In this case, then, $u_8$ breaks up into the square of the line $d$—the tangent to U at the point $(x_1 y_1 z_1)$—and the sextic

$$v_6 = 8b^2 - 9ac,$$

while, correspondingly, $u_6$ breaks up into the line $d$ and the quintic

$$v_5 = 2bc - 3ad,$$

which may be verified to be the first polar of $v_6$. For since

$$D_1 a = 4b, \quad D_1 b = 3c, \quad D_1 c = 2d, \quad D_1 d = e,$$

$$D_1 v_6 = 16bD_1 b - 9aD_1 c - 9cD_1 a$$

$$= 48bc - 18ab - 36bc = 6(2bc - 3ad) = 6v_5.$$

Again, $$D_1 v_5 = 2bD_1 c + 2cD_1 b - 3aD_1 d - 3dD_1 a$$

$$= 4bd + bc^2 - 12bd = 2(3c^2 - 4bd) = 2u_4;$$

so that $v_6$, $v_5$ have $(x_1 y_1 z_1)$ as a common quadruple point and four common tangents at that point; consequently they meet again in ten points only. The ten tangents which can be drawn from $(x_1 y_1 z_1)$ to U or $a$ touch $v_6$ at the same points, as is evident from the form of $v_6$; and being drawn from a quadruple point do not meet that curve again.

Prof. Cayley made a few remarks on Mr. Walker's paper, in the course of which he suggested the alteration subsequently adopted by the Author.

Mr. S. Roberts then read a paper

## *On the Order and Singularities of the Parallel of an Algebraical Curve.*

1. A Parallel of a curve is variously defined as the envelope of a circle of constant radius which touches the given curve, or has its centre thereon; and, as the locus of the centre of a circle of constant radius which touches the given curve.

The most obvious property of a Parallel is, that among the normal distances of any point on it from the primitive curve, there is always one of given length, which I call the Modulus of the Parallel. Moreover, if at any point on the primitive we draw a normal, two points will be determined on the parallel by taking upon the normal lengths equal to the Modulus in opposite directions from the point; and the normals to the Parallel at these points coincide with the corresponding normal of the Primitive.

The parallel, therefore, has the same normals as the primitive, but each is normal at two points. The latter curve is a parallel to itself to the modulus 0, but we shall have reason to remark that as the normal distances which determine points on the parallel are measured in two directions, on opposite sides of the corresponding branch of the primitive, this, as parallel to itself, must be reckoned twice.

Since the primitive is included in the family of parallels, they may be, in some points of view, preferably described as the system of involutes having a given evolute.

The variation of the modulus may affect the characteristics as well as the general shape and dimensions of the parallel. Hence the results which we shall be able to obtain must, in general, be held to refer to a parallel whose modulus is general.

2. Let $\phi(x, y) = 0$ be the equation of the primitive. The envelope of the circles $(x-\alpha)^2+(y-\beta)^2 = r^2, \quad \phi(\alpha, \beta) = 0$ ...............(a) is a parallel to the modulus $r$. Disregarding mere position, we may say that the envelope of a circle translated without rotation, so that a point of it describes the curve $\phi(x, y) = 0$ will be parallel to the curve with the radius of the circle as modulus.

If we write $\alpha'$ for $x-\alpha$, $\beta'$ for $y-\beta$, we get the system of curves,

$$\phi(x-\alpha', \; y-\beta') = 0,$$
$$\alpha'^2+\beta'^2-r^2 = 0,$$

and the envelope of these is the same parallel as before. Disregarding mere position, we are able to say, that if a curve $\phi(x, y) = 0$ is translated without rotation, so that a point of it (or indeed of its plane considered as moving with it) describes a circle whose radius is $r$, the envelope of the translated curve will be a parallel to the modulus $r$.

Thus if a curve be traced on the card of a mariner's compass, (the needle as usual being attached to the card,) and if the compass be moved with its centre on a circle and without oscillation, the envelope of the traced curve will be a parallel of it.

