On the Form of the Energy Integral in the Varying Motion of a Viscous Incompressible Fluid. *By* J. Brill, M.A. Received May 29th, 1895. Read June 13th, 1895. Received, in new form, September 11th, 1895.

1. In the varying motion of a viscous incompressible fluid, the energy integral can, in two special cases, be put into the same simple form as in the motion of the perfect fluid. These are the two-dimensional case and the case in which the motion is symmetrical about an axis. In the three-dimensional motion of the viscous fluid the energy integral is of a more complex form than in the corresponding case of motion of the perfect fluid.

2. We will first consider the case of two-dimensional motion. If we write

$$Q = \frac{p}{\rho} + V + \tfrac{1}{2} q^2,$$

the equations of motion may be written in the form

$$\left.\begin{aligned} \frac{\partial Q}{\partial x} + \frac{\partial u}{\partial t} - 2v\zeta + 2\nu \frac{\partial \zeta}{\partial y} = 0 \\ \frac{\partial Q}{\partial y} + \frac{\partial v}{\partial t} + 2u\zeta - 2\nu \frac{\partial \zeta}{\partial x} = 0 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

Eliminating Q from these equations, we obtain, for the equation controlling the vortex motion,

$$\frac{\partial \zeta}{\partial t} + \frac{\partial}{\partial x}\left(u\zeta - \nu \frac{\partial \zeta}{\partial x}\right) + \frac{\partial}{\partial y}\left(v\zeta - \nu \frac{\partial \zeta}{\partial y}\right) = 0 \quad \ldots\ldots\ldots\ldots(2).$$

Now, consider the differential equations

$$\frac{dx}{u\zeta - \nu \dfrac{\partial \zeta}{\partial x}} = \frac{dy}{v\zeta - \nu \dfrac{\partial \zeta}{\partial y}} = \frac{dt}{\zeta} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

If $m = \text{const.}$ and $\beta = \text{const.}$ be two independent integrals of these equations, we have

$$\frac{u\zeta - \nu \dfrac{\partial \zeta}{\partial x}}{\dfrac{\partial (m, \beta)}{\partial (y, t)}} = \frac{v\zeta - \nu \dfrac{\partial \zeta}{\partial y}}{\dfrac{\partial (m, \beta)}{\partial (t, x)}} = \frac{\zeta}{\dfrac{\partial (m, \beta)}{\partial (x, y)}}.$$

Also, in virtue of equation (2), we see that m and β may be so chosen that we may write

$$\left.\begin{aligned} 2\left(u\zeta-\nu\frac{\partial\zeta}{\partial x}\right) &= \frac{\partial(m,\beta)}{\partial(y,t)} \\ 2\left(v\zeta-\nu\frac{\partial\zeta}{\partial y}\right) &= \frac{\partial(m,\beta)}{\partial(t,x)} \\ 2\zeta &= \frac{\partial(m,\beta)}{\partial(x,y)} \end{aligned}\right\} \quad\text{.......................(4).}$$

But we have

$$\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}=2\zeta=\frac{\partial(m,\beta)}{\partial(x,y)}=\frac{\partial}{\partial x}\left(m\frac{\partial\beta}{\partial y}\right)-\frac{\partial}{\partial y}\left(m\frac{\partial\beta}{\partial x}\right);$$

from which it follows that there exists a certain function a, such that

$$\left.\begin{aligned} u &= \frac{\partial a}{\partial x}+m\frac{\partial\beta}{\partial x} \\ v &= \frac{\partial a}{\partial y}+m\frac{\partial\beta}{\partial y} \end{aligned}\right\} \quad\text{..............................(5).}$$

If we substitute from equations (4) and (5) in the first of equations (1), it becomes

$$\frac{\partial Q}{\partial x}+\frac{\partial}{\partial t}\left(\frac{\partial a}{\partial x}+m\frac{\partial\beta}{\partial x}\right)-\frac{\partial(m,\beta)}{\partial(t,x)}=0,$$

which reduces to $\dfrac{\partial Q}{\partial x}+\dfrac{\partial}{\partial x}\left(\dfrac{\partial a}{\partial t}+m\dfrac{\partial\beta}{\partial t}\right)=0$ ....(6).

