

A very good illustration of the buckling of a circular plate is frequently afforded by the lid of a circular canister, in which the thrust is due to the tension of the rim. The "dint" in such a lid can be readily pushed from one side to the other, but it is impossible to keep the surface flat, as that position is unstable.

The same principle is also illustrated in the "castanets," in which a "clicking" sound is produced by pushing a disc of metal from one side to the other of the unstable plane form.

13. In all the cases discussed in this paper, the stresses in the surface are proportional to β ; and, therefore, to the cube of the thickness of the plate. Since these stresses are distributed over the thickness of the plate, the strains they produce are proportional to the square of the thickness. If, therefore, the plate be thin, these strains will be small, and there will be no rupture of the material accompanying the buckling. This accords with the general results obtained in my paper "On the Stability of Elastic Systems."*

In a future paper, I hope to deal with further applications of the variational method, with special reference to the stability of a rectangular plate or strip in certain cases when the shear M does not vanish, and when the boundary conditions are different to those assumed in the present communication.

On the Application to Matrices of any Order of the Quaternion Symbols S and V . By HENRY TABER, Docent in Clark University, Worcester, Mass. U.S.A.

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1. *Properties of the Symbols S and V .*

The conception of scalar and vector parts of a quaternion, or matrix of the second order, may be extended to matrices of any order.† Regarded as a matrix, the scalar of any quaternion is one half the sum of its latent roots; following this analogy, I shall define the scalar of any matrix m of order ω as the ω^{th} part of the sum of

* *Camb. Phil. Proc.*, Vol. vi., p. 204.

† See paper by author on the "Theory of Matrices," *Amer. Journ. Math.*, Vol. xii.

its latent roots; and I shall denote this function of m , as in quaternions, by Sm . But then, since the sum of the latent roots of a matrix is the sum of the elements along its principal diagonal, it follows, if n is any other matrix of order ω , that

$$S(m+n) = Sm + Sn.$$

The vector part of m (denoted by Vm) is most simply defined as the matrix, less the ω -th part of the sum of its latent roots; and so defined V is also a distributive symbol of operation; i.e.,

$$\begin{aligned} V(m+n) &= (m+n) - S(m+n) = (m - Sm) + (n - Sn) \\ &= Vm + Vn. \end{aligned}$$

Following immediately from the distributive character of S and V , we have, as in quaternions,

$$dSm = Sdm, \quad dVm = Vdm.$$

Obviously, as in quaternions,

$$\begin{aligned} S.mn &= S(Sm + Vm)(Sn + Vn) \\ &= Sm.Sn + S.Vm.Vn.* \end{aligned}$$

$$\begin{aligned} \text{Therefore} \quad V.mn &= V(Sm + Vm)(Sn + Vn) \\ &= Sm.Vn + Sn.Vm + V.Vm.Vn. \end{aligned}$$

$$\text{If} \quad m = \begin{pmatrix} a_{11} & a_{12} & \&c. \\ a_{21} & a_{22} & \&c. \\ \&c. & & \end{pmatrix}, \quad n = \begin{pmatrix} b_{11} & b_{12} & \&c. \\ b_{21} & b_{22} & \&c. \\ \&c. & & \end{pmatrix},$$

$$\begin{aligned} \text{then} \quad S.mn &= \sum_r a_{1r} b_{r1} + \sum_r a_{2r} b_{r2} + \&c. \\ &= \sum_r a_{r1} b_{1r} + \sum_r a_{r2} b_{2r} + \&c. \\ &= S.nm. \end{aligned}$$

Consequently, the scalar of the product of any number of given matrices is unaltered by a cyclic interchange of these matrices.

$$\begin{aligned} \text{E.g.,} \quad S \left\{ \begin{pmatrix} ab & a'b' & a''b'' \\ cd & c'd' & c''d'' \end{pmatrix} \right\} \\ = (aa'a' + dd'd'') + (ab'c'' + a'b''c + a''b'c') + (bc''d' + b'cd'' + b''c'd). \end{aligned}$$

* For $SVm = S(m - Sm) = Sm - Sm = 0$.