It will be observed that the name is now used to denote a curve which may be so placed as to become a parallel in the limited sense. When we are discussing the characteristics of the family of curves, mere position is immaterial.

3. It is instructive to consider a more general case. Let $\phi(x, y) = 0$, $\psi(x, y) = 0$ represent any two curves $\phi$, $\psi$; then if $\phi$ be translated without rotation so that a point rigidly connected with it describes $\psi$, the envelope of the translated curve will be the same as if the offices of the curves were interchanged. For, in the first case, we have the system of curves $\phi(x-\alpha, \; y-\beta) = 0, \quad \psi(\alpha, \beta) = 0,$ and writing $\alpha'$ for $x-\alpha$, $\beta'$ for $y-\beta$, we get the system

$$\psi(x-\alpha', \; y-\beta') = 0, \quad \phi(\alpha', \beta') = 0;$$

the envelope is the same whichever system be taken.

Translation without rotation implies that the curve translated continues to pass through the same points at infinity, and a system of parallel tangents remains similarly directed. The points at infinity of the envelope are (with higher multiplicity) the points at infinity of the two curves. Let $m$, $n$ be the order and class of $\phi$; $m'$, $n'$ those of $\psi$. If we consider the generation of the envelope by intersections, suppose, of $\phi$ in successive positions as it is translated along a real branch of $\psi$, passing off to infinity, we have ultimately the curve $\phi$ sliding along its tangents parallel to the asymptote of the branch, while the distances between the points of contact of the tangents tend to vanish because all points at a finite distance from each other coalesce at infinity. The point at infinity in question becomes consequently a multiple point in the envelope of the order $n$. Hence the points where $\psi$ meets the line at infinity are multiple points of the order $n$, and similarly the points where $\phi$ meets the line at infinity are multiple points of the order $n'$, and the order of the envelope is in general $m'n+mn'$. The principle of continuity allows us to extend this conclusion to cases in which the points at infinity are imaginary.

4. As a particular case of the foregoing, the order of the parallel of $\phi$ is $2m+2n$; the points at infinity of $\phi$ are double points on the parallel, and the circular points are multiple points of the order $n$.

But from what precedes, we see at once that the line at infinity and the circular points have a special influence on the envelope. If the primitive touches the line at infinity, this line becomes part of the envelope, and the order is reduced. A reduction also takes place when the primitive passes through the circular points.

The parallel of a conic generally is of the 8th degree, that of a parabola is of the 6th. Hence the reduction is 2. And it may be readily inferred that for $p$ simple contacts, the reduction of the order is $2p$. Again, the parallel of a circle is of the 4th degree, and the reduction is 4; and hence we may infer, generally, that for ordinary multiplicity of the order $p$ at the circular points, the reduction is $4p$. Again, if there is a double point (or cusp) on the primitive, a circle, whose radius is the modulus described about the point as centre, satisfies the geometrical conditions.

The reduction for a double point is 4, and for a cusp 6, as appears from the expression for the order $2m+2n$.

(5.) From the form of the equations ($a$) it appears that if we make the first equation homogeneous in $x$, $y$, $z$, by introducing $z$ for unity, $z^2$ and $r^2$ enter the equation of the envelope together; and, consequently, if we make $r^2 = 0$, the remaining terms will be of the same order as the general equation, *i.e.*, will contain the highest terms in $x$, $y$. The order of the residuum can be determined without much difficulty.

In the first place, we have, as factor, the square of the function of the primitive, for we have seen this is the function of the parallel when the modulus is evanescent. Secondly, we have the factors which represent the circular asymptote through each of the real ordinary foci. For if $a$, $b$ are the coordinates of a focus, the normal distance from

$$(x-a)^2+(y-b)^2 = 0$$

satisfies the conditions, being evanescent. There are, in general, $n$ real ordinary foci, and the equation of the parallel is of the form

$$\chi_{2m+2n-2}\,r^2+\{\phi(x,y)\}^2\{(x-a_1)^2+(y-b_1)^2\}\ldots\{(x-a_n)^2+(y-b_n)\}=0.$$

If the primitive has $p$ simple contacts with the line at infinity, we have seen the reduction is $2p$. In this case, $p$ foci pass off to infinity, and the part of the equation of the parallel independent of $r^2$ remains of the same form, there being, however, only $n-p$ focal factors.