Similarly, the second of equations (1) may be reduced to the form

$$\frac{\partial Q}{\partial y}+\frac{\partial}{\partial y}\left(\frac{\partial a}{\partial t}+m\frac{\partial\beta}{\partial t}\right)=0 \quad\text{.......................(7).}$$

From equations (6) and (7) we immediately deduce the energy integral in the form

$$Q+\frac{\partial a}{\partial t}+m\frac{\partial\beta}{\partial t}=f(t) \quad\text{.......................(8).}$$

Now, if we write

$$u' = u - \nu \frac{\partial \log \zeta}{\partial x} = \frac{1}{\zeta} \left\{ u\zeta - \nu \frac{\partial \zeta}{\partial x} \right\},$$

$$v' = v - \nu \frac{\partial \log \zeta}{\partial y} = \frac{1}{\zeta} \left\{ v\zeta - \nu \frac{\partial \zeta}{\partial y} \right\},$$

then equations (3) assume the form

$$\frac{dx}{u'} = \frac{dy}{v'} = dt,$$

from which it follows that m and β satisfy the equations

$$\frac{\partial m}{\partial t} + u' \frac{\partial m}{\partial x} + v' \frac{\partial m}{\partial y} = 0,$$

$$\frac{\partial \beta}{\partial t} + u' \frac{\partial \beta}{\partial x} + v' \frac{\partial \beta}{\partial y} = 0.$$

3. In the case in which the motion is symmetrical about an axis, the equations of motion assume the form

$$\left.\begin{aligned} &\frac{\partial Q}{\partial r} + \frac{\partial U}{\partial t} - 2V\omega + 2\nu \frac{\partial \omega}{\partial z} = 0 \\ &\frac{\partial Q}{\partial z} + \frac{\partial V}{\partial t} + 2U\omega - 2\nu \left(\frac{\partial \omega}{\partial r} + \frac{\omega}{r} \right) = 0 \end{aligned}\right\} \text{...............(9).}$$

Eliminating Q, as before, we obtain

$$\frac{\partial \omega}{\partial t} + \frac{\partial}{\partial r} \left\{ U\omega - \nu \left(\frac{\partial \omega}{\partial r} + \frac{\omega}{r} \right) \right\} + \frac{\partial}{\partial z} \left(V\omega - \nu \frac{\partial \omega}{\partial z} \right) = 0 \text{......(10).}$$

Thus, if m and β be two independent integrals of the equations

$$\frac{dr}{U\omega - \nu \left(\frac{\partial \omega}{\partial r} + \frac{\omega}{r} \right)} = \frac{dz}{V\omega - \nu \frac{\partial \omega}{\partial z}} = \frac{dt}{\omega} \text{ ....(11),}$$

we have $$\frac{U\omega - \nu \left(\frac{\partial \omega}{\partial r} + \frac{\omega}{r} \right)}{\frac{\partial (m, \beta)}{\partial (z, t)}} = \frac{V\omega - \nu \frac{\partial \omega}{\partial z}}{\frac{\partial (m, \beta)}{\partial (t, r)}} = \frac{\omega}{\frac{\partial (m, \beta)}{\partial (r, z)}};$$

and, in virtue of equation (10), we see that m and β may be so chosen that we may write

$$\left.\begin{aligned} 2\left\{U\omega-\nu\left(\frac{\partial\omega}{\partial r}+\frac{\omega}{r}\right)\right\} &= \frac{\partial(m,\beta)}{\partial(z,t)} \\ 2\left(V\omega-\nu\frac{\partial\omega}{\partial z}\right) &= \frac{\partial(m,\beta)}{\partial(t,r)} \\ 2\omega &= \frac{\partial(m,\beta)}{\partial(r,z)} \end{aligned}\right\} \quad\ldots\ldots\ldots\ldots\ldots(12).$$

But $$\frac{\partial V}{\partial r}-\frac{\partial U}{\partial z}=2\omega=\frac{\partial(m,\beta)}{\partial(r,z)}=\frac{\partial}{\partial r}\left(m\frac{\partial\beta}{\partial z}\right)-\frac{\partial}{\partial z}\left(m\frac{\partial\beta}{\partial r}\right).$$