2. The catena of identical equations.

The term "catena of identical equations" was employed by Sylvester (*Johns Hopkins Univ. Circs.*, Vol. III.) to denote the $\omega+1$ identical relations (including the "identical equation" of either matrix) existing between two matrices of order ω which he obtained, with the aid of the latent function of the corpus of two matrices, by successive differentiations of the identical equation of either matrix. Thus, if m and n are two matrices of the second order, and the latent function of the corpus m, n , namely,

$$\text{Det. } (x + ym + zn) = x^2 + 2bxy + 2cxz + dy^2 + 2eyz + fz^2,$$

the catena of m, n consists of the three identities

$$m^2 - 2bm + d = 0,$$

$$mn + nm - 2bn - 2cn + 2e = 0,$$

$$n^2 - 2cn + f = 0.$$

The introduction into the general theory of matrices of the quaternion symbol S may be utilized to find the coefficients in the catena of equations without the aid of Sylvester's latent function of the corpus of the matrices which appear in the chain. In a paper to appear in No. 2, Vol. XIII., of the *Amer. Journ. Math.*, I have given, for matrices of the second and third orders, by employing the symbol S , the coefficients in the catena in terms of the scalars (or sums of the latent roots) of the products of powers of the matrices involved; and the method employed there may be extended to matrices of any order, though the extension of this method to matrices of the fifth or of higher orders is not very easy. In a somewhat different and simpler manner, I show here how to obtain the coefficients in the catena for matrices of the fourth order. Thus, if the identical equation in m is

$$m^4 - p_1 m^3 + p_2 m^2 - p_3 m + p_4 = 0,$$

we may replace the p 's by their values in terms of the sums of the powers of the latent roots of m . But since, by the law of latency, the latent roots of m^r are the r^{th} powers of the latent roots of m , hence $4Sm^r = \text{sum of the } r^{\text{th}} \text{ powers of the latent roots of } m$.

Consequently

$$p_1 = 4Sm,$$

$$1.2.p_2 = \begin{vmatrix} 4Sm, & 1 \\ 4Sm^2, & 4Sm \end{vmatrix} = 4(4S^2m - Sm^2),$$

$$1.2.3.p_3 = \begin{vmatrix} 4Sm, & 1, & 0 \\ 4Sm^2, & 4Sm, & 2 \\ 4Sm^3, & 4Sm^2, & 4Sm \end{vmatrix} = 8(8S^3m - 6Sm^2.Sm + Sm^3),$$

$$1.2.3.4.p_4 = \begin{vmatrix} 4Sm, & 1, & 0, & 0 \\ 4Sm^2, & 4Sm, & 2, & 0 \\ 4Sm^3, & 4Sm^2, & 4Sm, & 3 \\ 4Sm^4, & 4Sm^3, & 4Sm^2, & 4Sm \end{vmatrix} \\ = 8(32S^4m - 48S^3m.Sm^2 + 6S^2m^2 + 16Sm.Sm^3 - 3Sm^4),$$

where S^2m or $S^3(m)$, means $(Sm)^2$ as $\sin^2 x$ means $(\sin x)^2$.

It follows from a property of the symbol S , namely

$$dSm = Sdm,$$

that we may immediately differentiate the identity in m to obtain the first derived identical equation (the second of the chain). Thus, if $dm = n$, we have

$$\Sigma m^3n - (dp_1.m^3 + p_1\Sigma m^2n) + (dp_2m^2 + p_2\Sigma mn) \\ - (dp_3.m + p_3m) + dp_4 = 0,$$

where $\Sigma m^r n$ denotes the sum of all possible products of n with r m 's, i.e., Σm^3n means $nm^3 + mn^2 + m^2nm + m^3n$, and so in other cases; and

$$dp_1 = 4Sn,$$

$$dp_2 = 4(4Sm.Sn - S.mn),$$

$$dp_3 = 4(8S^2m.Sn - 4S.mn.Sm - 2Sm^2.Sn + S.m^2n),$$

$$dp_4 = \frac{4}{3}(32S^3m.Sn - 24S.mn.S^2m - 24Sm^2.Sm.Sn + 6Sm^3.S.mn \\ + 12S.m^2n.Sn + 4Sm^3.Sn - 3S.m^3n).$$

In the next differentiation, regarding n as 'constant, we may either put the new $dm = n$, and thus obtain the second equation of Sylvester's catena; or we may put the new $dm = p$, and proceeding

in this way ultimately obtain an identical relation between any four matrices.

