since Abelian integrals of the second kind (not canonical) may be so chosen as to

They have therefore the characteristic property of zeta-Fuchsian functions, as given by Poincaré (*Acta Mathematica*, Vol. v., pp. 227–8). The analytical expression given by him for these functions in the case of groups of the first two and sixth families* (pp. 232–5, 257–264) can be adapted to the present case as follows:—take $p = 2$, $H_1 = 0$. Let the generating substitutions be $S_1, S_2, S_3, \ldots$, and suppose any substitution S of the group to be $S_a^{a'} S_b^{b'} S_c^{c'}, \ldots$. Take a series of quantities $\mu_1, \mu_2, \mu_3, \ldots$, and suppose each to correspond to the substitution with the same suffix; corresponding to S take the quantity

$$\mu = a'\mu_a + b'\mu_b + c'\mu_c + \ldots .$$

Suppose also $\mu_1, \mu_2, \ldots$ to be such that when S is an identical or elliptic substitution μ is 0; this ensures that, if the same substitution S can be expressed in terms of $S_1, S_2, S_3, \ldots$ in two or more ways, there shall be no ambiguity in the value of μ. The group of linear substitutions, such as

$$\begin{matrix} 1 & \mu \\ 0 & 1 \end{matrix}$$

is then isomorphic to the Fuchsian group.†

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have any set of their moduli of periodicity zero, we are thus led to different self-conjugate sub-groups according to the particular set of moduli that are made to vanish. Any such sub-group will include all the elliptic substitutions; the number of its generating substitutions will usually be infinite.

* It is pointed out below (§ 9) that this restriction is unnecessary.

† The fundamental relations among the substitutions of a Fuchsian group are discussed by Poincaré (*Acta Mathematica*, Vol. I., pp. 45–7). He shows that one such relation arises from each cycle of the first category, and that there are no others. The number of relations that must be satisfied by the moduli $\mu_1, \mu_2, \ldots$ to secure the isomorphism is therefore the same as the number of cycles of the first category. With his notation we must, in fact, have

$$\lambda\,(\mu_{a_1} + \mu_{a_2} + \mu_{a_3} + \ldots) = 0,$$

that is to say, the modulus of periodicity for the elliptic substitution $\{z, F(z)\}$ must vanish. There will be one such relation among the moduli $\mu_1, \mu_2, \ldots$ for each cycle, but among these relations one will sometimes be a consequence of the rest, as pointed out later in the text.

Or thus: let S, T be any two substitutions of the group, μ, ν the corresponding moduli; then the modulus for the substitution TST^{-1} will be $\nu + \mu - \nu$, or the same as for S. Now, all the elliptic substitutions of the group can be expressed in the form TST^{-1}, where S, T belong to the group, and S is one of the finite number of elliptic substitutions that arise from the respective cycles of vertices of the original polygon. The same holds for parabolic substitutions.

In like manner all the identical relations among the operations of the group are combinations of transformations of a certain finite number.

Form the two series

$$-\Sigma\mu H(Sz\ \left(\frac{dSz}{dz}\right)^m = Z(z),$$

$$\Sigma H(Sz)\left(\frac{dSz}{dz}\right)^m = \Theta(z),$$

the summations being taken over all the substitutions of the group, and H denoting such a rational function that $\Theta(z)$ does not vanish. The absolute convergency of zeta-Fuchsian series for the first two families is established by Poincaré in the memoir referred to, on the supposition that m is great enough (p. 235), and that the zeta-Fuchsian functions whose expression is sought are of the first species (p. 258). Here the last is evidently true, since all the multipliers are unity, whether for critical, or other, substitutions.

Then, taking any substitution S_1, belonging to the group, we have, still following Poincaré (p. 232),

$$Z(S_1z) = -\Sigma\mu H(SS_1z)\left(\frac{dSS_1z}{dS_1z}\right)^m,$$

$$Z(z) = -\Sigma(\mu+\mu_1)H(SS_1z)\left(\frac{dSS_1z}{dz}\right)^m,$$

the last expression containing the same terms as the former series for $Z(z)$, but differently arranged. Hence

$$Z(z)-Z(S_1z)\left(\frac{dS_1z}{dz}\right)^m = -\mu_1\Theta(z),*$$

$$\frac{Z(z)}{\Theta(z)}-\frac{Z(S_1z)}{\Theta(S_1z)} = -\mu_1.$$

A like result holds for each of the other substitutions of the group. Hence $Z(z)/\Theta(z)$ is a uniform function of z, and its values at corresponding points of different polygons differ by multiples of the moduli $\mu_1, \mu_2, \ldots$, these multiples depending on the particular polygons in question only. Thus $Z(z)/\Theta(z)$ is an Abelian integral belonging to the Riemann surface. This investigation applies to the first, second, and sixth families.

