THE SIMULTANEOUS SYSTEM OF TWO QUADRATIC QUATERNARY FORMS

By H. W. TURNBULL.

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TABLE A.

The 125 Irreducible Forms of the System of Two Quaternary Quadratics $f = a_x^2$ and $f' = b_x^2$.

5 Invariants.	a^2_α, b^2_α, $(AB)^2$, a^2_β, b^2_β, *i.e.* Δ, Θ, Φ, Θ', Δ'.	§ 28 (ii)
5 Covariants.	f, $(A\beta x)^2$, $(B\alpha x)^2$, f' and the quartic	§ 28 (ii)
	$a_x b_x a_\beta b_\alpha (AB)(A\beta x)(B\alpha x)$.	§ 34
5 Contravariants.	u^2_α, $(Abu)^2$, $(Bau)^2$, u^2_β and the quartic	
	$u_\alpha u_\beta a_\beta b_\alpha (AB)(Abu)(Bau)$.	§ 34
16 Complexes.	$(Ap)^2$, $(Bp)^2$, $(abp)^2$, $(\alpha\beta p)^2$, $(AB)(Ap)(Bp)$,	§ 28 (ii)
	$(abp)(\alpha\beta p)\, a_\beta b_\alpha$,	§ 29
	$(abp)(a'bp)\, a_\beta a'_\beta\, (\equiv F_1^2)$, $(abp)(ab'p)\, b_\alpha b'_\alpha\, (\equiv F_2^2)$	§ 14
	{ $(abp)(aa'p)(a''bp)\, a'_\beta a''_\beta$, $(aa'B)(Bp)(a''bp)(abp)\, a'_\beta a''_\beta$,	§§ 37
	{ $(aa'p)(\alpha\beta p)(abp)\, a'_\beta b_\alpha$, $(\alpha\beta p)(aa'B)(abp)\, a'_\beta b_\alpha\, (Bp)$,	to 39
	and 4 similar forms.*	
20 Mixed Forms containing x and u.	$a_x a_\beta u_\beta$, $b_x b_\alpha u_\alpha$, $(AB)(aBu)\, a'_x$, (§ 35),	§ 28 (ii)
	$(aBu)(a'Bu)\, a_x a'_x$,	§ 35
	$(Abu)(A\beta x)\, b_x u_\beta$, $(Bau)(B\alpha x)\, a_x u_\alpha$,	§ 28 (iii)
	$(AB)(Abu)(Bau)\, a_x b_x$, $(AB)(A\beta x)(B\alpha x)\, u_\alpha u_\beta$.	§ 29
	$(Abu)(A\beta x)\, b_\alpha a_\beta u_\alpha a_x$, $(Bau)(B\alpha x)\, a_\beta b_\alpha u_\beta b_x$,	§ 31
	$(Abu)(A\beta x)\, b_\alpha u_\alpha u_\beta$ and a similar form	§ 29
	$(Abu)(A\beta x)\, a_\beta a_x b_x$,, ,,	,,
	$(AB)(Abu)(Bau)\, b_\alpha u_\alpha a_x$,, ,,	§ 31
	$(AB)(Abu)(Bau)\, a_\beta u_\beta b_x$,, ,,	,,
	$(a'a''bu)(Bau)(Ba'u)\, a_x b_x a''_x$,	§ 35
	$(a'a''\beta x)(B\alpha x)(Ba'u)\, u_\alpha u_\beta a''_x$.	,,

* By "similar form" in the above is meant a form in which all the symbols a, A, α are interchanged respectively with b, B, β.

In this, $\alpha = aa'a''$, $A = aa'$, $(\alpha\beta p) = (\alpha\beta uv) = u_\alpha v_\beta - v_\alpha u_\beta$.

$\beta = bb'b''$, $B = bb'$, $(A\beta x) = a_\beta a'_x - a'_\beta a_x$.

TABLE A.—*Continued.*

Mixed forms containing x and p:	$(abp)\,a_x b_x$, $(\alpha\beta p)\,a_\beta b_\alpha a_x b_x$, $(AB)(Bp)(A\beta x)\,a_\beta a_x$,	§ 28 (ii), § 29β, § 29
9 of orders 2 in x and 1 in p.	$(AB)(Ap)B\alpha x)\,b_\alpha b_x$, $(AB)(A\beta x)(B\alpha x)(\alpha\beta p)$,	§ 29
	$(Ap)(A\beta x)\,a_\beta a_x$, $(Bp)(B\alpha x)\,b_\alpha b_x$,	§ 28 (iii)
	$(abp)\,a'_\beta\,(A\beta x)\,b_x$, $(abp)\,b'_\alpha\,(B\alpha x)\,a_x$.	§ 39
4 of orders 2 in x, 2 in p.	$(\alpha\beta p)(A\beta x)(Ap)\,b_\alpha b_x$, and a similar form,	§ 29β
	$(abp)(AB)(B\alpha x)(Ap)\,b_\alpha a_x$,,	§ 31
1 of orders 2 in x, 3 in p.	$(\alpha\beta p)(A\beta x)(B\alpha x)(Ap)(Bp)$.	§ 29
Mixed forms containing u and p. Correlative of the above $9+4+1=14$ forms.	Re-write above set with u for x, a ,, α, α ,, a, β ,, b, b ,, β, and A, B, p unchanged.	§ 29
Mixed forms containing x, p, u:	$(Abu)(Ap)\,b$, $(A\beta x)(Ap)\,u_\beta$, and two similar forms,	§ 28(ii)
16 linear in all variables.	$(abp)\,a_x b_\alpha u_\alpha$, $(\alpha\beta p)\,a_\beta a_x u_\alpha$, ,, ,,	§ 28 (iii)
	$(AB)(Abu)(Bp)\,b_x$, $(AB)(A\beta x)(Bp)\,u_\beta$, ,,	,,
	$(abp)(Abu)(A\beta x)\,a_\beta$, and one similar form,	§ 29
	$(\alpha\beta p)(Abu)(A\beta x)\,b_\alpha$, ,, ,,	§ 29
26 of order 2 in p, linear in u and x.	$(Ap)(Bp)(Abu)\,b'_x$, $(AB)(\alpha\beta p)(Ap)(B\alpha u)\,a_\beta b_\alpha b_x$,	§ 35, § 34
	$(abp)(\alpha\beta p)\,a_\beta u_\alpha b_x$, $(abp)(\alpha\beta p)\,b_\alpha u_\beta a_x$,	§ 29β
	{ $(abp)(Abu)(Ap)\,a_x$, $(\alpha\beta p)(A\beta x)(Ap)\,u_\alpha$,	§ 28 (iii)
	{ $(AB)(abp)(Abu)(Bp)\,a_x$, $(AB)(\alpha\beta p)(A\beta x)(Bp)\,u_\alpha$,	§ 29
	{ $(\alpha\beta p)(Abu)(Ap)\,a_\beta b_\alpha a_x$,	§ 31
	and five similar forms.	
	{ $(abp)\,a'_\beta b_x u_\beta\,(Ap)$, $(abp)\,a'_\beta\,(Abu)(A\beta x)(A'p)$,	§ 37, § 38
	{ $(AB)(abp)\,a'_\beta b_x u_\beta\,(Bp)$, $(abp)\,a'_\beta(Abu)\,(A\beta x)(A'B)(Bp)$,	,,
	{ $(abp)\,a'_\beta\,(a''bp)\,a''_\beta\,(Ab'u)\,b'_x$, $(abp)\,a'_\beta b_\alpha\,(\alpha\beta p)(A\beta' x)\,u_{\beta'}$,	§ 38
	and six similar forms.	
4 forms of orders (2, 1, 2).	$(Abu)(B\alpha u)(Ap)\,b'_x a_x$, $(Abu)(B\alpha x)(Ap)\,b'_x u_\alpha$,	§ 35
	and two similar forms.	