This method of deriving the coefficients of the catena is, I think, somewhat simpler than to derive them from the latent function of the corpus of the matrices which appear in the catena; and has this advantage over that method, that these coefficients are expressed immediately as simple functions of the sums of the latent roots of the products of the powers of the matrices involved in the catena.

Obviously this process may be extended to matrices of any order.

3. Application of the symbol S to a problem of Sylvester's.

In the *Johns Hopkins Univ. Circs.*, Vol. I., p. 241, and Vol. III., p. 7, Sylvester has considered the problem to determine the conditions necessary and sufficient that for two nonions m and n ,

$$m^3 = n^3 = 1, \quad nm = \lambda mn,$$

where λ is an imaginary cube root of unity. Sylvester finds the necessary and sufficient conditions to be

$$\text{Det. } (x + ym + zn) = x^3 + y^3 + z^3$$

(involving nine conditions), and that the coefficients of mn in the identical equation of mn shall vanish. Sylvester has also given a special solution.* These ten conditions leave $18 - 10 = 8$ arbitrary constants in the general solution. But if m, n are expressions linear in terms of $(1, i, i^2 \mathfrak{X} 1, j, j^2)$ (i and j being special solutions), so that

$$m = (a_1, a_2, \dots a_6 \mathfrak{X} 1, i, i^2 \mathfrak{X} 1, j, j^2),$$

$$n = (b_1, b_2, \dots b_6 \mathfrak{X} 1, i, i^2 \mathfrak{X} 1, j, j^2),$$

Sylvester states that he is unable to express the conditions in terms

* If i and j are the two special solutions, so that

$$i = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda^2 \end{pmatrix}, \quad j = \begin{pmatrix} 0 & 0 & 1 \\ \lambda & 0 & 0 \\ 0 & \lambda^2 & 0 \end{pmatrix},$$

the general solution is given by $m = \varpi i \varpi^{-1}$, $n = \varpi j \varpi^{-1}$, where $|\varpi| \neq 0$ (see *American Journal of Mathematics*, Vol. XII., p. 390).

of these coefficients*. In what follows I shall show how this may be accomplished.

If $m^3 = 1$, $n^3 = 1$, hence $Sm^3 = 1$, $Sn^3 = 1$. If, in addition, $nm = \lambda mn$, hence $m = \lambda n^2 mn$, and $m^3 = \lambda n^2 mnm = \lambda^2 n^2 m^2 n$; and, consequently,

$$Sm = \lambda S \cdot n^2 mn = \lambda S \cdot n^3 m = \lambda Sm,$$

$$Sm^2 = \lambda^2 S \cdot n^2 m^2 n = \lambda^2 S \cdot n^3 m^2 = \lambda^2 Sm^2;$$

i.e., $Sm = 0$, $Sm^2 = 0$. Similarly, $Sn = 0$, $Sn^2 = 0$. Since $m = Vm$, $n = Vn$, it seems preferable to denote m and n , as in quaternions, by Greek letters, so I shall put $m = \alpha$, $n = \beta$; i.e., the problem is: find α , β cubic matrices, such that $\alpha^3 = 1$, $\beta^3 = 1$, $\beta\alpha = \lambda\alpha\beta$, λ being an imaginary cube root of unity. From $\beta\alpha = \lambda\alpha\beta$, whence follows $S \cdot \alpha\beta = S \cdot \beta\alpha = 0$, we also get

$$\beta\alpha \cdot \beta\alpha = \lambda^2 \alpha\beta \cdot \alpha\beta, \quad \beta\alpha^2 = \lambda^2 \alpha^2 \beta, \quad \beta^2 \alpha = \lambda^2 \alpha \beta^2;$$

$$\therefore S(\beta\alpha)^2 = S(\alpha\beta)^2 = 0, \quad S \cdot \alpha^2 \beta = 0, \quad S \cdot \alpha\beta^2 = 0.$$

I shall presently show that the ten conditions

$$S\alpha = 0, \quad S\alpha^2 = 0, \quad S\alpha^3 = 1,$$

$$S\beta = 0, \quad S\beta^2 = 0, \quad S\beta^3 = 1,$$

$$S \cdot \alpha\beta = 0, \quad S \cdot \alpha^2 \beta = 0, \quad S \cdot \alpha\beta^2 = 0, \quad S(\alpha\beta)^2 = 0,$$

are sufficient as well as necessary. From

$$S\alpha = 0, \quad S\alpha^2 = 0, \quad S\alpha^3 = 1, \quad \text{we get} \quad \alpha^3 = 1 \dagger;$$

and from

$$S\beta = 0, \quad S\beta^2 = 0, \quad S\beta^3 = 1, \quad \text{we get} \quad \beta^3 = 1.$$