The modulus for a parabolic substitution need not vanish, although

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* Thus, if $\Theta(z)$ vanishes identically, $Z(z)$ is a theta-Fuchsian function.

the principal point of such a substitution forms a cycle, or at least may be made to; suppose this to be done. Then the substitution changes one of the sides drawn from this vertex of the polygon into the other. A line drawn between these sides corresponds to a circuit round the corresponding point on the Riemann surface; hence, if the modulus for this substitution does not vanish, the Abelian integral has a logarithmic discontinuity at the corresponding point on the Riemann surface.

The Abelian integrals of the first two kinds will therefore have the moduli corresponding to the parabolic substitutions zero.

If the fundamental polygon is not simply connected, some irreducible circuits on the Riemann surface will correspond to contours round the holes, and an Abelian integral will therefore only be a uniform function of z when its moduli for all such circuits vanish.

4. Take a Fuchsian group of the first, second, or sixth family, giving functions with a natural boundary. The fundamental polygon is here simply connected; the number of moduli is the number of pairs of corresponding sides; the number of relations connecting them is the number of cycles diminished by one, since the aggregate of all the cycles contains each substitution once, and also its inverse once. This is on the supposition that the modulus for each parabolic substitution vanishes. Hence the number of arbitrary moduli is the same as that of irreducible circuits on the Riemann surface,* and the modulus for each irreducible circuit is, in fact, arbitrary.

The difference of two functions whose moduli are the same will clearly be a Fuchsian function, that is, a uniform function of position on the Riemann surface.

Let c be the number of irreducible circuits. Then c linearly independent functions can be formed having certain assigned poles, amongst others, and having different sets of moduli. The poles can be assigned by making them zeroes of $\Theta(z)$ which need not be finite everywhere, and can therefore be made to have any assigned zeroes. Let $p+1$ be the least number of arbitrary poles that can be assigned for a Fuchsian function which is to have no others. Then by subtraction of Fuchsian functions the above set of c functions can be made to have p arbitrary poles and no others. By combining them linearly we can form $c-p$ functions whose residues at these poles

* Compare *Acta Mathematica*, Vol. I., p. 229, lines 1-6.

are zero, and which are therefore everywhere finite. But now consider the integral $\int w\,du$, where w is a Fuchsian function, and u one of these functions that are everywhere finite. The value of this integral round the contour of the polygon is zero, and thus we have a linear relation connecting the residues of w at its different poles. There will be one such relation for each of the $c-p$ functions that may be taken in the place of u, so that, if the residues are not all to vanish, there must be at least $c-p+1$ of them, and we conclude that $p = c-p$ or $c = 2p$. This argument is, of course, loose; but, as the result is so well known, there is no object in labouring the point. The c (or $2p$) independent functions with p arbitrary poles and with moduli as above can thus be combined so as to give p functions without poles and p others having one pole each; these are, of course, the usual Abelian integrals of the first two kinds.

It is clear that in general an Abelian integral of the third kind is not a uniform function of the automorphic argument. If, however, the points of logarithmic discontinuity fall at vertices of the polygon belonging to parabolic cycles, this statement ceases to be true. In fact it was seen above that the function $Z(z)/\Theta(z)$ was an Abelian integral of the third kind if its modulus for any parabolic substitution was other than zero. Now, in the construction of § 1, we may add any points we please to the list of branch-points $a_1, a_2, \ldots$, without affecting the argument, and thus ensure that the logarithmic discontinuities of any finite number of Abelian integrals of the third kind shall fall on the bounding circle; hence we may, if it is desired, suppose the automorphic representation so chosen that any finite number of specified integrals of the third kind are uniform functions of the automorphic argument.