I.

Preliminary Reductions.

1. In the *Mathematische Annalen*, Bd. 56, Gordan has worked out a system of invariants for two quaternary quadratics. In the following pages it is proposed to shew how Gordan's system can be very much

simplified. In fact, we shall reduce the system to 125 forms instead of Gordan's 580 forms.

The theorems established in Sections I and II of this paper are substantially the same as in Chapters 1 and 2 of Gordan's work. They are only reproduced here to make the investigation complete.

2. Let

$$\left.\begin{aligned} f &= a_x^2 = a_x'^2 = \dots \\ f' &= b_x^2 = b_x'^2 = \dots \end{aligned}\right\}, \tag{1}$$

and

be the two quadratics.

Then any invariant is expressible as a product of factors d_x, (dd_1p), (dd_1d_2u), $(dd_1d_2d_3)$, where d stands for a or b and each different d occurs exactly twice in the form. The symbols x, p, u are the variables,

u	denoting	plane	coordinates,
$p = (uv)$	,,	line	,, ,
$x = (uvw)$	,,	point	,, ,

so that $u_x = v_x = w_x = 0$ always holds good.

To be more precise, we define v by the relations

$$u_i v_k - v_i u_k = p_{ik} \quad (i, k = 1, 2, 3, 4).$$

Any invariant J is then a function of the variables x, p, u: but for purposes of reduction it is sometimes well to break up p into its elements u, v; in which case we may have, in addition to the four types of factors d_x, (dd_1p), ... above, the new factor

$$[(dd_1d_2u)(d_3d_4d_5v)-(dd_1d_2v)(d_3d_4d_5u)].$$

This factor is equal to

$$(dd_1d_2d_3)(d_4d_5uv)+(dd_1d_2d_4)(d_5d_3uv)+(dd_1d_2d_5)(d_3d_4uv),$$

which contains p explicitly.

The Fundamental Identities of Quaternary Forms.

3. Neglecting reducible terms containing u_x or v_x, we have three identities

$$(abcd)\,e_x = (abce)\,d_x+(abed)\,c_x+(aecd)\,b_x+(ebcd)\,a_x, \tag{I}$$

$$(abcu)\,d_x = (abdu)\,c_x+(adbu)\,b_x+(dbcu)\,a_x, \tag{II}$$

$$(abp)\,c_x = (acp)\,b_x+(cbp)\,a_x. \tag{III}$$

In these, let g_i denote the bracket in the left-hand member. Then a useful reference symbol is

$$(\dot{g}_i,\ \dot{e}_x) \text{ for identity (I)},$$

$$(\dot{g}_i,\ \dot{d}_x) \quad \text{,,} \quad \text{(II)},$$

$$(\dot{g}_i,\ \dot{c}_x) \quad \text{,,} \quad \text{(III)}.$$

These identities follow from the identical vanishing of a determinant

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 & a_x \\ b_1 & & \dots & & \dots \\ c_1 & & \dots & & \dots \\ d_1 & & \dots & & \dots \\ e_1 & & \dots & & \dots \end{vmatrix}.$$

Determinantal Permutations.

4. Let a_δ denote a_x or else a bracket containing a. Then, if a_δ, b_ϵ is a pair of such symbolic factors, let

$$\dot{a}_\delta \dot{b}_\epsilon \text{ denote } a_\delta b_\epsilon - b_\delta a_\epsilon;$$

but let $(\dot{a}\dot{b}kl)$ leave $(abkl)$ unchanged.

This is a determinantal permutation of a, b. We extend the operation to any number μ of such symbols, $a, b, c, \ldots$, the dotted letters undergoing all interchanges, as in the development of a determinant. Thus

$$\dot{a}_\delta \dot{b}_\epsilon \dot{c}_\zeta \text{ denotes the 6 terms } a_\delta b_\epsilon c_\zeta - a_\delta c_\epsilon b_\zeta + \ldots,$$

while $\quad (\dot{a}\dot{b}kl)\,\dot{c}_\delta \quad$,, $\quad$ 3 $\quad$,, $\quad (abkl)\,c_\delta + (bckl)\,a_\delta + (cakl)\,b_\delta.$

This allows a very concise notation for the symbolic identities. In fact,

if $\mu = 2$, we have the identity $(\dot{a}kmn)\,\dot{b}_\delta = (ab\dot{m}\dot{n})\,\dot{k}_\delta$; (IV)

,, $\mu = 3$, ,, , besides this, $(\dot{a}kmn)\,\dot{b}_\delta \dot{c}_\epsilon = (abc\dot{n})\,\dot{k}_\delta \dot{m}_\epsilon$, (V)

with its 6 terms on each side;

if $\mu = 4$, we have $\dot{a}_\delta \dot{b}_\epsilon \dot{c}_\zeta \dot{d}_\eta = (abcd)(\delta\epsilon\zeta\eta)$

$$= (abcd)\,\dot{d}_\epsilon \dot{d}'_\zeta \dot{d}''_\eta, \text{ where } \delta = (dd'd''), \quad \text{(VI)}$$

$$(\dot{a}\dot{b}mn)\,\dot{c}_\delta \dot{d}_\epsilon = (abcd)\,\dot{m}_\delta \dot{n}_\epsilon, \quad \text{(VII)}$$

$$(\dot{a}\dot{b}kl)(\dot{c}\dot{d}mn) = (abcd)(klmn); \quad \text{(VIII)}$$

if $\mu > 4$, the result is zero, *e.g.*

$$(\dot{a}\dot{b}\dot{c}\dot{d})\,\dot{e}_x = 0.$$

Each identity can easily be verified,* being deduced from Identity (I).

II.

Application of these Identities to Two Quadrics.

5. Suppose that in the above identities, $a, b, c, \ldots$ refer to one quadric f. Complementary to these there will be another set of symbols $a, b, c, \ldots$ in each symbolic product P representing an invariant of the system. Now let these μ complementary symbols lie in one bracket, g, where $\mu \not> 4$. We may then typify the product P as

$$(abc \ldots kd_1 d_2 \ldots d_{4-\mu})[a, b, c, \ldots, k],$$

Now permute $[a, b, c, \ldots, k]$ determinantally. Then from one or other of the identities, we have

$$(abc \ldots kd_1 d_2 \ldots d_{4-\mu})[\dot{a}, \dot{b}, \dot{c}, \ldots, \dot{k}] = \Sigma g g_1 g_2 \ldots,$$

where g_1 is a bracket containing $(abc \ldots k \ldots)$, and where g denotes the initial bracket on the left. But each term of the series on the left may immediately be reduced back to $+P$, since all the symbols $a, b, \ldots, k$ are equivalent. We may therefore say that

$$P \equiv 0 \mod gg_1,$$

where both g and g_1 contain $(abc \ldots k)$.