Similarly,

$$S \cdot \alpha\beta = 0, \quad S(\alpha\beta)^2 = 0, \quad \text{give} \quad (\alpha\beta)^3 - S(\alpha\beta)^3 = 0;$$

* Sylvester remarks that "the solution of this problem would seem to involve some unknown expansion of the idea of orthogonalism." Evidently the introduction into the theory of nonions, and into the theory of matrices in general, of the symbol S furnishes such an expansion of the idea of orthogonalism. Thus, just as in quaternions, two quaternion square roots of unity (i.e., unit vectors) α and β are normal if $S\alpha\beta = 0$, when we have $\beta\alpha = -\alpha\beta$; so in nonions, two nonion cube roots of unity α , β may be regarded as completely normal, if $S\alpha\beta = 0$, $S(\alpha\beta)^2 = 0$, $S\alpha^2\beta = 0$, $S\alpha\beta^2 = 0$, when we have $\beta\alpha = \lambda\alpha\beta$, λ being an imaginary cube root of unity. It should be observed that the nonions α , β are not the most general forms of nonion unit vectors; the most general form is $xi + yi^2$, where i is a nonion cube root of unity, and $x^3 + y^3 = 1$.

† This is at once evident from the form of the identical equation of any nonion m ,

$$m^3 - 3Sm \cdot m^2 + \frac{3}{2}(3S^2m - Sm^2) \cdot m + (\frac{3}{2}S^3m - \frac{9}{2}Sm \cdot Sm^2 + Sm^3) = 0.$$

but $S(\alpha\beta)^3$ is here the explicit scalar term in the identical equation of $\alpha\beta$, and so is

$$\text{Det.}(\alpha\beta) = \text{Det.}(\alpha) \times \text{Det.}(\beta) = S\alpha^3 \cdot S\beta^3 = 1;$$

hence $(\alpha\beta)^3 = 1$. We may now derive the equation $S \cdot \alpha^2\beta^2 = 0$,* involved in the ten conditions, and necessary to the proof. For, since $(\alpha\beta)^3 = 1$, hence $\beta\alpha \cdot \beta\alpha = \alpha^2 (\alpha\beta)^3 \beta^2 = \alpha^2 \beta^3$; and consequently

$$S \cdot \alpha^2\beta^3 = S(\beta\alpha)^2 = S(\alpha\beta)^2 = 0.$$

The proof that the ten conditions are sufficient depends upon the lemma that the involutant of α , β does not vanish.† This may be shown very simply as follows. If

$$[a, \beta] \equiv (a_0 + a_1\alpha + a_2\alpha^2) + (b_0 + b_1\alpha + b_2\alpha^2)\beta + (c_0 + c_1\alpha + c_2\alpha^2)\beta^2 = 0,$$

then, since the scalar of all these products of powers of α and β is zero, $a_0 = S[a, \beta] = 0$; and multiplying successively $[a, \beta] = 0$ by α^2 and α , we get

$$a_1 = S \cdot \alpha^2[a, \beta] = 0, \quad \text{and} \quad a_2 = S \cdot \alpha[a, \beta] = 0;$$

similarly, multiplying $[a, \beta]$ successively by 1 , α^2 , α , and into β^2 , we get

$$b_0 = S \cdot [a, \beta]\beta^2 = 0, \quad b_1 = S \cdot \alpha^2[a, \beta]\beta^2 = 0, \quad b_2 = S \cdot \alpha[a, \beta]\beta^2 = 0.$$

In the same way we may show that the c 's are zero.