5. Now take an integral of the first kind u, and another of the second kind v_c, having in the polygon a single pole for the value $z = c$, and such that

$$\operatorname*{Lim}_{z=c}\,(z-c)\,v_c\,\frac{du}{dz} = 1.$$

Then $\exp\int_{z_0} v_c\,\frac{du}{dz}\,dz$ is a uniform function of z, having simple zeroes at c and the corresponding points in other polygons only, and having no infinity except at the essential singularities. Let us consider the effect on it of any substitution S of the group. Suppose μ_c to be the corresponding modulus for v_c; then

$$\int_{z_0}^{Sz} v_c \frac{du}{dz} dz = \int_{z_0}^{Sz_0} v_c \frac{du}{dz} dz + \int_{Sz_0}^{Sz} v_c \frac{du}{dz} dz$$

$$= \int_{z}^{Sz_0} v_c \frac{du}{dz} dz + \int_{z_0}^{z} (v_c + \mu_c) \frac{du}{dz} dz$$

$$= \int_{z_0} v_c \frac{du}{dz} dz + \mu_c (u - u_0) + \int_{z_0}^{Sz_0} v_c \frac{du}{dz} dz.$$

If, then, we write $\dfrac{\vartheta_c(z)}{\vartheta_c(z_0)}$ for $\exp \int_{z}^{z} v \dfrac{du}{dz} dz$, we have

$$\vartheta_c(Sz) = \vartheta_c(z) \frac{\vartheta_c(Sz_0)}{\vartheta_c(z_0)} \exp \mu_c (u - u_0).$$

Suppose now that v_c is a multiple of the normal integral of the second kind, and take $2(p+1)$ distinct points $b_1, b_2, \ldots, b_{p+1}$, $c_1, c_2, \ldots, c_{p+1}$. The function

$$\Pi\vartheta_b(z) \div \Pi\vartheta\ (z)$$

is a uniform function of z, and by the change of z into Sz it is multiplied by

$$\exp(\Sigma\mu_b - \Sigma\mu_c)(u - u_0).$$

If, then, $$\Sigma\mu_b = \Sigma\mu_c$$

for each substitution, that is to say, if p conditions are satisfied,

$$\Pi\vartheta_b(z) \div \Pi\vartheta_c(z)$$

is a uniform function of position on the Riemann surface, having, say, $p+1$ arbitrary poles and one arbitrary zero.

The function ϑ_c is not unique, since u may be any integral of the first kind. The quotient of two such functions, having a common vanishing point, will therefore be a uniform function of z, neither vanishing nor becoming infinite except at an essential singularity; also, if v is an integral of the first kind, $\exp \int v du$ will have the same properties.

The function $\vartheta_c(z)$ is clearly very like the prime function of Schottky and Klein. (See, for instance, Prof. Burnside's paper, *Proc. Lond. Math. Soc.*, Vol. XXIII., pp. 289–293.) It differs, however, in not having a pole at infinity, as is to be expected, since infinity is outside the region in which it exists.

Factorial Functions.

6. Pseudo-automorphic functions of the factorial class can also be considered as zeta-Fuchsian or zeta-Kleinian functions. Let the multiplier corresponding to the substitution

$$S \equiv S_a^{a'} S_b^{b'} S_c^{c'} \dots$$

be

$$M = M_a^{a'} M_b^{b'} M_c^{c'} \dots .$$

Suppose also that, when S is an elliptic substitution of period k, M is a k^{th} root of unity; that, when S is a parabolic substitution, the modulus $|M|$ is unity (see *Acta Mathematica*, Vol. v., pp. 258, 269); and that, if $S \equiv 1$, then $M = 1$. Form the series

$$\Sigma M^{-1} H(Sz) \left(\frac{dSz}{dz}\right)^m = \phi(z),$$

$$\Sigma H(Sz) \left(\frac{dSz}{dz}\right)^m = \Theta(z),$$

the summations being taken over all the substitutions of the group. The series $\phi(z)$ is a zeta-Fuchsian series. We may therefore use Poincaré's results as to the convergency of such series, already quoted. The subject is further discussed below (§ 9). Then, if S_1 is a substitution belonging to the group, we have

$$\phi(S_1 z) = \Sigma M^{-1} H(SS_1 z) \left(\frac{dSS_1 z}{dS_1 z}\right)^m,$$

$$\phi(z) = \Sigma M_1^{-1} M^{-1} H(SS_1 z) \left(\frac{dSS_1 z}{dz}\right)^m,$$

so that $\phi(S_1 z) \div \phi(z) = M_1 \left(\dfrac{dS_1 z}{dz}\right)^{-m} = M_1 \Theta(S_1 z) \div \Theta(z).$

Thus $\phi(z) \div \Theta(z)$ is a factorial pseudo-automorphic function. The restrictions on the multipliers are very much like those on the moduli in the former case; if, however, the multipliers for parabolic substitutions are unity, those for irreducible circuits will be arbitrary, and the roots of unity chosen for the elliptic substitutions must satisfy a relation which may compel them all to be unity itself.