Moreover we may select for g_1, any bracket whatever, except g, of the original form P. What happens is that the contents of g_1 are diverted to the positions in P which were originally held by the μ symbols $a, b, \ldots, k$.

Suppose now $\mu = 4$. Then gg_1 is equal to the invariant $(abcd)^2$. Hence if P contain

$$(aa_1 a_2 a_3) \quad \text{or} \quad (bb_1 b_2 b_3),$$

it is reducible. (2)

* For a formal proof in the general case, see the author's paper, "Quadratics in n Variables", pp. 201-4, *Camb. Phil. Trans.*, Vol. 21, No. 8.

The Characters c_1, c_2, c_{12}, ...

6. Let $c_1 =$ the degree of P in the coefficients of f,

$c_2 =$,, ,, f',

$c_{1\nu} =$ the total of factor pairs gg_1 containing $a_1 a_2 \ldots a_\nu$,

$c_{2\nu} =$,, ,, $b_1 b_2 \ldots b_\nu$.

It follows that, if $c_{14} > 0$ or $c_{24} > 0$, P is reducible.

Then we consider the forms P in the following order:—

1. P_1 comes before P_2 if c_1 or c_2 in P_1 is less than in P_2, while the other c_2 or c_1 is not greater.

2. If P_1 and P_2 have the same c_1 and c_2, take P_1 before P_2 if c_{13} is greater in P_1, while c_{23} is not less; or *vice versa* for c_{13}, c_{23}.

3. If c_1, c_2, c_{13}, c_{23} are the same, take P_1 before P_2, if c_{12} is greater in P_1, while c_{22} is not less; or *vice versa* for c_{12}, c_{22}.

The Modular Notation.

7. If P contains a factor g which implies reducibility, we write either $P \equiv 0$, or $g \equiv 0$, indifferently.

There may be groups of equivalent forms $P_1, P_2, P_3, \ldots$, which are such that each form of the group differs from another by reducible terms. Then we write

$$P_1 \equiv P_2 \equiv P_3 \ldots .$$

The Contracted Symbolic Notation.

8. Let α denote three equivalent symbols aa_1a_2 bracketed twice.

β ,, ,, bb_1b_2 ,,

A two ,, aa_1 ,,

B ,, ,, bb_1 ,,

Then c_{13} denotes the number of different symbols α in the form. And it is clear from § 5 that

$$a_\alpha \equiv 0, \quad b_\beta \equiv 0. \tag{3}$$

The Symbols α, β.

9. If P contains $(a_1 a_2 a_3 u)(a_i a_j a_k v)$, where $i, j, k = 1, 2, 3$ or any other suffix, P is reducible.

For, if the complementary a_1 is absent from the second bracket g_2, by using the process $(\dot{g}_2, \dot{a}_1)$ of § 3, we express this as 3 terms

$$(a_1 a_2 a_3 u)(a_1 a_j a_k v) \bmod (a_i a_j a_k a_1), \text{ i.e. } c_{14}.$$

Repeating the process, twice if necessary, we obtain

$$(a_1 a_2 a_3 u)(a_1 a_2 a_3 v) \bmod c_{14},$$

i.e.

$$u_\alpha v_\alpha .$$

As this increases c_{18}, with $c_1 c_2$ unaltered, P is therefore reducible. Further, it implies that all the symbols a, a_1, a_2 of P are interchangeable in every way whatever. We may therefore drop the suffix 1, 2, ..., and call them all α. Similarly for β.

In the above, u and v denote any symbol.

The Factor $(a_1 a_2 a_3 d) \equiv 0$.

10. If $a_1 a_2 a_3$ are not twice bracketed, their complements are found either in brackets or in factors a_{1_β}, a_{1_x}. If the tag suffix is the same in two cases, P is zero.

Hence there is at least one bracket g_1. By a determinantal permutation we may bracket the three complements $a_1 a_2 a_3$ in g_1. This expresses P in terms of products where α can be written twice for $(a_1 a_2 a_3)$. So P is reduced. But we must include a new type of bracket defined as

$$(\alpha\beta p) = (u_\alpha v_\beta - v_\alpha u_\beta) = \dot{u}_\alpha \dot{v}_\beta. \tag{4}$$

For if in the above reduction g_1 contained p or (uv), then necessarily uv are separated when $a_1 a_2 a_3$ all come into g_1. Let them stand in $u_\delta v_{\delta'}$. Since uv started from one bracket g_1, they will now be found in the combination

$$\dot{u}_\delta \dot{v}_{\delta'},$$

where δ, δ' stand for α, β, x or (dp), $dd_1 u$, $dd_1 d_2$. In these cases uv may be rebracketed in one or other factor u_δ or $v_{\delta'}$, thereby regaining the lost symbol p, unless both δ, δ' are α, β or x. But u_x, v_x are both 0, which only leaves the new type $(\alpha\beta p)$.

To sum up the present results, we must consider forms made of factors

$$u_\alpha,\ u_\beta,\ a_x,\ b_x,\ a_\beta,\ b_\alpha,\ (\alpha\beta p),\ (abp),\ (aa_1 p),\ (aa_1 bu),\ (bb_1 p),\ (abb_1 u),\ (aa_1 bb_1). \tag{5}$$

Factors g with two Symbols a or b.

11. The product $(aa_1 kl)(a_i a_j mn)$, where $k, l, m, n = u, b$, or p, is reducible. As before, by permuting the complements of a, a_1, we express this in terms of

$$(aa_1 kl)(aa_1 mn),$$

and terms with more than two symbols a in the second bracket. We write this as

$$(Akl)(Amn).$$

Thus if there are an even number of brackets each with two a's, we can pair off symbols $AA, A_1A_1, \ldots$ to fit them. Clearly the permutation $\dot{A}, \dot{A}_1$ would now shew that any pair of A's can be interchanged. The same can be done for B. We have then possibly one pair (aa_1) left over, which we do not call A until its complements are explicitly bracketed. As for the other A's, we may drop their suffixes. We deal with (bb_1) in like manner.

We have now to consider these factors

$$(aa_1 bb_1),\ (abb_1 u),\ (baa_1 u),\ (abp),\ (AB),\ (aBu),\ (bAu),\ (Ap),\ (Bp),\ a_\beta,\ \text{etc.} \quad (6)$$

The Factor $(aa_1 ij)$.