Again, if the primitive passes simply through the circular points, and $a$, $b$ are the coordinates of the corresponding double focus, the equation of the general parallel will be divisible by

$$\{(x-a)^2+(y-b)^2\}^2;$$

for the normal distance of the double focus may be taken as $r$, since for the circular points we have

$$(x-a)^2+(y-b)^2=r^2.$$

The factor will be squared, because the focus is a double focus, *i.e.*, takes the place of two ordinary foci.

In like manner, if the primitive has $\mu$ foci due to tangents at the circular points and taking the place of $\nu$ ordinary foci, we have a reduction of the order of the parallel by $2\nu$. In this case also, if we make $r^2=0$ after removing the factors due to the foci formed by tangents at the circular points, the residuum remains of the same form containing the function of the primitive squared and the ordinary focal factors.

Thus it appears that the order of the parallel generally is $2m+2M$ where M is the number of real ordinary foci.

Suppose that the primitive passes through the circular points which are multiple points of the order $p$ with $s$ coincident tangents.

A multiple point of the order $p$ with $s$ coincident tangents is equivalent to $\frac{p(p-1)}{2}-s+1$ double points, and $s-1$ cusps. The number of tangents which can be drawn from such a point is

$$m^2-m-2\left(\delta-\frac{p(p-1)}{2}+s-1\right)-3(\kappa-s+1)-p(p+1);$$

$\delta$, $\kappa$ representing the numbers of the double points and cusps, or their equivalents on the primitive.

The above expression is, in fact,

$$n-2p+s-1.$$

The reduction of the order of the parallel is $2(2p-s+1)$.

Again, if the circular points are double points on the primitive, and the line at infinity is a common tangent, we can draw from each circular point $\quad m^2-m-2(\delta-1)-3\kappa-2\,.\,3-1$

tangents. Hence the order of the parallel is

$$2m+2n-10.$$

If the circular points are cusps, and the line at infinity is the common tangent, we have, for the number of tangents which can be drawn, as before, $m^2-m-2\delta-3(\kappa-1)-2.3-1$,
and the order of the parallel is $2m+2n-8$.

6. Practically, when a primitive is given, we can ascertain the number and position of real double points and cusps, by determining the positions in which a circle whose radius is the modulus will doubly touch or osculate the curve. Thus we see that a parallel of an ellipse can be made to show two real double points and four real cusps. A parallel of a parabola can be drawn to show one real double point and two real cusps, and that of a hyperbola may show four real double points and four real cusps.

An easy way is suggested by the foregoing remark of determining the number of cusps on the parallel of a curve without singularities, and without special relation to the line at infinity. We have, in fact, simply the conditions [writing $\phi$ for $\phi(x, y)$],

$$\phi=0, \quad (n-1)^4\left\{\left(\frac{d\phi}{dx}\right)^2+\left(\frac{d\phi}{dy}\right)^2\right\}^3-r^2\mathrm{H}^2=0,$$

where H represents the Hessian. The latter equation is obtained immediately from the expression for the radius of curvature.

The number of intersections of these curves is

$$6m(m-1)=6n.$$

Knowing, then, the order $2n+2m$, the class $2n$, and the cusps $6n$, in this case, the other characteristics follow from Plücker's equations.

For such a primitive, and even for one more general, having contacts with the line at infinity, we may use a still more simple consideration.

There being no double points or cusps on the primitive, the only way in which points of inflexion can arise on the parallel will be by means of the points of inflexion on the primitive; and evidently to each such point will correspond two on the parallel. Hence the number of inflexions on the parallel is $2i$, $i$ being the corresponding number for the primitive. And this reasoning probably holds good when the primitive passes simply through the circular points.