If we further choose roots of unity for the arbitrary multipliers, we have a class of factorial functions that will be of great importance in the transformation theory of automorphic functions. (Compare Ritter, *Math. Annalen*, Vol. XLI., pp. 31, 58.)

7. Other expressions for the Abelian integrals have been given, as, for instance, by Prof. Burnside (*Proc. Lond. Math. Soc.*, Vol. XXIII., pp. 66–9), and it is a point worthy of notice that his function

$$\theta(z, a) = \Sigma \frac{1}{Sz-a} \frac{dSz}{dz},$$

if considered as a function of a, z being constant, is an Abelian integral, and, in fact, is practically the normal function of the second kind.* Prof. Burnside's paper treats mainly of functions of the first class; whereas the method used in the present paper is only completely applicable to those of the second class.

8. The following investigations relate to the convergency of some of the series that are used in the paper.

To prove the convergency of the series

$$\Sigma \mu H(Sz) \left(\frac{dSz}{dz}\right)^m,$$

and to show that this convergency is uniform when z varies within proper limits, it is enough to prove the convergency of the series

$$\Sigma \left| \frac{\mu}{\gamma^{2m}} \right|,$$

where $$Sz = \frac{az+\beta}{\gamma z+\delta}, \quad a\delta-\beta\gamma = 1.$$

The function H is supposed to have no pole at any essential singularity of the group, and z is supposed not to fall at any point that is transformed by a substitution of the group into ∞ or one of the poles of $H(z)$. Thus a superior limit can be fixed for $|H(Sz)|$ and superior and inferior limits to $\left| z+\dfrac{\delta}{\gamma} \right|$, if we suppose, as we may, that one of the polygons outside the fundamental circle contains the point ∞, so that γ does not vanish for any of the substitutions except the identical one. Such an exception affecting only an isolated term may be ignored.

Let κ be the greatest among the absolute values of the moduli of periodicity μ for the generating substitutions, and let n be the exponent of the substitution S, that is the least value of $a'+b'+c'+\ldots$,

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* See H. F. Baker, *Abelian Functions*, p. 357, Ex. ii.

when S is expressed in the form $S_a^{a'} S_b^{b'} S_c^{c'} \ldots$ in terms of the generating substitutions and their inverses in such a way that $a', b', c' \ldots$ are all positive whole numbers. Then

$$|\mu| \ngtr n\kappa.$$

Thus we have to consider the series $\Sigma n\,|\gamma|^{-2m}$. Now, since $\left|z+\dfrac{\delta}{\gamma}\right|$ has both a superior and an inferior limit, the series

$$\Sigma\,|\gamma|^{-2m}, \quad \Sigma\,|\gamma z+\delta|^{-2m}$$

converge or diverge together. (The identical substitution for which $\gamma = 0$ is, of course, left out here.) All the terms are positive; so, if V_n is the sum of all the terms in $\Sigma\,|\gamma|^{-2m}$ corresponding to substitutions of exponent n, the series

$$V_1+V_2+\ldots+V_n+\ldots$$

is convergent, when $m>1$. Thus, when $m>1$,

$$V_n < \frac{1}{n\log n}$$

from and after some term in the series.

Hence $$nV_n^2 < \frac{1}{n(\log n)^2}$$

after this term, and therefore

$$V_1^2+2V_2^2+\ldots+nV_n^2+\ldots$$

is convergent. Now V_n^2 is greater than the sum of the terms in $\Sigma\,|\gamma|^{-4m}$ which correspond to substitutions of exponent n, and therefore $\Sigma n\,|\gamma|^{-4m}$ is convergent if $m>1$, or

$$\Sigma\mu H(Sz)\left(\frac{dSz}{dz}\right)^m$$

is absolutely and uniformly convergent if $m>2$.

9. M. Poincaré's discussions of the convergency of theta-Fuchsian and zeta-Fuchsian series may be adapted to the case of Kleinian groups as follows.