12. If P contain this odd factor $(aa_1 ij)$, the complements a, a_1 must occur in factors of type

$$(abb_1 u),\ (abp),\ (aBu),\ a_\beta,\ a_x, \text{ or } g_1, g_2, g_3, g_4, g_5 \text{ say}.$$

If g_1 occurs, we may bracket the complements a, a_1 in g_1 and reduce P. The same process avails if a, a_1 occur in two factors like g_2, or $g_2 g_5$ or $g_3 g_4$, or $g_3 g_3$. Two factors $g_4 g_4$ or $g_5 g_5$ are clearly reducible. We have then left over

$$g_2 g_3,\ g_2 g_4,\ g_3 g_5,\ g_4 g_5.$$

Now
$$\begin{aligned} g_2 g_3 &= (abp)(a_1 Bu) \\ &= \tfrac{1}{2}(\dot{a}bp)(\dot{a}_1 Bu) \quad \text{(since } \overline{aa_1} \text{ is in another bracket)} \\ &= \tfrac{1}{2}\Sigma(b''bp)(Ab'u) \quad (\text{if } B = b'b'') \\ &= \tfrac{1}{2}\Sigma(Bp)(Abu) \quad (\S\ 11) \\ &= 0 \quad (\text{for } c_{12} \text{ is increased}). \end{aligned}$$

We have left over $g_2 g_4$, $g_3 g_5$, $g_4 g_5$, *i.e.* forms containing

$$(aa_1 ij))(\dot{a}bp)\,\dot{a}_{1_\beta}, \quad (aa_1 ij)(\dot{a}Bu)\,\dot{a}_{1_x}, \quad (aa_1 ij)\,\dot{a}_\beta \dot{a}_{1_x}.$$

Let us denote these by

$$\left.\begin{aligned} F_1 &= -(\dot{a}bp)\,\dot{a}_{1_\beta}, \\ (AB)' &= (\dot{a}Bu)\,\dot{a}_{1_x} = -(\dot{b}Au)\dot{b}_{1_x} \text{ [by (I) of § 3]}, \\ \text{and} \qquad (A\beta x) &= \dot{a}_\beta \dot{a}_{1_x}. \end{aligned}\right\} \qquad (7)$$

13. Similarly for an odd factor (bb_1ij), only now we have to add to the five possibilities corresponding to $g_1, g_2, \ldots, g_5$, a sixth

$$g_6 = F_1 = -(\dot{a}bp)\,\dot{a}_{1_\beta}.$$

If b, b_1 stand in F_1 and another p-bracket, or F_1 and b_{1_x}, we may at once bracket bb_1 in F_1 and reduce it. The other two cases are

$$F_1(b_1Au) \quad \text{and} \quad F_1 b_{1_a}.$$

Of these, the first is reduced as in g_2g_3 just above. The second is substantially

$$\begin{aligned} P &= (aa_1ij)(bb_1kl)(abp)\,a_{1_\beta} b_{1_a} \quad (\text{where } p = uv) \\ &= \tfrac{1}{2} \quad ,, \quad ,, \quad (a\dot{b}p)\,a_{1_\beta}\dot{b}_{1_a} \\ &\equiv \tfrac{1}{2} \quad ,, \quad ,, \quad (abb_1\dot{v})\,a_{1_\beta}\dot{u}_a \quad \text{[by (I) of § 3]} \\ &\equiv \tfrac{1}{4} \quad ,, \quad (Bkl)(\dot{a}B\dot{v})\,\dot{a}_{1_\beta}\dot{u}_a \end{aligned}$$

(where the dots go in two pairs, aa_1 being permuted separately from uv)

$$\begin{aligned} &\equiv \tfrac{1}{4} \quad ,, \quad ,, \quad (AB)\,\dot{v}_\beta \dot{u}_a \\ &\equiv \tfrac{1}{4}(Aij)(Bkl)(AB)(a\beta p) \\ &\equiv 0. \end{aligned} \qquad (8)$$

14. This completes the preliminary investigation. All the pairs aa_1, bb_1 are replaced by A, B.

Any form P of the system is composed of factors of types (ϕ), $(\phi\psi)$, and F; where

$$(\phi) = a_x,\ b_x,\ u_a,\ u_\beta,\ (Ap),\ (Bp),$$

$$(\phi\psi) = (abp),\ a_a,\ a_\beta,\ b_a,\ b_\beta,\ (aBu),\ (bAu),\ (AB),$$
$$(AB)',\ (A\beta x),\ (Bax),\ (a\beta p).$$

$$F_1 = (bap)\,a'_\beta - (ba'p)\,a_\beta = (b\dot{a}p)\,\dot{a}'_\beta = (A\beta\overline{bp}),$$

$$F_2 = (abp)\,b'_a - (ab'p)\,b_a = (a\dot{b}p)\,\dot{b}'_a = (Ba\overline{ap}).$$

Moreover, each symbol can be interchanged with any equivalent symbol, a with a', A with A', and so on.

Gordan proceeds to arrange the system in 6 classes J.

$J^{(1)}$	contains the squares of these factors including $(AB)(AB)'$	....	21	forms
$J^{(2)}$	,,	only factors $(\phi\psi)$	23	,,
$J^{(3)}$	,,	,, $(\phi\psi)$, (ϕ)	186	,,
$J^{(4)}$	,,	F_1 but not F_2	134	,,
$J^{(5)}$	,,	F_2 ,, F_1	134	,,
$J^{(6)}$	,,	F_1F_2	82	,,
		Total......	580	,,

We shall prove that these sets contain at most

$$21+7+73+12+12+0 = 125$$

irreducible forms.

III.

The 1, 2, 3 *Notation.*

15. To discriminate between forms which are not equivalent it is well to change our notation, using a device employed in dealing with systems of ternary quadratics.*

Let every form P be rewritten according to the following notation :—

for a_x write 1_ξ or ${}_\xi 1$, for b_x write ω_1 or ${}_1\omega$,

,, (Ap) ,, 2_ξ ,, ${}_\xi 2$, ,, (Bp) ,, ω_2 ,, ${}_2\omega$,

,, u_α ,, 3_ξ ,, ${}_\xi 3$, ,, u_β ,, ω_3 ,, ${}_3\omega$;

for (abp) write 1_1 or ${}_1 1$,

,, (aBu) ,, 1_2 ,, ${}_2 1$,

,, (AB) ,, 2_2 ,, ${}_2 2$,

,, $(AB)'$,, $2'_2$,, ${}_2 2'$,

,, $(\alpha\beta p)$,, 3_3 ,, ${}_3 3$,

and so on.

Thus 1, 2, 3 denote a, A, α, while 1, 2, 3 used as suffices denote b, B, β, respectively.

* Cf. Turnbull, "Ternary Quadratic Types", *Proc. London Math. Soc.*, Ser. 2, Vol. 9, p. 83.

The factors F_1 and F_2 are however retained.

Let the symbols $i, j, k, \ldots$ denote 1, 2, 3 indiscriminately.

*Chains and Tags.**

Further, let such a product as

$$i_j \,.\, k_j \,.\, k_l \,.\, m_l \ldots$$

be abbreviated to

$$\begin{pmatrix} \ldots i & & k & & m \ldots \\ & j & & l & \end{pmatrix}.$$

If this expression start with i and end with ξ or ω, it is called a *tag*. We denote it by (i). Otherwise it is a *chain*, when both the extreme symbols are of type ξ or ω, or else neither ξ, ω appear at all. We denote a chain by its extreme symbols

$$(i, j), \quad \begin{pmatrix} i, & \\ & j \end{pmatrix}, \quad \begin{pmatrix} & \omega \\ \xi, & \end{pmatrix}, \quad \text{etc.}$$

Properties of Chains and Tags.

16. (1) In one chain or tag all the upper symbols i, k, m must differ, except possibly the two extremes of a chain.

