except the first, when n is odd. Hence we may enunciate the multiplication theorem thus,

$$a\,|1,\,2,\,\dots\,r|\quad \times b\,|1,\,2,\,\dots\,r|^n = c\,|1,\,2,\,\dots\,r|^{m+n-2},$$

where
$$c_{i\dots jk\dots l} = \sum_{p=1}^{p=r} a_{i\dots p\dots j}\, b_{k\dots p\dots l}\,;$$

the p being the s^{th} subscript to a, and the t^{th} subscript to b. But s must not be 1 when m is odd; nor may t be 1 when n is odd.

Notes on Plane Curves. *By* J. J. Walker, M.A.

[*Read June* 12*th*, 1879.]

I. *Reduction of a Ternary Cubic to the sum of Four Cubes.*

Some time back, when occupied about another question, I was led to observe the method of actually reducing a ternary cubic to the quadrinomial form $ax^3+by^3+cz^3+dw^3=0$ ....(1),

which has not, as far as I am aware, been given by those geometers who have employed this form. It may be worth while to give it.

Let $x'y'z'$ be any point on the right line

$$lx+my+nz=0 \quad \text{.............................(2)};$$

its polar conic with respect to a cubic, $u=0$, is

$$x'\frac{du}{dx}+y'\frac{du}{dy}+z'\frac{du}{dz}=0,$$

which, since $x'y'z'$ satisfy (2), may be written

$$y'\left(l\frac{du}{dy}-m\frac{du}{dx}\right)+z'\left(l\frac{du}{dz}-n\frac{du}{dx}\right)=0,$$

showing that it passes through the four points common to

$$n\frac{du}{dy}-m\frac{du}{dz}=0,\quad l\frac{du}{dz}-n\frac{du}{dx}=0,\quad m\frac{du}{dx}-l\frac{du}{dy}=0;$$

i.e., if $u \equiv a_1x^3+b_2y^3+c_3z^3+3a_2x^2y+3a_3x^2z+3b_1y^2x+3b_3y^2z$
$\qquad +3c_1z^2x+3c_2z^2y+6dxyz,$

through the four points common to

$$(na_2-ma_3)\,x^2+(nb_2-mb_3)\,y^2+(nc_2-mc_3)\,z^2+2\,(nb_3-mc_2)\,yz$$
$$+2\,(nd-mc_1)\,zx+2\,(nb_1-md)\,xy=0,$$

$$(la_3-na_1)\,x^2+(lb_3-nb_1)\,y^2+(lc_3-nc_1)\,z^2+2\,(lc_2-nd)\,yz$$
$$+2\,(lc_1-na_3)\,zx+2\,(ld-na_2)\,xy=0,$$

$$(ma_1-la_2)\,x^2+(mb_1-lb_2)\,y^2+(mc_1-lc_2)\,z^2+2\,(md-lb_3)\,yz$$
$$+2\,(ma_3-ld)\,zx+2\,(ma_2-lb_1)\,xy=0.$$

If the triangle of reference is that which is self-conjugate with respect to these conics, then the coefficients of yz, zx, xy in the above three equations must vanish simultaneously, for which conditions it is necessary and sufficient that

$$a_2=\frac{ld}{n},\quad a_3=\frac{ld}{n},\quad b_1=\frac{md}{n},\quad b_3=\frac{md}{l},\quad c_1=\frac{nd}{m},\quad c_2=\frac{nd}{l};$$

whence u is of the form

$$u=a_1x^3+b_2y^3+c_3z^3$$
$$+d\left(3\frac{l}{n}x^2y+3\frac{l}{m}x^2z+3\frac{m}{n}y^2x+3\frac{m}{l}y^2z+3\frac{n}{m}z^2x+3\frac{n}{l}z^2y+6xyz\right)=0;$$