Thus we see that there exists a function α, such that

$$\left.\begin{aligned} U &= \frac{\partial\alpha}{\partial r}+m\frac{\partial\beta}{\partial r} \\ V &= \frac{\partial\alpha}{\partial z}+m\frac{\partial\beta}{\partial z} \end{aligned}\right\} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(13).$$

Substituting from equations (12) and (13) in equations (9), we readily find that they reduce to the form

$$\frac{\partial Q}{\partial r}+\frac{\partial}{\partial r}\left(\frac{\partial\alpha}{\partial t}+m\frac{\partial\beta}{\partial t}\right)=0,$$

$$\frac{\partial Q}{\partial z}+\frac{\partial}{\partial z}\left(\frac{\partial\alpha}{\partial t}+m\frac{\partial\beta}{\partial t}\right)=0.$$

These equations at once give us the energy integral in the simple form

$$Q+\frac{\partial\alpha}{\partial t}+m\frac{\partial\beta}{\partial t}=f(t) \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(14),$$

which is exactly like equation (8).

Further, if we write

$$U'=U-\nu\frac{\partial}{\partial r}\log r\omega=\frac{1}{\omega}\left\{U\omega-\nu\left(\frac{\partial\omega}{\partial r}+\frac{\omega}{r}\right)\right\},$$

$$V'=V-\nu\frac{\partial}{\partial z}\log r\omega=\frac{1}{\omega}\left(V\omega-\nu\frac{\partial\omega}{\partial z}\right),$$

we see that equations (11) assume the form

$$\frac{dr}{U'} = \frac{dz}{V'} = dt.$$

Thus m and β satisfy the equations

$$\frac{\partial m}{\partial t} + U'\frac{\partial m}{\partial r} + V'\frac{\partial m}{\partial z} = 0,$$

$$\frac{\partial \beta}{\partial t} + U'\frac{\partial \beta}{\partial r} + V'\frac{\partial \beta}{\partial z} = 0.$$

4. We now come to the discussion of the three-dimensional case. We will write

$$u = \frac{\partial \alpha}{\partial x} + m\frac{\partial \beta}{\partial x}, \quad v = \frac{\partial \alpha}{\partial y} + m\frac{\partial \beta}{\partial y}, \quad w = \frac{\partial \alpha}{\partial z} + m\frac{\partial \beta}{\partial z},$$

from which it follows that

$$2\xi = \frac{\partial (m, \beta)}{\partial (y, z)}, \quad 2\eta = \frac{\partial (m, \beta)}{\partial (z, x)}, \quad 2\zeta = \frac{\partial (m, \beta)}{\partial (x, y)}.$$

The equations which control the vortex motion in this case are

$$\left.\begin{aligned}
\frac{\partial \xi}{\partial t} + \frac{\partial}{\partial y}\left\{v\xi - u\eta + \nu\left(\frac{\partial \eta}{\partial x} - \frac{\partial \xi}{\partial y}\right)\right\} - \frac{\partial}{\partial z}\left\{u\zeta - w\xi + \nu\left(\frac{\partial \xi}{\partial z} - \frac{\partial \zeta}{\partial x}\right)\right\} &= 0 \\
\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial z}\left\{w\eta - v\zeta + \nu\left(\frac{\partial \zeta}{\partial y} - \frac{\partial \eta}{\partial z}\right)\right\} - \frac{\partial}{\partial x}\left\{v\xi - u\eta + \nu\left(\frac{\partial \eta}{\partial x} - \frac{\partial \xi}{\partial y}\right)\right\} &= 0 \\
\frac{\partial \zeta}{\partial t} + \frac{\partial}{\partial x}\left\{u\zeta - w\xi + \nu\left(\frac{\partial \xi}{\partial z} - \frac{\partial \zeta}{\partial x}\right)\right\} - \frac{\partial}{\partial y}\left\{w\eta - v\zeta + \nu\left(\frac{\partial \zeta}{\partial y} - \frac{\partial \eta}{\partial z}\right)\right\} &= 0
\end{aligned}\right\} \quad \text{..............(15).}$$