(2) So also must all the lower symbols j, l differ. Otherwise the form is reducible. Thus the chain

$$\begin{pmatrix} & 1 & & 3 & & 1 & & \omega \\ \xi & & 2 & & 1 & & 3 & \end{pmatrix} \text{ contains the factor } \begin{pmatrix} 1 & & 3 & & 1 \\ & 2 & & 1 & \end{pmatrix}.$$

(3) In a tag (i), each symbol $j, k, \ldots$ is paired, but i only occurs once.

(4) In a chain (ξ, ξ) or (ω, ω) each symbol is paired.

(5) In a chain (ξ, ω), (i, j), or $\begin{pmatrix} i, & \\ & j \end{pmatrix}$, each symbol except the two extremes is paired.

(6) The symbols ξ, ω can only occur as extreme symbols.

* See note on previous page.

The Set of Forms $J^{(1)}$, $J^{(2)}$, $J^{(3)}$.

17. It is obvious that all forms P which are made up of symbols i_ξ, ${}_j\omega$, k_l, but which have no factor F_1 or F_2, are only of two kinds, viz. :—

$$\text{(I) chains such as } \binom{i\,\dots}{\xi\qquad\xi},\ \binom{i\,\dots\,\omega}{\xi\qquad},\ \binom{\omega\,\dots\,\omega}{i};$$

$$\text{(II) chains such as } \binom{i\ k\,\dots\,i}{j\qquad},\ \binom{k\,\dots\,j}{j\qquad}.$$

For P consists of chains and tags only, and each extreme symbol of a chain or tag must be paired with some other extreme symbol elsewhere. The resultant of such can only be one chain.

Reducibility.

18. If by any process we can transform P to terms involving (1) either less symbols 1, 2, 3, or (2) more symbols ξ, ω, then P is reducible.

Proof.—(1) The number of pairs of symbols 1, 2, or 3 in P is the grade of P. To diminish the number of these symbols is to lower the grade. All we have to do is to deal with the forms in ascending value of grade.

(2) If we consider the chains (I) before chains (II) of the same grade, then P is reduced if we increase the number of ξ, ω symbols in P.

IV.

Formulae of Reduction.

19. The above principles will now be illustrated by establishing the following formulæ :—

$$\text{(A)}\quad 2'_2 \,.\, 3_1 \equiv 2_1 \,.\, 3_2 + 2_2 \,.\, 3_\xi \,.\, {}_1\omega,$$

$$2'_2 \,.\, 1_3 \equiv -\,2_3 \,.\, 1_2 - 2_2 \,.\, 1_\xi \,.\, {}_3\omega,$$

$$\text{(B)}\quad 2'_2 \,.\, 1_1 \equiv -\,2_1 \,.\, 1_\xi \,.\, {}_2\omega - 1_2 \,.\, {}_1\omega \,.\, 2_\xi,$$

$$2'_2 \,.\, 3_3 \equiv +\,2_3 \,.\, 3_\xi \,.\, {}_2\omega + 3_2 \,.\, 2_\xi \,.\, {}_3\omega,$$

$$\text{(C)}\quad F_2 \,.\, {}_3\omega \equiv 3_3 \,.\, 1_2 + 1_3 \,.\, 3_\xi \,.\, {}_2\omega,$$

$$F_1 \,.\, 3_\xi \equiv -\,3_3 \,.\, 2_1 + 3_1 \,.\, 2_\xi \,.\, {}_3\omega,$$

(D) $F_2 \,.\, {}_1\omega \equiv -1_1 \,.\, 3_2 - 3_1 \,.\, 1_\xi \,.\, {}_2\omega,$

$F_1 \,.\, 1_\xi \equiv 1_1 \,.\, \dot{2}_3 - 1_3 \,.\, 2_\xi \,.\, {}_1\omega,$

(E) $F_2 \,.\, 2_1 \equiv 2_\xi \,.\, 3_1 \,.\, 1_2 + 1_1 \,.\, 3_\xi \,.\, 2_2,$

$F_1 \,.\, 1_2 \equiv \omega_2 \,.\, 1_3 \,.\, 2_1 - 1_1 \,.\, \omega_3 \,.\, 2_2,$

(F) $F_2 \,.\, 2_3 \equiv -2_2 \,.\, 3_3 \,.\, 1_\xi - 2_\xi \,.\, 1_3 \,.\, 3_2,$

$F_1 \,.\, 3_2 \equiv 2_2 \,.\, 3_3 \,.\, \omega_1 - \omega_2 \,.\, 3_1 \,.\, 2_3,$

(G) $2_1 \,.\, 1_2 \,.\, 3_3 \equiv 0,$

(H) $2_3 \,.\, 3_2 \,.\, 1_1 \equiv 0,$

(J) $F_1 F_2 \equiv -2_2 \,.\, 3_3 \,.\, 1_1 + 2_\xi \,.\, {}_2\omega \,.\, 1_3 \,.\, 3_1,$

(K) $F_2 \,.\, 2'_2 \equiv +2_\xi \,.\, 3_2 \,.\, 1_2 - 2_2 \,.\, 3_\xi \,.\, {}_2\omega \,.\, 1_\xi,$

$F_1 \,.\, 2'_2 \equiv 0,$

(L) $2_1 \,.\, 3_3 \,.\, 1_\xi + 1_1 \,.\, 3_\xi \,.\, 2_3 \equiv 2_\xi \,.\, 3_\xi \,.\, 1_3 \,.\, {}_1\omega + 1_\xi \,.\, {}_3\omega \,.\, 3_1 \,.\, 2_\xi,$

(M) $1_2 \,.\, 3_3 \,.\, {}_1\omega + 1_1 \,.\, {}_3\omega \,.\, 3_2 \equiv 0.$

Proofs of the Formulae.

(A).

20. $$2_1 \,.\, 3_2 = (Abu)(Bax) = (Abu)\, \dot{b}'_a \dot{b}''_x \text{ (say)}$$

$$\equiv (Ab\dot{b}')u_a \dot{b}''_x + (A\dot{b}'u)\, b_a \dot{b}''_x \text{ (mod } a_a) \quad \text{(by 1d. 1)}$$

$$\equiv (Ab''b')\, u_a b_x + (AB)'\, b_a$$

(and terms with $bb'b''$ bracketed, which are reducible by § 10)

$$\equiv -(AB)\, u_a \,.\, b_x + (AB)'\, b_x$$

$$\equiv -2_2 \,.\, 3_\xi \,.\, {}_1\omega + 2'_2 \,.\, 3_1.$$

Interchange the symbols of the two quadratics f and f' throughout, and the second formula is established. The signs are changed because

$$(BA)' = -(AB)'.$$

(B).