Hence we may put

$$\beta\alpha = (x_0 + x_1\alpha + x_2\alpha^2) + (y_0 + y_1\alpha + y_2\alpha^2)\beta + (z_0 + z_1\alpha + z_2\alpha^2)\beta^2;$$

* The equation $S \cdot \alpha^2\beta^2 = 0$ may be substituted for the condition $S(\alpha\beta)^2 = 0$, which may be derived from it and seven of the other nine conditions. Thus

$$\begin{aligned} \beta\alpha \cdot \beta\alpha &= \alpha^2(\alpha\beta)^3\beta^2 = \alpha^2 \left\{ 3S \cdot \alpha\beta \cdot (\alpha\beta)^2 - \frac{3}{2}(3S^2 \cdot \alpha\beta - S\bar{\alpha}\bar{\beta}^2) \cdot \alpha\beta + S(\alpha\beta)^3 \right\} \beta^2 \\ &= \frac{3}{2}S \cdot (\alpha\beta)^2 + S \cdot (\alpha\beta)^3 \cdot \alpha^2\beta^2, \end{aligned}$$

if $\alpha^3 = \beta^3 = 1$ and $S \cdot \alpha\beta = 0$. Hence, if $S\alpha^2\beta^2 = 0$,

$$S(\alpha\beta)^2 = S(\beta\alpha)^2 = \frac{3}{2}S(\alpha\beta)^2 + S(\alpha\beta)^3 \cdot S \cdot \alpha^2\beta^2 = \frac{3}{2}S(\alpha\beta)^2;$$

i.e.,

$$S(\alpha\beta)^2 = 0.$$

† The involutant of m , n , two matrices of order ω , is the resultant of the ω^2 scalar equations obtained by equating to zero a linear function with scalar coefficients of the ω^2 matrices which result from multiplying $1, m, m^2, \dots m^{\omega-1}$ into $1, n, n^2, \dots n^{\omega-1}$. Obviously, if the involutant of (m, n) does not vanish, these ω^2 products are linearly independent, and any matrix of order ω may be expressed linearly in terms of them. In general, the m, n involutant (i.e., the involutant of m, n) differs from the n, m involutant (i.e., the involutant of n, m).

and, since $S\beta\alpha = 0$, hence $x_0 = 0$. Moreover,

$$x_1 = S \cdot \alpha^2 (\beta\alpha) = S \cdot \alpha^3 \beta = S\beta = 0,$$

$$x_2 = S \cdot \alpha (\beta\alpha) = S \cdot \alpha^2 \beta = 0,$$

$$y_0 = S \cdot (\beta\alpha) \beta^2 = S \cdot \alpha\beta^3 = S\alpha = 0,$$

$$z_0 = S \cdot (\beta\alpha) \beta = S \cdot \alpha\beta^2 = 0,$$

$$z_1 = S \cdot \alpha (\beta\alpha) \beta = S(\alpha\beta)^2 = 0;$$

therefore $\beta\alpha = y_1\alpha\beta + y_2\alpha\beta^2 + z_1\alpha^2\beta;$

therefore $1 = (\beta\alpha)^3 = \beta\alpha (y_1\alpha\beta + y_2\alpha\beta^2 + z_1\alpha^2\beta) \beta\alpha$
 $= y_1\beta\alpha^3\beta^2\alpha + y_2\beta + z_1\alpha;$

therefore $0 = S\beta^3 = y_1S\beta^3 + y_2 + z_1S \cdot \alpha\beta^2 = y_2,$

$$0 = S\alpha^3 = y_1S_1\alpha^3 + y_2S \cdot \alpha^3\beta + z_1 = z_1;$$

therefore $\beta\alpha = y_1\alpha\beta.$

Since $(\alpha\beta)^3 = (\beta\alpha)^3 = 1$, y_1 is a cube root of unity; and since

$$(1 + y_1 + y_1^2) \alpha^3\beta = \alpha^3\beta + \alpha\beta\alpha + \beta\alpha^2 = 0,*$$

hence y_1 is an imaginary cube root of unity.

If now i, j are Sylvester's special solutions (since the i, j involutant does not vanish), we may put for α, β (the matrices of the general solution),

$$\alpha = (a_0 + a_1 i + a_2 i^2) + (b_0 + b_1 i + b_2 i^2) j + (c_0 + c_1 i + c_2 i^2) j^2,$$

$$\beta = (a'_0 + a'_1 i + a'_2 i^2) + (b'_0 + b'_1 i + b'_2 i^2) j + (c'_0 + c'_1 i + c'_2 i^2) j^2;$$

and if we write

$$\alpha^2 = (A_0 + A_1 i + A_2 i^2) + (B_0 + B_1 i + B_2 i^2) j + (C_0 + C_1 i + C_2 i^2) j^2,$$

$$\beta^2 = (A'_0 + A'_1 i + A'_2 i^2) + (B'_0 + B'_1 i + B'_2 i^2) j + (C'_0 + C'_1 i + C'_2 i^2) j^2,$$

$$\alpha\beta = (A''_0 + A''_1 i + A''_2 i^2) + (B''_0 + B''_1 i + B''_2 i^2) j + (C''_0 + C''_1 i + C''_2 i^2) j^2,$$