Guided by our former work, we will now make the assumptions

$$2\left\{w\eta - v\zeta + \nu\left(\frac{\partial \zeta}{\partial y} - \frac{\partial \eta}{\partial z}\right)\right\} = \frac{\partial (m, \beta)}{\partial (x, t)} + a,$$

$$2\left\{u\zeta - w\xi + \nu\left(\frac{\partial \xi}{\partial z} - \frac{\partial \zeta}{\partial x}\right)\right\} = \frac{\partial (m, \beta)}{\partial (y, t)} + b,$$

$$2\left\{v\xi - u\eta + \nu\left(\frac{\partial \eta}{\partial x} - \frac{\partial \xi}{\partial y}\right)\right\} = \frac{\partial (m, \beta)}{\partial (z, t)} + c.$$

Substituting these values in equations (15), and taking account of the values for ξ, η, ζ given above, we obtain

$$\frac{\partial c}{\partial y}-\frac{\partial b}{\partial z}=\frac{\partial a}{\partial z}-\frac{\partial c}{\partial x}=\frac{\partial b}{\partial x}-\frac{\partial a}{\partial y}=0.$$

These equations indicate the existence of a function ϑ, such that

$$a=\frac{\partial \vartheta}{\partial x},\quad b=\frac{\partial \vartheta}{\partial y},\quad c=\frac{\partial \vartheta}{\partial z}.$$

Now the equations of motion of the fluid may be written in the forms

$$\frac{\partial Q}{\partial x}+\frac{\partial u}{\partial t}+2\,(w\eta-v\zeta)+2\nu\left(\frac{\partial \zeta}{\partial y}-\frac{\partial \eta}{\partial z}\right)=0,$$

$$\frac{\partial Q}{\partial y}+\frac{\partial v}{\partial t}+2\,(u\zeta-w\xi)+2\nu\left(\frac{\partial \xi}{\partial z}-\frac{\partial \zeta}{\partial x}\right)=0,$$

$$\frac{\partial Q}{\partial z}+\frac{\partial w}{\partial t}+2\,(v\xi-u\eta)+2\nu\left(\frac{\partial \eta}{\partial x}-\frac{\partial \xi}{\partial y}\right)=0$$

By means of the results given above, these equations may be easily reduced to the forms

$$\frac{\partial Q}{\partial x}+\frac{\partial}{\partial x}\left(\frac{\partial \alpha}{\partial t}+m\,\frac{\partial \beta}{\partial t}\right)+\frac{\partial \vartheta}{\partial x}=0,$$

$$\frac{\partial Q}{\partial y}+\frac{\partial}{\partial y}\left(\frac{\partial \alpha}{\partial t}+m\,\frac{\partial \beta}{\partial t}\right)+\frac{\partial \vartheta}{\partial y}=0,$$

$$\frac{\partial Q}{\partial z}+\frac{\partial}{\partial z}\left(\frac{\partial \alpha}{\partial t}+m\,\frac{\partial \beta}{\partial t}\right)+\frac{\partial \vartheta}{\partial z}=0.$$

Hence we obtain for the form of the energy integral

$$Q+\vartheta+\frac{\partial \alpha}{\partial t}+m\,\frac{\partial \beta}{\partial t}=f\,(t).$$

The function ϑ satisfies the equation

$$\xi\,\frac{\partial \vartheta}{\partial x}+\eta\,\frac{\partial \vartheta}{\partial y}+\zeta\,\frac{\partial \vartheta}{\partial z}=2\nu\left\{\xi\left(\frac{\partial \zeta}{\partial y}-\frac{\partial \eta}{\partial z}\right)+\eta\left(\frac{\partial \xi}{\partial z}-\frac{\partial \zeta}{\partial x}\right)+\zeta\left(\frac{\partial \eta}{\partial x}-\frac{\partial \xi}{\partial y}\right)\right\}$$

...............(16).

From equation (16) we see that the energy integral can only become reduced to the simple form that obtains for the motion of the perfect fluid if the vortex lines can continually be cut orthogonally by a family of surfaces. This is necessarily so in the two special cases we have considered. It, however, indicates a state of affairs that must be very rare in cases of three-dimensional motion.