When the primitive has $n$ real ordinary foci, the circular points are, as we have seen, multiple points of the order $n$ on the parallel. There are consequently no real ordinary finite foci on the general parallel. This indeed appears to be true always for a general modulus, and might be anticipated from the manner in which focal factors enter the equation of a parallel.

When the primitive is compound, the parallel manifestly breaks up into the parallels of the compound curves. The parallel of the modulus $s$ of a parallel to the modulus $r$ consists of a compound of parallels. In fact, since parallels of a parallel are parallels, and the general parallel is of determinate order, the result of repetitions of the generation must be compounded of parallels to various moduli.

The result of taking parallels of parallels, where certain simple relations exist between the moduli, can be readily traced. Thus a curve may become part parallel to itself for other than evanescent values of the modulus.

7. In order to obtain the general characteristics of a parallel, I shall now make use of the substitution $\frac{1}{\rho}+r$ for $\frac{1}{\rho}$ in the reciprocal of the primitive, which substitution gives the reciprocal of the parallel. (Salmon's Higher Plane Curves, p. 273.)

Let a point curve of the degree $n$ be represented by the equation

$$(n)\,\rho^n+(n-1)\,\rho^{n-1}+\dots+(2)\,\rho^2+(1)\,\rho+(0)=0 \;\dots\dots\dots\; (b),$$

where $\rho$ is the radius vector; (0), (1), (2) represent A, $\mathrm{B}\cos\theta+\mathrm{C}\sin b$, $\mathrm{D}\cos^2\theta+\mathrm{E}\cos\theta\sin\theta+\mathrm{F}\sin^2\theta$; and generally $(s)$ stands for a homogeneous function of $\cos\theta$, $\sin\theta$ of the order $s$.

The substitution in question gives, after multiplication by $\rho^n$,

$$(n)\,\rho^n+(n-1)\,\rho^{n-1}(1+r\rho)+\dots+(2)\,\rho^2(1+r\rho)^{n-2}+(1)\,\rho\,(1-r\rho)^{n-1}$$
$$+(0)\,(1+r\rho)^n=0 \;\dots\dots\dots\dots\; (c).$$

If $\chi\,(x, y, 1)$ is the equivalent of $(b)$, in $x, y$ coordinates, the above result is

$$\chi\,(x, y,\ 1+r\rho)=0 \;\dots\dots\dots\dots\dots\dots\dots\dots\; (d).$$

And since the value of $\rho$ is radical, we have also

$$\chi\,(x, y,\ 1-r\rho)=0 \;\dots\dots\dots\dots\dots\dots\dots\dots\; (e).$$

The product of $(d)$, $(e)$ is the reciprocal of the parallel of the reciprocal of $(b)$, or its tangential equation, if we choose so to regard it.

Writing $\chi$ for $\chi\,(x, y, z)$ where $z$ represents unity, we have the form

$$\left\{\chi+\frac{d^2\chi}{dz^2}\frac{r^2\rho^2}{1.2}+\frac{d^4\chi}{dz^4}\frac{r^4\rho^4}{1.2.3.4}+\dots\right\}^2$$
$$-\,r^2\rho^2\left\{\frac{d\chi}{dz}+\frac{d^3\chi}{dz^3}\frac{r^2\rho^2}{1.2.3}+\dots\right\}^2=0 \;\dots\dots\; (f).$$

As long as $\rho^2$ does not vanish for a singular point,—and in the general case we may make this supposition,—a double point on either of the part curves $(d)$, $(e)$ implies a corresponding singular point of the same nature on the primitive. If, therefore, the curve $(b)$ has no singularities, the singularities of $(f)$ will consist of the intersections of

$$\chi+\frac{d^2\chi}{dz^2}r^2\rho^2+\dots=0,\ \text{ and }\ \frac{d\chi}{dz}+\frac{d^3\chi}{dz^3}\frac{r^2\rho^2}{1.2.3}+\dots=0.$$

The curve $(f)$ is in this case of the degree $2n$, and has $n\,(n-1)$ double points. We are, therefore, in a position to determine the characteristics of the parallel of a curve of the degree $n\,(n-1)$ and class $n$, having no special relation to the line at infinity.