21. $$2'_2 \,.\, 1_1 = (aa'\dot{b}u)\,\dot{b}'_x\,(a''b''p)\,;$$

and using 1d. II, remembering $(up) = 0$, this

$$= (aa'b''u)\,\dot{b}'_x\,(a''\dot{b}p) + (\dot{a}b''\dot{b}u)\,\dot{b}'_x\,(a''\dot{a}'p)$$

(where a is only permuted with a' and b with b')

$$\equiv (Ab''u)\,a''_x\,(b'bp) + (a''b''\dot{b}u)\,\dot{b}'_x\,(aa'p)$$

$$\equiv \quad ,, \quad + (a''b'bu)\,b''_x\,(aa'p)$$

$$\equiv -2_1 \,.\, 1_\xi \,.\, {}_2\omega - 1_2 \,.\, {}_1\omega \,.\, 2_\xi.$$

The second formula (B) follows from the first by the principle of duality, which is effected by interchanging u with x, a with α, and b with β: or simply 1 with 3. But we give a direct proof: we have

$$2'_2 \,.\, 3_3 = -(\dot{a}Bu)\,\dot{a}'_x\,(u_\alpha v_\beta - u_\beta v_\alpha).$$

Using 1d. I upon (aBu), v_β, this product

$$\equiv -(vBu)\,\dot{a}'_x u_\alpha \dot{a}_\beta - (\dot{a}Bv)\,\dot{a}'_x u_\alpha u_\beta + (\dot{a}Bu)\,\dot{a}'_x u_\beta v_\alpha.$$

The first term $\equiv (Bp)\,\dot{a}'_x \dot{a}_\beta u_\alpha$ [since $p = (uv)$],

$$\equiv (Bp)(A\beta x)\,u_\alpha \quad (i.e.\ {}_2\omega \,.\, 3_\xi \,.\, 2_3).$$

The other two terms $= -(\dot{a}B\dot{v})\,\dot{a}'_x \dot{u}_\alpha u_\beta$ (as they stand)

$$\equiv -(aa'\dot{b}'\dot{v})\,\dot{b}_x \dot{u}_\alpha u_\beta \quad (\text{since } v_x = 0)\,;$$

and now using 1d. I on u_α and this bracket, this

$$\equiv -(aa'uv)\,\dot{b}_x \dot{b}'_\alpha u_\beta$$

$$\equiv (Ap)(B\alpha x)\,u_\beta \quad (i.e.\ 2_\xi \,.\, 3_2 \,.\, {}_3\omega).$$

Hence $$2'_2 \,.\, 3_3 \equiv {}_2\omega \,.\, 3_\xi \,.\, 2_3 + 2_\xi \,.\, 3_2 \,.\, {}_3\omega.$$

(C).

22. $$F_2 \,.\, {}_3\omega = (a\dot{b}p)\,\dot{b}'_\alpha \,.\, u_\beta = (a\dot{b}uv)\,\dot{b}'_\alpha \,.\, u_\beta$$

$$\equiv (abb'v)\,u_\alpha u_\beta - (abb'u)\,v_\alpha u_\beta.$$

Now using 1d. I upon the first term, this

$$\equiv (ubb'v)\, u_\alpha a_\beta + (abb'u)\, u_\alpha v_\beta - (abb'u)\, v_\alpha u_\beta$$

$$\equiv (Bp)\, u_\alpha a_\beta + (aBu)(a\beta p)$$

$$\equiv {}_2\omega \,.\, 3_\xi \,.\, 1_3 + 1_2 \,.\, 3_3.$$

Interchange f and f', and the second formula follows.

(D).

These are the reciprocals of (C):

$$F_2 \,.\, {}_1\omega = (a\dot{b}p)\, \dot{b}'_\alpha b''_x$$

$$\equiv (b''\dot{b}p)\, \dot{b}'_\alpha a_x + (ab''p)\, \dot{b}'_\alpha \dot{b}_x \quad \text{(by 1d. III)}$$

$$\equiv (b'bp)\, b''_\alpha a_x + (ab''p)(Bxa) \quad [\text{mod } (bb'b'')]$$

$$\equiv -(Bp)\, b''_\alpha \,.\, a_x - (ab''p)(Bax)$$

$$\equiv -\omega_2 \,.\, 3_1 \,.\, 1_\xi - 1_1 \,.\, 3_2.$$

The second formula is analogous to this.

(E).

23. $$F_2 \,.\, 2_1 = (a\dot{b}p)\, \dot{b}'_\alpha (b''Au)$$

$$\equiv (a\dot{b}p)\, b''_\alpha (\dot{b}'Au) + (a\dot{b}p)\, u_\alpha (b''A\dot{b}')$$

$$\equiv (a\dot{a}'p)\, b''_\alpha (b'b\dot{a}''u) + (ab''p)\, u_\alpha (bAb')$$

[on collecting bb' into the second bracket and omitting reducing factors $(b''B)$ or (aA), A being $(a'a'')$]

$$\equiv (a''a'p)\, b''_\alpha (b'bau) + (ab''p)\, u_\alpha (Abb')$$

$$\equiv (Ap)\, b''_\alpha (aBu) + (ab''p)\, u_\alpha (AB)$$

$$\equiv 2_\xi \,.\, 3_1 \,.\, 1_2 + 1_1 \,.\, 3_\xi \,.\, 2_2.$$

Similarly for $F_1 \,.\, 1_2$.

(F).

These are the reciprocals of (E):

$$F_2 \,.\, 2_3 = (a\dot{b}p)\,\dot{b}'_\alpha \,.\, \dot{a}'_\beta \dot{a}''_x$$

$$\equiv (\dot{a}''\dot{b}p)\,\dot{b}'_\alpha \dot{a}'_\beta a_x + (a\dot{a}''p)\,\dot{b}'_\alpha \dot{a}'_\beta \dot{b}_x \quad \text{(by 1d. III)}$$

$$\equiv -(aa'bb')(\alpha\beta p)\,a_x + (a'a''p)\,\dot{b}'_\alpha a_\beta \dot{b}_x \quad [\text{cf. § 13, (8)}]$$

$$\equiv -2_2 \,.\, 3_3 \,.\, 1_\xi - 2_\xi \,.\, 3_2 \,.\, 1_3.$$

Similarly for $F_1 \,.\, 3_2$.

(G).

24. Multiply the first formulæ (C) and (E) by 2_1 and ${}_3\omega$ respectively and equate results. This expresses $2_1 \,.\, 1_2 \,.\, 3_3$ in terms each involving ξ, ω symbols, and so it is reducible (§ 18).

Likewise (H) follows from formulæ (D) and (F).

(J).

25. Here

$$F_1 F_2 = (a\dot{b}p)\,\dot{b}'_\alpha\,(b''\dot{a}'p)\,\dot{a}''_\beta$$

$$\equiv (\dot{a}'\dot{b}p)\,\dot{b}'_\alpha\,(b''ap)\,\dot{a}''_\beta + (a\dot{a}'p)\,\dot{b}'_\alpha\,(b''\dot{b}p)\,\dot{a}''_\beta$$

$$\equiv -(AB)(\alpha\beta p)(abp) + (Ap)(Bp)\,b_\alpha a_\beta \quad [\text{as in formula (F)}]$$

$$\equiv -2_2 \,.\, 3_3 \,.\, 1_1 + 2_\xi \,.\, 2\omega \,.\, 1_3 \,.\, 3_1.$$

(K).