If we write $u = u'+f, \quad v = v'+g, \quad w = w'+h,$

and determine f, g, h so as to satisfy the equations

$$\left.\begin{aligned} 2\left\{h\eta-g\zeta+\nu\left(\frac{\partial\zeta}{\partial y}-\frac{\partial\eta}{\partial z}\right)\right\} &= \frac{\partial\vartheta}{\partial x} \\ 2\left\{f\zeta-h\xi+\nu\left(\frac{\partial\xi}{\partial z}-\frac{\partial\zeta}{\partial x}\right)\right\} &= \frac{\partial\vartheta}{\partial y} \\ 2\left\{g\xi-f\eta+\nu\left(\frac{\partial\eta}{\partial x}-\frac{\partial\xi}{\partial y}\right)\right\} &= \frac{\partial\vartheta}{\partial z} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots(17);$$

then we have
$$2(w'\eta-v'\zeta) = \frac{\partial(m,\beta)}{\partial(x,t)},$$
$$2(u'\zeta-w'\xi) = \frac{\partial(m,\beta)}{\partial(y,t)},$$
$$2(v'\xi-u'\eta) = \frac{\partial(m,\beta)}{\partial(z,t)}.$$

From these equations, we obtain

$$u'\frac{\partial(m,\beta)}{\partial(x,t)}+v'\frac{\partial(m,\beta)}{\partial(y,t)}+w'\frac{\partial(m,\beta)}{\partial(z,t)} = 0,$$

which may be replaced by the two equations

$$k\frac{\partial m}{\partial t}+u'\frac{\partial m}{\partial x}+v'\frac{\partial m}{\partial y}+w'\frac{\partial m}{\partial z} = 0,$$

$$k\frac{\partial\beta}{\partial t}+u'\frac{\partial\beta}{\partial x}+v'\frac{\partial\beta}{\partial y}+w'\frac{\partial\beta}{\partial z} = 0.$$

From these we easily deduce

$$k\frac{\partial(m,\beta)}{\partial(x,t)} = w'\frac{\partial(m,\beta)}{\partial(z,x)}-v'\frac{\partial(m,\beta)}{\partial(x,y)} = 2(w'\eta-v'\zeta).$$

Comparing this with the equation given above, we see that $k = 1$, and the above equations assume the form

$$\frac{\partial m}{\partial t} + u' \frac{\partial m}{\partial x} + v' \frac{\partial m}{\partial y} + w' \frac{\partial m}{\partial z} = 0,$$

$$\frac{\partial \beta}{\partial t} + u' \frac{\partial \beta}{\partial x} + v' \frac{\partial \beta}{\partial y} + w' \frac{\partial \beta}{\partial z} = 0.$$

Thus m and β are solutions of the equations

$$\frac{dx}{u'} = \frac{dy}{v'} = \frac{dz}{w'} = dt.$$

It is to be noted that equations (20) allow one degree of freedom in the choice of the quantities f, g, h, as should be the case.

On an Expansion of the Potential Function $1/R^{\kappa-1}$ in Legendre's Functions. By E. J. ROUTH. Received May 29th, 1895. Read June 13th, 1895.

1. When we require the potential of a body attracting according to the inverse square of the distance, we use Legendre's series

$$\frac{1}{R} = \Sigma P_n h^n \quad \text{..............................(1)},$$

where $\qquad R^2 = 1 - 2ph + h^2.$

But, when the law of attraction is the inverse κ^{th} power of the distance, we require the expansion of $1/R^{\kappa-1}$. There are two ways of extending Legendre's series.

First, we may continue to make the expansion in powers of h^n, and put

$$\frac{1}{R^{\kappa-1}} = \Sigma P'_n h^n \quad \text{..............................(2)}.$$

If $\kappa - 1$ is an odd integer equal to $2m+1$, we have

$$P'_n = \frac{1}{1.3.5 \ldots (2m-1)} \frac{d^m}{dp^m} P_{m+n} \quad \text{..............(3)}.$$