8. Let us now suppose that the curve $(b)$ has $\alpha$ double points and $\beta$ cusps, we shall have in general $2\alpha$ double points and $2\beta$ cusps in the

transformed curve; for to each double point or cusp $x_1$, $y_1$, in (*b*), correspond two double points or cusps $\frac{x}{1 \pm r\rho} = x_1$, $\frac{y}{1 \pm r\rho} = y_1$, in (*f*).

It will be seen presently that, in special cases, there may be a modification of the characteristics, which makes it convenient to add $\epsilon$ to the number of double points, and $\eta$ to the number of cusps in the general formulæ. For $2\epsilon + 3\eta$ I write Q, and

M for the order,
N „ class,
D „ number of double points,
C „ „ „ cusps,
I „ „ stationary tangents,
T „ „ double tangents.

It is to be understood, however, that we must expect only the equivalents of these elements. For instance, generally, the circular points at infinity are multiple points of this order $\frac{M-2q}{2}$, where $q$ is the number of points, not circular points, at infinity on the primitive. It would, I think, be convenient to adopt a less definite phrase than "number of double points," &c., such as "nodal characteristic," "cuspidal characteristic," &c. I shall also call $\delta + \kappa$ the *multiplicity*, and $2\delta + 3\kappa$ the *distribution*. This phraseology may be used with reference to an isolated singularity, or to the sum total of the singularities, of a curve. In the latter case, we have

$$\frac{(m-1)\,(m-2)}{2} - \text{multiplicity} = \text{deficiency},$$

a term introduced, I think, by Prof. Cayley.

Let the primitive have $\delta$ double points, and $\kappa$ cusps, then for the parallel we have

$$\begin{aligned}
N &= 2n,^* \\
I &= 6n - 6m + 2\kappa + \eta, \\
T &= (n-m)(n+m-9) + 2\delta + n\,(n-1) + \epsilon, \\
M &= 2m + 2n - Q, \\
C &= 6n + 2\kappa - 3Q + \eta, \\
D &= 2(m+n)^2 - 11(m+n) + 10m - 3\kappa \\
&\quad + \frac{Q^2}{2} - Q\left(2m + 2n - \frac{9}{2}\right) + \epsilon.
\end{aligned}$$

If $i$, $t$ are the numbers of stationary tangents and double tangents on the primitive, we have

$$\begin{aligned}
I &= 2i + \eta, \\
T &= 2t + n\,(n-1) + \epsilon.
\end{aligned}$$

* The value of the class follows at once from the consideration, that for each of any set of parallel tangents to the primitive we must have two tangents to the parallel similarly directed.

When the primitive has no special relation to the line at infinity,

$$\eta = \epsilon = Q = 0.$$

It appears, then, that I remains equal to $2i$ when there are singular points on the primitive; but this is not so manifest, *à priori*, as to give the characteristic at once.*

9. We have now to consider cases in which a singularity of the form ($f$) satisfies $\rho^2 = 0$. These cases relate to passages through the circular points and contact with the line at infinity.

Let $u_s$, $w_s$, $p_s$ represent homogenous functions of $x$, $y$ of the order $s$. The form $\{u_n + u_{n-1} + \dots + u_p\}^2 - p_1 q_1 \{w_{n-1} + w_{n-2} + \dots w_{p-1}\} = 0$ represents a curve, having at the origin a multiple point of the order $2p$, and $n(n-1) - p(p-1)$ other double points. Removing $w_{p-1}$ altogether, we make $p$ of the double points adjacent to the origin, which remains a multiple point of the order $2p$. If, again, we make $w_p$ identical with $u_p$, except as to a constant factor, we bring another set of $p$ double points into contiguity with the origin. We have, therefore, the equivalent of

$$n(n-1) - p(p-1) + \frac{2p(2p-1)}{2} = n(n-1) + p(p-1) + p$$

double points. Or we may get the same result thus:—If, in the form

$$(u_n + u_{n-1} + \dots u_p)^2 - p_1 q_1 (w_{n-1} + \dots + w_{p+1} + u_p)^2 = 0 \quad \dots\dots\dots (g),$$

we suppose $p_1$, $q_1$ to coincide, we have the equivalent of $n^2$ intersections of two component curves of the order $n$; and we add $n-p$ points to the multiplicity, since $p_1$, $q_1$ pass through the origin which is of the order $p$ on $u_n + \dots = 0$, $w_{n-1} + \dots = 0$.