Suppose the plane of the variable z which is unchanged by the operations of the generalized Kleinian group (*Acta Mathematica*, Vol. III., pp. 52–6) to be turned into a sphere by stereographic projection. Take this to be the sphere

$$x^2+y^2+z^2 = 1.$$

Then any operation of the group turns the point P (x, y, z) into the point P' (x', y', z'), where

$$\frac{x'^2+y'^2+z'^2-1}{x^2+y^2+z^2-1} = \frac{x'}{A} = \frac{y'}{B} = \frac{z'}{C} = \frac{1}{E},$$

A, B, C, E being linear in $x^2+y^2+z^2$, x, y, z, 1. Also, if ds is any infinitesimal arc at the point P, $\dfrac{ds}{x^2+y^2+z^2-1}$ is unchanged by any inversion which leaves the sphere $x^2+y^2+z^2=1$ unchanged, and therefore by every operation of the group. Hence the linear magnification $\dfrac{ds'}{ds}$ is equal to $\dfrac{x'^2+y'^2+z'^2-1}{x^2+y^2+z^2-1}$ and to $\dfrac{1}{E}$.

Now, if P' goes to infinity, P approaches a definite limiting position K, and therefore E can only be a constant multiple of PK^2, which will be positive here, since the inside and outside of the sphere $x^2+y^2+z^2=1$ are not interchanged by any of the operations. We may then write

$$E = \eta^2 . PK^2.$$

Take, then, the series $\Sigma \dfrac{1}{E^m}$ and $\Sigma \dfrac{1}{\eta^{2m}}$, the summation being over all the operations of the generalized Kleinian group, and the x, y, z which occur in E being the coordinates of a point P which is inside one of the polyhedra, and does not coincide with the point in that polyhedron which corresponds to ∞. We shall suppose that ∞, and therefore also the origin, is within one of the polyhedra; this can be secured by proper choice of the vertex of the stereographic projection. Thus $E \div \eta^2$ has a superior and an inferior limit, for every position of the point P, including the origin, except those which correspond to ∞ in the different polyhedra. The series $\Sigma \dfrac{1}{E^m}$ and $\Sigma \dfrac{1}{\eta^{2m}}$ will therefore converge or diverge together. Now, if P lies on the sphere $x^2+y^2+z^2=1$, and within one of the polyhedra, the series $\Sigma \dfrac{1}{E^2}$ converges, as in M. Poincaré's first proof for the theta-Fuchsian series, since the whole surface of the sphere is finite. Therefore the series $\Sigma \dfrac{1}{\eta^4}$ converges, and so does $\Sigma \dfrac{1}{E^2}$ for all positions of P, including the origin. Let E_0 be the value of E when P is at the origin; then, since

$$\frac{1}{E} = \frac{x'^2+y'^2+z'^2-1}{x^2+y^2+z^2-1},$$

we have

$$\frac{1}{E_0} = 1-r^2,$$

r being the distance from the origin of the point into which it is transformed by the substitution in question. That is,

$$E_0 = \cosh^2 R,$$

R being the non-Euclidian distance of this point from the origin, so that

$$R = \int_0^r \frac{d\rho}{1-\rho^2} = \tfrac{1}{2}\log\frac{1+r}{1-r}.$$

Now the arguments used by M. Poincaré (*Acta Mathematica*, Vol. v., pp. 233–235, 259–264) apply here, with the substitution of polyhedra and spheres for polygons and circles, and show that, if there is a group of homogeneous linear substitutions isomorphic to the Kleinian group, the coefficients A in any substitution of the new group are all less in absolute value than

$$e^{aR},$$

where R is the quantity just now indicated, and a is a constant. It is, of course, still necessary to suppose that all the multipliers in a substitution of the new group which corresponds to a parabolic substitution in the Kleinian group have modulus unity.

Hence the series $\Sigma \frac{A}{E_0^m}$, and therefore $\Sigma \frac{A}{\eta^{2m}}$, $\Sigma \frac{A}{E^m}$ will be convergent if

$$2m > 4+a,$$

since $\Sigma \frac{1}{E_0^2}$ is known to be convergent.

The ratio of the magnification on the original plane to that on the sphere has finite limits, superior and inferior, if we suppose ∞ on the plane to be contained within one of the polygons. Hence the convergency of the zeta-Kleinian series is assured for values of m exceeding $2+\frac{1}{2}a$.

M. Poincaré's arguments here referred to apply to all families; the absence of closed cycles would, in fact, simplify the discussion. Closed hyperbolic cycles must be removed, as at *Acta Mathematica*, Vol. III., pp. 71, 72.