* This follows immediately from the form of the second equation in the catena α, β , namely,

$$\begin{aligned} 0 &= (\alpha^2\beta + \alpha\beta\alpha + \beta\alpha^2) - 3S\beta \cdot \alpha^2 - 3S\alpha \cdot (\alpha\beta + \beta\alpha) \\ &\quad + 3(3S\alpha \cdot S\beta - S \cdot \alpha\beta) \cdot \alpha + \frac{2}{3}(3S^2\alpha - S\alpha^3) \cdot \beta \\ &\quad - (\frac{2}{3}S^2\alpha \cdot S\beta - \frac{2}{3}S\beta \cdot S\alpha^2 - 9S\alpha \cdot S \cdot \alpha\beta + 3S \cdot \alpha^2\beta) \\ &= \alpha^2\beta + \alpha\beta\alpha + \beta\alpha^2. \end{aligned}$$

then the necessary and sufficient conditions expressed in terms of these coefficients that α and β shall also be solutions, are

$$\begin{aligned}
 a_0 &= S\alpha = 0, & A_0 &= S\alpha^2 = 0, \\
 b_0 &= S\beta = 0, & B_0 &= S\beta^2 = 0, \\
 a_0a'_0 + (a_1a'_2 + a'_1a_2) + (b_0c'_0 + b'_0c_0) + \lambda^2(b_1c'_2 + b'_1c_2) \\
 &+ \lambda(b_2c'_1 + b'_2c_1) = S \cdot \alpha\beta = 0, \\
 A_0a'_0 + (A_1a'_2 + A'_1A_2) + (B_0c'_0 + B'_0C_0) + \lambda^2(B_1c'_2 + b'_1C_2) \\
 &+ \lambda(B_2c'_1 + b'_2C_1) = S \cdot \alpha^2\beta = 0, \\
 a_0A'_0 + (a_1A'_2 + A'_1a_2) + (b_0C'_0 + B'_0c_0) + \lambda^2(b_1C'_2 + B'_1c_2) \\
 &+ \lambda(b_2C'_1 + B'_2c_1) = S \cdot \alpha\beta^2 = 0, \\
 a_0A_0 + (a_1A_2 + a_2A_1) + (b_0C_0 + B_0c_0) + \lambda^2(b_1C_2 + B_1c_2) \\
 &+ \lambda(b_2C_1 + B_2c_1) = S \cdot \alpha^3 = 1, \\
 a'_0A'_0 + (a'_1A'_2 + a'_2A'_1) + (b'_0C'_0 + B'_0c'_0) + \lambda^2(b'_1C'_2 + B'_1c'_2) \\
 &+ \lambda(b'_2C'_1 + B'_2c'_1) = S \cdot \beta^3 = 1, \\
 A''_0A''_0 + 2A''_1A''_2 + 2B''_0C''_0 + 2\lambda^2B''_1C''_2 + \lambda B''_2C''_1 = S(\alpha\beta)^2 = 0.
 \end{aligned}$$

If the latent function of the corpus m, n is

Det. $(x + ym + zn)$

$$\begin{aligned}
 &= x^3 + 3Bx^2y + 3Ox^2z + 3Dxy^2 + 6Exyz + 3Fxx^2 + Gy^3 + 3Hy^2z \\
 &\quad + 3Kyz^2 + Lz^3,
 \end{aligned}$$

then

$$\begin{aligned}
 B &= Sm, & O &= Sn, \\
 D &= \frac{1}{2}(3S^2m - Sm^2), & E &= \frac{1}{2}(3Sm \cdot Sn - S \cdot mn), \\
 F &= \frac{1}{2}(3S^2m - Sn^2), & G &= \frac{2}{3}S^3m - \frac{2}{3}Sm \cdot Sm^2 + Sm^3, \\
 H &= \frac{2}{3}S^3m \cdot Sn - \frac{2}{3}Sn \cdot Sm^2 - 3Sm \cdot S \cdot mn + S \cdot m^2n, \\
 K &= \frac{2}{3}Sm \cdot S^2n - 3Sn \cdot S \cdot mn - \frac{2}{3}Sm \cdot Sn^2 + S \cdot mn^2, \\
 L &= \frac{2}{3}S^3n - \frac{2}{3}Sn \cdot Sn^2 + Sn^3.
 \end{aligned}$$