Let $\Delta$ be the multiplicity of the form ($g$), then we have

$$\Delta + n - p = n^2 - p^2 + \frac{2p(2p-1)}{2},$$

$$\Delta = n(n-1) + p(p-1) + p.$$

10. If the primitive has $p$ simple contacts with the line at infinity, the origin is a multiple point of the order $p$ on the reciprocal. The transformed curve becomes of the form

$$(u_n + u_{n-1} + \dots + u_p)^2 - r^2 \rho^2 (w_{n-1} + \dots + u_p)^2 = 0 \dots\dots\dots\dots (g').$$

Hence there are due to the tangency $p(p-1) + p$ double points, of which $p(p-1)$ are due to the multiple tangent as such, and the re-

* Since the present paper was read before the Society, I have had the advantage of comparing my formulæ with those of Professor Cayley, given in the sequel of his paper on Evolutes and Parallel Curves (April number of the "Quarterly Journal of Mathematics"). Our corresponding results seem to be generally accordant. As to the point referred to in the text, Professor Cayley had concluded that $I = 2i$ in all cases. On my drawing his attention to the case of the tricusped hypocycloid, the parallel of which does not break up as it ought if the above equation held good, he recognised that the assumption is too general. It follows that the parallels of curves having no foci do not necessarily break up.

N.B.: Pages 217 and 218 reversed in reprinting; please turn next page.

For simplicity, let $p = 2$, $n = 4$; the reciprocal of the primitive will be of the form $\rho^2(u_2+u_1+u_0)+(w_2+w_1+w_0)^2 = 0$,

and the transformed equation is of the form

$$\{\rho^2\chi_2+(w_2+w_1+w_0)^2\}^2+r^2\rho^2\{\rho^2\xi_1+2(w_1+2w_0)(w_2+w_1+w_0)\}^2 = 0.$$

By the preceding article, we have two additional double points on $x+\sqrt{-1}\,y = 0$ and two on $x-\sqrt{-1}\,y = 0$; so that, in the formulæ of (8), we have $\eta=0$, $\epsilon=4$, $Q=8$. Generally, for $p$ passages through the circular points, we have $\eta=0$, $\epsilon=2p$, $Q=4p$.

13. Let the circular points be cusps on the primitive. For simplicity, let $n=3$; the reciprocal may be written in the form

$$\rho^2\chi_1+Z^3 = 0,$$

and the transformed equation is

$$(\rho^2\xi_1+Z^3)^2-r^2\rho^2(k\rho^2+lZ^2)^2 = 0.$$

We have, for double points or cusps on $\rho^2=0$,

$$\rho^2 = 0,\quad Z^3 = 0;$$

but the form gives on the face of it 3 . 2 double points including the double points $\rho^2 = 0,\quad Z^2 = 0.$

We have, therefore, for the multiplicity of the addition,

$$\Delta+K = 2.$$

But we know, by (5), that the distribution is

$$2\Delta+3K = 6.$$

Hence $K = 2,\quad \Delta = 0.$

In the formulæ of (8), we must write $\eta=2$, $\epsilon=0$, $Q=6$. We may generalize this equation by means of a compound primitive.

14. If the primitive touches the line at infinity at the circular points, the origin will be a double point on the reciprocal, and the circular asymptotes through the origin will be its tangents. For to the line at infinity corresponds the origin, and to the circular points correspond the circular asymptotes through the origin. Hence we have to consider the form $(u_n+u_{n-1}+\ldots+\rho^2)^2-r^2\rho^2(w_{n-1}+\ldots+\rho^2)^2 = 0.$

This is a particular case of the form ($g'$) where $p=2$, restricted by the coincidence of $u_2$ with $\rho^2$.