26. $$F_2 \,.\, 2'_2 \equiv (a\dot{b}p)\,\dot{b}'_\alpha\,(\dot{a}''Bu)\,\dot{a}'_x$$

$$\equiv (\dot{a}''\dot{b}p)\,\dot{b}'_\alpha\,(aBu)\,\dot{a}'_x + (a\dot{b}u\dot{a}'')\,\dot{b}'_\alpha\,(vBu)\,\dot{a}'_x \quad \text{(by 1d. I)}$$

$$\equiv (a''a'p)\,\dot{b}'_\alpha\,(aBu)\,\dot{b}_x + (abb'\dot{a}'')\,u_\alpha\,(Bvu)\,\dot{a}'$$

$$\equiv 2_\xi \,.\, 1_2 \,.\, 3_2 + (a'bb'a'')\,u_\alpha\,(Bvu)\,a_x$$

$$\equiv 2_\xi \,.\, 1_2 \,.\, 3_2 - 2_3 \,.\, 3_\xi \,.\, {}_2\omega \,.\, 1_x.$$

Similarly for $F_1 \,.\, 2'_2$.

(L).

27. This follows by combining formulæ (C) and (D), eliminating F_1 as in the case of (H). In formula (L) the two right members contain more ξ, ω factors than the left. They are therefore reducible. Lastly (M) follows by eliminating F_2 from (C) and (D).

V.

The Complete System.

28. We take the forms in the following order :—

(1) forms with no factor $2_2'$, F_1, F_2,

(2) ,, $2_2'$ but no F_1, F_2,

(3) ,, F_1, F_2.

Let these be denoted by $K^{(1)}$, $K^{(2)}$, $K^{(3)}$ respectively.

The System $K^{(1)}$.

We enumerate these in ascending grade.

(i) Grade 1 has six forms

$$f,\ f',\ (Ap)^2,\ (Bp)^2,\ u_\alpha^2,\ u_\beta^2.$$

(ii) Grade 2 has nine forms

$$\begin{pmatrix} i & \omega \\ \xi & j \end{pmatrix},$$

i, j being 1, 2, 3 in all their arrangements; and nine forms

$$\begin{pmatrix} i \\ j \end{pmatrix}^2.$$

(iii) Grade 3 has eighteen forms

$$\begin{pmatrix} & i & k & \\ \xi & j & & \xi \end{pmatrix}, \qquad \begin{pmatrix} \omega & j & & \omega \\ & i & k & \end{pmatrix},$$

where $i \neq k$, but i, j, k are all other possible permutations of 1, 2, 3.

Note that these forms are exactly the same if i, k are interchanged. No obvious reductions exist for any of these forms.

(iv) Grade 4 has two types

$$\begin{pmatrix} i & k & \omega \\ \xi & j & l \end{pmatrix} \text{ and } \begin{pmatrix} i & k & i \\ & j & l \end{pmatrix},$$

where i, j, k, l must contain a repetition.

(a) Let i, j, k, l include a repetition of 2,

(β) " " " 1 or 3.

(a).

29. We may have

$$\begin{pmatrix} i & 2 & \omega \\ \xi & j & 2 \end{pmatrix}, \quad \begin{pmatrix} i & 2 & \omega \\ \xi & 2 & j \end{pmatrix}, \quad \begin{pmatrix} 2 & i & \omega \\ \xi & 2 & j \end{pmatrix}, \quad \begin{pmatrix} 2 & j & \omega \\ \xi & i & 2 \end{pmatrix},$$

where i, j denote 1, 3.

The second and fourth of these reduce by formula (A), if $i \neq j$. We must omit these cases.

Proof.—The chain $\dots 1_2 2_3 \dots = 1_2 \,.\, 2_2 \,.\, 2_3$ (in full)

$$\equiv 0 \;(\text{modd } 2_2 \times 2_2,\; 2_2 \times 2'_2) \;[\text{by (A)}].$$

(β).

Besides $\begin{pmatrix} i & j & \omega \\ \xi & j & i \end{pmatrix}$, where 2 does not appear, we may have

$$\begin{pmatrix} 2 & i & \omega \\ \xi & i & j \end{pmatrix}, \quad \begin{pmatrix} 2 & j & \omega \\ \xi & i & j \end{pmatrix}, \quad \begin{pmatrix} i & 2 & \omega \\ \xi & i & j \end{pmatrix}, \quad \begin{pmatrix} j & 2 & \omega \\ \xi & i & j \end{pmatrix},$$

and four similar forms with 2 as a suffix.

By squaring formulæ (C) and (D) we reduce the second of these four forms. For example,

$$F_1^2 \begin{pmatrix} 3 \\ \xi \end{pmatrix}^2 \equiv \begin{pmatrix} 3 \\ 3 \end{pmatrix}^2 \begin{pmatrix} 2 \\ 1 \end{pmatrix}^2 + \begin{pmatrix} 3 \\ 1 \end{pmatrix} \begin{pmatrix} 2 \\ \xi \end{pmatrix}^2 \begin{pmatrix} \omega \\ 3 \end{pmatrix}^2 - 2 \begin{pmatrix} 2 & 3 & \omega \\ \xi & 1 & 3 \end{pmatrix}.$$

Again, the third form

$$\begin{pmatrix} i & 2 & \omega \\ \xi & i & j \end{pmatrix} = i_\xi \,.\, i_i \,.\, 2_i \,.\, 2_j \,.\, \omega_j \equiv i_\xi \,.\, 2_i \,.\, \omega_j \,(\pm F_1 \,.\, i_\xi \pm i_j \,.\, 2_\xi \,.\, {}_i\omega),$$

by (C) or (D), and each term has a reducing factor $\begin{pmatrix} i \\ \xi \end{pmatrix}^2$ or $\begin{pmatrix} i & \omega \\ \xi & j \end{pmatrix}$.

The first and fourth forms, however, do not reduce.

The second type, for Grade 4, viz. $\begin{pmatrix} i & k & i \\ j & & l \end{pmatrix}$, is unaltered by interchanging i and k or j and l, as is seen by writing it in full.

There are seven irreducible forms

$$\begin{pmatrix} 1 & 2 & 1 \\ 2 & & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & 1 \\ 1 & & 3 \end{pmatrix}, \quad \begin{pmatrix} 1 & 3 & 1 \\ 1 & & 2 \end{pmatrix}, \quad \begin{pmatrix} 1 & 3 & 1 \\ 1 & & 3 \end{pmatrix}, \quad \begin{pmatrix} 1 & 3 & 1 \\ 2 & & 3 \end{pmatrix}, \quad \begin{pmatrix} 3 & 2 & 3 \\ 1 & & 3 \end{pmatrix}, \quad \begin{pmatrix} 3 & 2 & 3 \\ 2 & & 3 \end{pmatrix},$$

omitting the form $\begin{pmatrix} i & 2 & i \\ 2 & & j \end{pmatrix}$, which reduces by formula (A).

Grade 5.

30. There are two sets of correlative forms

$$\begin{pmatrix} & i & j & k & \\ \xi & l & & m & \xi \end{pmatrix} \text{ and } \begin{pmatrix} \omega & l & m & \omega \\ & i & j & k & \end{pmatrix},$$

where i, j, k are 1, 2, 3 in some order.

(α) First take forms with 2_2 as a factor, and let i, j denote 1, 3.