As given by Sylvester, the first 9 conditions are

$$B = 0, \quad O = 0, \quad D = 0, \quad E = 0, \quad F = 0, \quad H = 0, \quad K = 0,$$

and

$$G = 1, \quad L = 1.$$

This gives

$$\text{Det. } (x + ym + zn) = x^3 + y^3 + z^3.$$

It is evident, from the above values of the coefficients of the latent function, that these conditions are equivalent to 9 of the conditions as given by me. Sylvester's 10th condition is that the coefficient of mn in the identical equation of mn shall vanish. This coefficient is, however, $\frac{3}{2} [3S^2 \cdot mn - S(mn)^2]$; and, as $S \cdot mn = 0$ (which follows from $B = 0, C = 0, E = 0$), Sylvester's 10th condition reduces to my 10th condition, namely,

$$S \cdot (mn)^2 = 0.$$

4. *Expression for the involutants* of two matrices in terms of the sums of the latent roots of products of powers of the matrices.*

For simplicity, I consider the involutants of the two quaternions q and r ; but the form of the involutants for matrices of higher order is at once evident, and the same demonstration applies in any case. If the q, r involutant vanishes, then

$$Q \equiv (a + bq) + (c + dq)r = 0,$$

for some value of the scalars a, b, c, d . Therefore

$$\left. \begin{aligned} SQ = 0, \quad S \cdot qQ = 0 \\ S \cdot rQ = 0, \quad S \cdot qrQ = 0 \end{aligned} \right\} \dots\dots\dots (A).$$

The resultant of equations (A) is

$$I \equiv \begin{vmatrix} 1, & Sq, & Sr, & S \cdot qr \\ Sq, & Sq^2, & S \cdot qr, & S \cdot q(qr) \\ Sr, & S \cdot rq, & Sr^2, & S \cdot r(qr) \\ S \cdot qr, & S \cdot (qr)q, & S \cdot (qr)r, & S \cdot (qr)^2 \end{vmatrix}.$$

I is, of course, a multiple of the q, r involutant.

It is readily seen that the equations (A) do not all coexist, unless $Q = 0$. For, otherwise, from equations (A), we obtain

$$S [\{x + yq + zr + w(qr)\} Q] \equiv 0$$

* See note †, p. 73.

for any values of the scalars x, y, z, w ; and, if $Q \neq 0$, $x + yq + zr + w(qr)$ may be any quaternion; hence

$$S(RQ) = 0$$

for any quaternion R , which is impossible unless

$$Q = 0.$$

For let the two quaternions i, j be such that

$$i^2 = j^2 = 1, \quad ji = -ij;$$

then the involutant of i, j does not vanish.* Let the expression for Q in terms of $(1, i, j, 1, j)$ be

$$Q = (A + Bi) + (C + Di)j;$$

$$\text{then } A = SQ = 0, \quad B = S.iQ = 0,$$

$$C = S.Qj = S.jQ = 0, \quad D = S.iQj = S(ji)Q = 0.$$

Hence I vanishes, if and only if $Q = 0$; and consequently I is some power of the involutant of q, r .

The involutant of r, q is similarly the resultant of the equations

$$SQ' = 0, \quad S.rQ' = 0, \quad S.qQ' = 0, \quad S.rqQ' = 0,$$

$$\text{where } Q' = (a' + b'r) + (c' + d'r)q,$$

a', b', c', d' being scalars. If J denotes this resultant, then

$$J \equiv \begin{vmatrix} 1, & Sr, & Sq, & S.qr \\ Sr, & Sr^2, & S.rq, & S.r(qr) \\ Sq, & S.qr, & Sq^2, & S.q(qr) \\ S.rq, & S.(rq)r, & S.(rq)q, & S.(qr)^2 \end{vmatrix}.$$

* The same method employed in § 3 to show that the involutant of i, j does not vanish for the two nonions i and j , for which

$$Si = Si^2 = 0, \quad Sj = Sj^2 = 0, \quad S.ij = S.i^2j = S.j^2i = 0,$$

may be applied to show that the i, j involutant of two matrices i, j of order ω does not vanish, if

$$S.i^p j^q = 0,$$

for all values of p and q from 0 to $\omega - 1$, except $p = q = 0$.