The multiplicity of the addition is 2, or

$$\Delta+K = 2.$$

But we know, by (5), that the distribution is $2\Delta+3K = 6$, so that $K = 2,\quad \Delta = 0.$

We must write, in the formulæ of (8), $\eta=2$, $\epsilon=0$, $Q=6$.

The difficulty of the subject rapidly increases as the relation of the primitive to the circular points becomes complex. These points, for instance, may be cusps with a common tangent altogether at infinity, mainder $p$ are due to the position of the tangent. Hence, in the formulæ of (8), we must write $\eta = 0$, $\epsilon = p$, $Q = 2p$.

If some of the contacts coalesce, the effect of the tangency on the characteristics remains the same.

For suppose $u_p$ is of the form $v_1^s u_{p-s}$. We have at the origin, in each of the intersecting curves $u_n + \ldots = 0$, $w_{n-1} + \ldots = 0$, the equivalent of $(s-1)$ cusps, and on the curve $(g')$ the equivalent of $2(s-1)$ cusps, but these are due to the multiple tangent as such, so that the addition on account of its position is unaltered.

The fact that multiple points of the primitive lie at infinity does not of itself alter the characteristics. But if a double point is at infinity, and one of its tangents coincides with the line at infinity, the origin will be a point on the reciprocal curve. We shall have to apply the same reasoning as in the case of a simple contact with the line at infinity. The reduction of the order will be 2; and in like manner the reduction on account of $p$ such tangencies will be $2p$. If there is a multiple point at infinity of the order $p$, with $s$ tangents altogether at infinity, the reducing effect of the tangency is separate and equal to $s$ simple tangencies.

11. Let the form

$$(xy\phi_{n-2} + \psi_p^2 \chi_{n-2p})^2 + xy\,\{xy\xi_{n-3} + \psi_p \zeta_{n-p-1}\}^2 = 0 \;\ldots\ldots\ldots\ldots (h),$$

or

$$\Phi^2 + xy\Psi^2 = 0,$$

be homogeneous in $x$, $y$, $z$.

For double points we have

$$2\Phi\frac{d\Phi}{dx} + y\Psi^2 + 2xy\Psi\frac{d\Psi}{dx} = 0,$$

$$2\Phi\frac{d\Phi}{dy} + x\Psi^2 + 2xy\Psi\frac{d\Psi}{dy} = 0,$$

$$2\Phi\frac{d\Phi}{dz} + 2xy\Psi\frac{d\Psi}{dz} = 0.$$

We have, therefore, upon $xy = 0$, the double points

$$x = 0, \quad \psi_p^2 = 0,$$
$$y = 0, \quad \psi_p^2 = 0.$$

On the face of the form $(h)$ and independently of the factor $xy$, we have $n(n-1)$ double points, including the systems

$$\left.\begin{array}{l} x = 0, \quad \psi_p = 0 \\ y = 0, \quad \psi_p = 0 \end{array}\right\} \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (k).$$

Consequently, by virtue of the factor $xy$, we have $p$ double points on $x = 0$ and $p$ double points on $y = 0$ adjacent to the double points $(k)$.

12. If the circular points at infinity are multiple points of the order $p$ on the primitive, we have the circular asymptotes through the origin multiple tangents to the reciprocal touching at $p$ points.

&c. I have not been able to satisfy myself as to the effect of such singularities generally.*

15. As a partial verification of the formulæ of (8), we will consider the parallel of the parallel of a primitive with $\delta$ double points and $\kappa$ cusps, and having no special relation to the line at infinity.

We have seen that, the circular points being multiple points of the order $n$ on the first parallel, give rise to $2n$ additional double points on the reciprocal of the second parallel. Making $\eta=0$, $\epsilon=2n$, $Q=4n$, and otherwise appropriately substituting in the formulæ of (8), we have

$$M = 4(m+n),$$
$$N = 4n,$$
$$D = 4(m+n)^2+2\{2(m+n)^2-11(m+n)+10m-3\kappa\}.$$

The term $4(m+n)^2$ in the last expression is due to the intersections of the two parallels of which the second parallel is compounded. The numbers of the other characteristics are, as they should be, the numbers of the characteristics of the first parallel doubled.