or, adding and subtracting

$$d\left(\frac{l^2}{mn}x^3+\frac{m^2}{nl}y^3+\frac{n^2}{lm}z^3\right),$$

$$u=\left(a_1-\frac{l^2d}{mn}\right)x^3+\left(b_2-\frac{m^2d}{nl}\right)y^3+\left(c_3-\frac{n^2d}{lm}\right)z^3$$
$$+\frac{d}{lmn}\,(l^3x^3+m^3y^3+n^3z^3+3l^2mx^2y+3l^2nx^2z+3m^2ly^2x+3m^2ny^2z$$
$$+3n^2lz^2x+3n^2mz^2y+6lmnxyz);$$

i.e., multiplying by lmn, u is of the form

$$ax^3+by^3+cz^3+dw^3,$$

where $a=(a_1mn-dl^2)\,l,\quad b=(b_2nl-dm^2)\,m,\quad c=(c_3lm-dn^2)\,n,$

$$w=lx+my+nz.$$

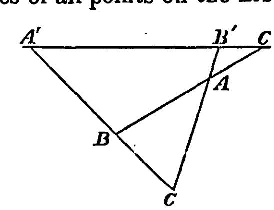
Hence, to reduce a cubic to the sum of four cubes: take any arbitrary line, which determines four points, viz., those common to the polar conics of all points on it; taking this as one of the four lines of reference, the others will be the connectors of the points of intersection of the three pairs of lines passing through the four points so determined.

The Hessian of (1) is

$$d\,(bclyz+camzx+abnxy)(lx+my+nz)-abcxyz;$$

the form of which shows that the corners of the triangle (xyz) lie on it, and that the third points in which the Hessian is met by the sides lie on the line $lx+my+nz$. Thus arises another representation of the reduction: An arbitrary line determines a triangle, inscribed in the Hessian, the three sides of which again meet that curve at the points in which it is cut by the line; the three sides of the triangle, and the line determining it, form the four lines of reference.

But the relation among the three sides of the triangle and the arbitrary line is obviously symmetrical; viz., taking one of those sides as the arbitrary line, then the other three form the triangle, which is self-conjugate with respect to the polar conics of all points on the first. Thus, in the accompanying figure, the line $A'B'C'$, cutting the Hessian in those points, and the sides of the triangle ABC, may be taken as representing the four lines of reference, or the line CAB' with the sides of the triangle $C'A'B$, and so on.



In fact, the points AA', BB', CC' are three pairs of Prof. Cayley's "conjugate poles"; *i.e.*, the line-pair forming the polar conic of one of each pair of points intersect in the other.

Consider the point A $(y=0,\ z=0)$; its polar conic is

$$ax^2+ldw^2;$$

i.e., a pair of lines meeting in the point common to x and w; viz., A'.

Thus, finally, arises another method of determining a system of four lines of reference. Take any line cutting the Hessian in three points; the conjugate poles of these with respect to the cubic form with them a system of six points lying three and three on four lines; which being taken as lines of reference, the equation of the cubic is reduced to the form

$$ax^3+by^3+cz^3+dw^3=0.$$

II. *On a certain Derived Curve.*

In connexion with Cotes's Theorem, the following seems worthy of being recorded:—

Consider a transversal $L\,(lx+my+nz=0)$, cutting a given line $R\,(ax+\beta y+\gamma z=0)$ in the point P, and the curve (of order p) $u=0$ in the p points $O_1, O_2 \dots O_p$; there are $p-1$ points $O, O', O'' \dots$ on L, for which

$$\frac{1}{OO_1}+\frac{1}{OO_2}+\dots+\frac{1}{OO_p}=\frac{p}{OP},$$

$$\frac{1}{O'O_1}+\frac{1}{O'O_2}+\dots+\frac{1}{O'O_p}=\frac{p}{OP},$$

$$\dots \quad \dots \quad \dots \quad \dots$$

The locus of these $p-1$ points, as the transversal L revolves about a fixed point $x_1y_1z_1$, may be shown to be another curve of order p, passing through $x_1y_1z_1$, and through the $(p-1)^2$ poles of the line R with respect to u; and consequently, when $x_1y_1z_1$ coincides with one of those poles, having it as a double point.