78 Mr. Henry Taber on the Application to Matrices, &c. [Dec. 11,

And, since $S.qr = S.rq$, $S.r(qr) = S.r(rq)$, &c.,

hence $I = J$.

It is of interest to observe that the involutant of the two quaternions vectors α, β is

$$\{S(\alpha\beta)^2 - S^2.\alpha\beta\}(\alpha^2\beta^2 - S^2.\alpha\beta) = V^4.\alpha\beta = T^4V.\alpha\beta.$$

It is, of course, otherwise evident that the involutant must be some multiple of a power of $TV\alpha\beta$.

The m, n involutant of the two nonions m and n is the resultant of the equations

$$\begin{aligned} SM &= 0, & S.mM &= 0, & S.m^2M &= 0, \\ S.nM &= 0, & S.mnM &= 0, & S.m^2nM &= 0, \\ S.n^2M &= 0, & S.mn^2M &= 0, & S.m^2n^2M &= 0, \end{aligned}$$

where $M = (a_0, a_1, a_2, \dots a_8 \text{ } \text{ } 1, m, m^2 \text{ } 1, n, n^2)$;

and the n, m involutant is the resultant of the equations

$$\begin{aligned} SM' &= 0, & S.mM' &= 0, & S.m^2M' &= 0, \\ S.nM' &= 0, & S.nmM' &= 0, & S.nm^2M' &= 0, \\ S.n^2M' &= 0, & S.n^2mM' &= 0, & S.n^2m^2M' &= 0, \end{aligned}$$

where $M' = (a'_0, a'_1, a'_2, \dots a'_8 \text{ } 1, n, n^2 \text{ } 1, m, m^2)$.

Both these involutants are also symmetric matrices, and so in general. In general, the m, n involutant I of two matrices, m and n , of the third or higher order, is not equal to the n, m involutant J of the matrices. For the two nonions m and n , I find

$$S(m^2n^2mn) = S(n^2m^2nm)$$

to be a sufficient condition that $I = J$; for, then, on comparing the determinants I and J , it will be found that every constituent of the one is equal to the corresponding constituent of the other.

5. Proof of the theorem that the latent roots of mn and nm are identical.

This theorem has been given by Sylvester (*Johns Hopkins Circs.*, Vol. III., p. 8) for matrices in general, and earlier by Tait (*Elements*,

Chap. v., p. 103), and probably by Hamilton, for matrices of the third order.

It is, of course, only necessary to show that the sum of the k^{th} powers of the latent roots of mn [*i.e.*, the sum of the latent roots of $(mn)^k$] is equal to the sum of the k^{th} powers of the latent roots of nm [*i.e.*, the sum of the latent roots of $(nm)^k$]. But

$$S(mn)^k = S.m(nm)^{k-1}n = S.nm(nm)^{k-1} = S(nm)^k.$$

Note.—By an oversight the word “involutant” has been used above in § 4 for “square of the involutant.” Thus, the above resultants I and J are the *squares* of the corresponding involutants.

On the Reversion of Partial Differential Expressions with two Independent and two Dependent Variables. By E. B. ELLIOTT.

[Read Dec. 11th, 1890.]

1. The theory of the interchange of the dependent and independent variables, in functions of the derivatives of one variable with regard to another of which it is by supposition a function, has of late years been studied with great elaboration on the basis of Prof. Sylvester's new discovery of reciprocants. The last five volumes of our *Proceedings* abound in valuable contributions to the study.

Mr. A. Berry, in his paper on “Simultaneous Reciprocants” (*Quarterly Journal*, 1888, p. 260), has initiated an analogous theory as to the reversion, by interchange of x and y and of x' and y' , of functions of the two sets of ordinary derivatives

$$\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, \dots, \frac{dy'}{dx'}, \frac{d^2y'}{dx'^2}, \frac{d^3y'}{dx'^3}, \dots;$$

his supposition being that x, y are connected by one relation, and x', y' by another.

It would appear that the theory of the interchange of the dependent and independent pairs in functions of the partial derivatives of two