16. I give some results relative to the parallels of a few of the simpler curves :—

The parallel of a conic, generally, is of the 8th degree, with 8 double points and 12 cusps.

The parallel of a parabola is of the 6th degree, with 4 double points and 6 cusps. It is unicursal.

The parallel of a tricusped circular quartic is of the degree 10, with 24 double points and 12 cusps. It is unicursal.

The parallel of a cardioid is of the degree 8, with 13 double points and 8 cusps. It is unicursal.

The parallel of a bicircular quartic with a cusp is of the degree 10, with 28 double points and 8 cusps. It is unicursal.

The parallel of a tricusped hypocycloid is of the degree 8, with 13 double points and 8 cusps. It is unicursal.

In tracing parallels, it is well to observe that the curve will not always exhibit its full degree. Manifestly, a small modulus makes the form of the parallel approach to that of the primitive doubled. Attention must be paid to this, if it is desired to exhibit all the real singularities of which a parallel is susceptible.

Whilst giving his account of the methods employed in the above paper, Mr. Roberts expressed his desire to record a construction which he had omitted to see at the proper moment, though at different times he had noted two conclusions which, when brought together, make it obvious enough. Stated as a theorem, it is as follows :—In a plane, if

* The difficulty arises from the circumstance that $I=2i$ does not necessarily hold in these cases. Otherwise, knowing the order, the class, and the number of inflexions, we should have the other singularities by Plücker's equations. Even in the simple cases of §§ 13, 14, I give the results with diffidence.

a limited straight line whose length is equal to the distance between the centres of two equal circles, moves with an extremity on each, the locus of any point rigidly connected with the line will consist of a circle and a bicircular quartic with a third node.

This construction is mechanically more convenient than the one discussed in Mr. Roberts' paper "On the Pedals of Conics," and the two constructions are intimately connected with each other, and belong to the same theory.

Prof. Peirce, of Harvard University, laid before the Society some details of the methods made use of by him in his work on "Linear Associative Algebra."

The President conveyed to the Author the thanks of the Society for his interesting communication; and, at the request of Dr. Hirst, Prof. Peirce presented a copy of his work to the Library of the Society.

The following presents were received:—

"Linear Associative Algebra," by Benjamin Peirce, LL.D., Perkins Professor of Mathematics and Astronomy in Harvard University: from the author.

"Transactions of the Connecticut Academy of Arts and Sciences," vol. I., part 1; vol. II., part 1: from the Connecticut Academy.

"Nautical Almanac for 1874:" from the Nautical Almanac Office.

"Monatsbericht" for August, September, and October, 1870.

"Crelle," 72 Band, 4$^{es}$ Heft.

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*Feb. 9th,* 1871.

W. SPOTTISWOODE, Esq., F.R.S., President, in the Chair.

Mr. C. R. Hodgson, B.A. Lond., was proposed for election, and the Rev. J. Wolstenholme, M.A., and Mr. R. B. Hayward, M.A., were elected Members.

Prof. Cayley made a communication

## *On an Analytical Theorem from a New Point of View.*

The theorem is a well known one, derived from the equation

$$(az^2+2bz+c)\,w^2+2\,(a'z^2+2b'z+c')\,w+a''z^2+2b''z+c''=0;$$

viz., considering this equation as establishing a relation between the variables $z$ and $w$, and writing it in the forms

$$2u = \text{A}w^2+2\text{B}w+\text{C} = \text{A}'z^2+2\text{B}'z+\text{C}' = 0,$$

(where, of course, A, B, C are quadric functions of $z$, and A$'$, B$'$, C$'$ quadric functions of $w$,) we have

$$0 = \frac{du}{dw}\,dw + \frac{du}{dz}\,dz = (\text{A}w+\text{B})\,dw + (\text{A}'z+\text{B}')\,dz;.$$