I recall the equation by which the segments on the transversal between (xyz), and the p points in which it meets $u=0$, may be determined ("Proceedings," Vol. IX., p. 227); viz., these will be equal to the values of $\rho\ (l^2+m^2+n^2-2mn\cos A-2nl\cos B-2lm\cos C)^{\frac{1}{2}}$, from

$$u+p\rho Du+\frac{p\,.\,p-1}{1\,.\,2}\rho^2D^2u\ldots+\rho^pD^pu=0,$$

where $pDu=(m\sin C-n\sin B)\dfrac{du}{dx}+(n\sin A-l\sin C)\dfrac{du}{dy}$

$$+(l\sin B-m\sin A)\frac{du}{dz};$$

from which, first making $u=\alpha x+\beta y+\gamma z=R$, to the same factor *près*,

$$\frac{1}{OP}=\frac{(\beta\sin C-\gamma\sin B)\,l+(\gamma\sin A-\alpha\sin C)\,m+(\alpha\sin B-\beta\sin A)\,n}{R},$$

and $\Sigma\dfrac{1}{OO_1}=$

$$\frac{\left(\dfrac{du}{dy}\sin C-\dfrac{du}{dz}\sin B\right)l+\left(\dfrac{du}{dz}\sin A-\dfrac{du}{dx}\sin C\right)m+\left(\dfrac{du}{dx}\sin B-\dfrac{du}{dy}\sin A\right)n}{u}$$

so that the condition determining $O,\ O',\ O''\ldots$, gives

$$\frac{(\beta\sin C-\gamma\sin B)\,l+\ldots}{R}=\frac{(u''\sin C-u'''\sin B)\,l+\ldots}{u},$$

where $u'=\dfrac{1}{p}\dfrac{du}{dx},\ \ u''=\dfrac{1}{p}\dfrac{du}{dy},\ \ u'''=\dfrac{1}{p}\dfrac{du}{dz}$; or

$$\sin A\,\{(m\gamma-n\beta)\,u-(mu'''-nu'')\,R\}+\sin B\,\{\ldots\}+\sin C\,\{\ldots\}=0.$$

But, since xyz, $x_1y_1z_1$ are points on $lx+my+nz=0$,

$$l:m:n=yz_1-zy_1:zx_1-xz_1:xy_1-yx_1;$$

by which substitutions, and those of $xu'+yu''+zu'''$ for u, $\alpha x+\beta y+\gamma z$ for R, the equation above reduces identically to

$$x\sin A\,\{(\alpha x+\beta y+\gamma z)\,(x_1u'+y_1u''+z_1u''')$$
$$-(\alpha x_1+\beta y_1+\gamma z_1)\,(xu'+yu''+zu''')\}$$
$$+y\sin B\,\{\ldots\}+z\sin C\,\{\ldots\}=0.$$

This, divided by $x\sin A+y\sin B+z\sin C$, gives, as the required locus,

$$v\equiv(\alpha x+\beta y+\gamma z)(x_1u'+y_1u''+z_1u''')-(\alpha x_1+\beta y_1+\gamma z_1)\,u=0;$$

or, in another form,

$$v\equiv(yz_1-zy_1)(\gamma u''-\beta u''')+(zx_1-xz_1)(\alpha u'''-\gamma u')$$
$$+(xy_1-yx_1)(\beta u'-\alpha u'')=0;$$

which, plainly, passes through the point $(x_1y_1z_1)$, and through the $(p-1)^2$ points of intersection of the system

$$\gamma\frac{du}{dy}-\beta\frac{du}{dz}=0,\quad \alpha\frac{du}{dz}-\gamma\frac{du}{dx}=0,\quad \beta\frac{du}{dx}-\alpha\frac{du}{dy}=0.$$

Further, if $x_1y_1z_1$ coincides with one of the points common to this system, then it is evidently a double point on v; in fact, for $x=x_1$,

$y=y_1$, $z=z_1$, $\quad \dfrac{dv}{dx}=y_1(\beta u'-\alpha u'')-z_1(\alpha u'''-\gamma u');$

and therefore vanishes if $x_1y_1z_1$ is one of the points common to

$$\beta\frac{du}{dx}-\alpha\frac{du}{dy},\quad \alpha\frac{du}{dz}-\gamma\frac{du}{dx};$$

and, similarly, $\dfrac{dv}{dy}$, $\dfrac{dv}{dz}$.

III. *Envelope of Diameters of a Cubic.*

In the case of u being a cubic (p. 183),

$$u+3\rho Du+3\rho^2D^2u+\rho^3D^3u=0$$

determines the intercepts on any transversal between a point xyz in it, and the points in which the transversal meets u, when it is drawn parallel to a fixed line $lx+my+nz=0$. Hence the equation of the rectilinear *diameter* to a system of chords parallel to $lx+my+nz=0$, in the sense of the locus of a point O on any chord of the system, meeting the cubic in the points O_1, O_2, O_3, such that $OO_1+OO_2+OO_3=0$, is

$$(m\sin C-n\sin B)^2\frac{d^2u}{dx^2}+(n\sin A-l\sin C)^2\frac{d^2u}{dy^2}+(l\sin B-m\sin A)^2\frac{d^2u}{dz^2}$$
$$+2\,(n\sin A-l\sin C)\,(l\sin B-m\sin A)\,\frac{d^2u}{dy\,dz}$$
$$+2\,(l\sin B-m\sin A)\,(m\sin C-n\sin B)\,\frac{d^2u}{dz\,dx}$$
$$+2\,(m\sin C-n\sin B)\,(n\sin A-l\sin C)\,\frac{d^2u}{dx\,dy}=0;$$

and it is a question worth considering, what is the envelope of these diameters?

Writing, for shortness,

$$m \sin C - n \sin B = \lambda,$$
$$n \sin A - l \sin C = \mu,$$
$$l \sin B - m \sin A = \nu;$$

then, identically, $\lambda \sin A + \mu \sin B + \nu \sin C = 0.$

If, further, a, b, c, f, g, h be written for $\frac{d^2u}{dx^2} \dots \frac{d^2u}{dx\,dy}$, then the equation to a diameter is

$$a\lambda^2 + b\mu^2 + c\nu^2 + 2f\mu\nu + 2g\nu\lambda + 2h\lambda\mu = 0,$$

and the envelope of this line, where the parameters λ, μ, ν vary, subject to the above relation, is at once seen to be

$$(bc-f^2)\sin^2 A + (ca-g^2)\sin^2 B + (ab-h^2)\sin^2 C + 2(gh-af)\sin B \sin C$$
$$+ 2(hf-bg)\sin C \sin A + 2(fg-ch)\sin A \sin B = 0\,;$$

which, in relation to these "diameters," may be called the "central conic."

The preceding method may obviously be generalised. The equation to any "diametral curve" is, whatever the order of u,

$$\lambda^r \frac{d^r u}{dx^r} + \dots + r\lambda^r \mu \frac{d^r u}{dx^{r-1}\,dy} + \dots = 0,$$

and the envelope of this, subject to

$$\lambda \sin A + \mu \sin B + \nu \sin C = 0,$$

may similarly be sought.

The "diametral conics" of a cubic, in particular, pass through the four fixed points common to the polar conics of all points on the line at infinity.

On the Twenty-one Coordinates of a Conic in Space.

By WILLIAM SPOTTISWOODE, P.R.S.

[*Read June 12th*, 1879.]

In a Note published in the Report of the British Association for 1878 (Dublin Meeting), I gave a short account of the coordinates (then described as 18 in number) of a conic in space, and of the equations of condition by which they were connected. A modification of the notation there used will perhaps place these quantities in a clearer light.