There are two sorts, $\begin{pmatrix} & 2 & \dots & \\ \xi & 2 & & \xi \end{pmatrix}$ and $\begin{pmatrix} & i & 2 & \dots & \\ \xi & 2 & & & \xi \end{pmatrix}$, since the chain may be written backwards or forwards, $\begin{pmatrix} & i & j & k & \\ \xi & l & & m & \xi \end{pmatrix}$ being the same as

$$\begin{pmatrix} & k & j & i & \\ \xi & m & & l & \xi \end{pmatrix}.$$

Thus we have the four types

$$\begin{pmatrix} & 2 & i & j & \\ \xi & 2 & & i & \xi \end{pmatrix}, \quad \begin{pmatrix} & i & 2 & j & \\ \xi & 2 & & j & \xi \end{pmatrix}, \quad \begin{pmatrix} & 2 & i & j & \\ \xi & 2 & & j & \xi \end{pmatrix}, \quad \begin{pmatrix} & i & 2 & j & \\ \xi & 2 & & i & \xi \end{pmatrix}.$$

By squaring each of the formulæ (E), (F), we reduce the first of these four types.

The second reduces by formula (A).

The other two lead to four irreducible forms.

(β) Next take forms with no factor 2_2.

With 2 in the middle of the top row we may have

$$\begin{pmatrix} & 1 & 2 & 3 & \\ \xi & 3 & & 1 & \xi \end{pmatrix}, \quad \begin{pmatrix} & 1 & 2 & 3 & \\ \xi & 1 & & 3 & \xi \end{pmatrix},$$

of which the latter is reducible by formula (D), as it contains the combination $1_\xi \,.\, 1_1 \,.\, 2_3$.

With 2 at the end of the top row, we may have

$$\begin{pmatrix} 2 & i & j & \\ \xi & i & 2 & \xi \end{pmatrix}, \quad \begin{pmatrix} 2 & j & i & \\ \xi & i & 2 & \xi \end{pmatrix}, \quad \begin{pmatrix} 2 & i & j & \\ \xi & i & j & \xi \end{pmatrix}, \quad \begin{pmatrix} 2 & j & i & \\ \xi & i & j & \xi \end{pmatrix},$$

The first reduces by (C) or (D), as it contains $i_i \,.\, j_2$ and ${}_i2_\xi$. The second, containing, $2_i \,.\, j_2$ reduces by (A). The third, containing $2_i \,.\, j_j$ with j_ξ, reduces by (C) or (D). The fourth, representing two forms, involves the relation

$$\begin{pmatrix} 2 & 3 & 1 & \\ \xi & 1 & 3 & \xi \end{pmatrix} + \begin{pmatrix} 2 & 1 & 3 & \\ \xi & 3 & 1 & \xi \end{pmatrix} \equiv 0,$$

which is proved by multiplying formula (L) by $\xi^2 \,.\, 1^3 \,.\, 3^1$.

31. Summing up, Grade 5 has six forms ξ,

$$\begin{pmatrix} 2 & i & j & \\ \xi & 2 & j & \xi \end{pmatrix}, \quad \begin{pmatrix} i & 2 & j & \\ \xi & 2 & i & \xi \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & 3 & \\ \xi & 3 & 1 & \xi \end{pmatrix},$$

and *one* of

$$\begin{pmatrix} 2 & i & j & \\ \xi & j & i & \xi \end{pmatrix}.$$

Together with six ω forms, this makes twelve forms in all.

Grade 6.

32. There are two kinds

$$\text{I} \quad \begin{pmatrix} & i & j & k & \omega \\ \xi & i_1 & j_1 & k_1 & \end{pmatrix},$$

$$\text{II} \quad \begin{pmatrix} i & j & k & i \\ i_1 & j_1 & k_1 & \end{pmatrix},$$

i, j, k as well as i_1, j_1, k_1 being 1, 2, 3 in some order.

I $\begin{pmatrix} & i & j & k & \omega \\ \xi & i_1 & j_1 & k_1 & \end{pmatrix}$.—First let $j = 2$. If $i_1 = 2$, then, by formula (A), j_1 and i cannot differ, so that $j_1 = i$. This gives $\begin{pmatrix} & i & 2 & k & \omega \\ \xi & 2 & i & k & \end{pmatrix}$, which reduces at once by (G) or (H). Hence i_1 cannot be 2.

It follows that either $j_1 = 2$ or $k_1 = 2$.

If $j_1 = 2$, then by formula (A) i_1 and k cannot differ.

This gives the form $\begin{pmatrix} i & 2 & j & \omega \\ \xi & j & 2 & i \end{pmatrix}$. (9)

Again, if $k_1 = 2$, *i.e.* if we have

$$\begin{pmatrix} i & 2 & k & \omega \\ \xi & i_1 & j_1 & 2 \end{pmatrix},$$

then $j_1 = k$ by formula (A). This gives

$$\begin{pmatrix} i & 2 & k & \omega \\ \xi & i & k & 2 \end{pmatrix},$$

which reduces by (G) or (H).

This exhausts the case when $j = 2$. Correlatively it exhausts the case when $j_1 = 2$. Take now the remaining cases, when 2 is found among i, k as well as i_1, k_1:

$$(a)\ \begin{pmatrix} 2 & j & k & \omega \\ \xi & 2 & j_1 & k_1 \end{pmatrix}, \qquad (b)\ \begin{pmatrix} 2 & j & k & \omega \\ \xi & i_1 & j_1 & 2 \end{pmatrix},$$

$$(c)\ \begin{pmatrix} i & j & 2 & \omega \\ \xi & 2 & j_1 & k_1 \end{pmatrix}, \qquad (d)\ \begin{pmatrix} i & j & 2 & \omega \\ \xi & i_1 & j_1 & 2 \end{pmatrix}.$$

In these (*a*) and (*d*) are correlative cases.

(*a*) If $j = j_1$, then $k = k_1$, and the form reduces, as it contains $j_2 . k_k . {}_k\omega$, by the first of formulæ (C) or (D).

This leaves the case $\begin{pmatrix} 2 & j & k & \omega \\ \xi & 2 & k & j \end{pmatrix}$. Similarly for (*d*).

(*b*) If $j = j_1$, the form reduces by formula (G) or (H), as it contains $2_i . j_j . i_2$. Similarly for (*c*).

But if $j \neq j_1$, then $i_1 \neq k$, and the form reduces by formula (A), as it contains 2_i, k_2 $(i \neq k)$. Similarly for (*c*).

Altogether we have then $\begin{pmatrix} 2 & j & k & \omega \\ \xi & 2 & k & j \end{pmatrix}$ and $\begin{pmatrix} j & k & 2 & \omega \\ \xi & k & j & 2 \end{pmatrix}$. But multiplying formula (L) by $1_3 . 3_1 . 2_2 . {}_2\omega$, we get

$$\begin{pmatrix} 1 & 3 & 2 & \omega \\ \xi & 3 & 1 & 2 \end{pmatrix} + \begin{pmatrix} 3 & 1 & 2 & \omega \\ \xi & 1 & 3 & 2 \end{pmatrix} \equiv 0.$$

Similarly for $\begin{pmatrix} 2 & j & k & \omega \\ \xi & 2 & k & j \end{pmatrix}$.

Again $\begin{pmatrix} 2 & 1 & 3 & \omega \\ \xi & 2 & 3 & 1 \end{pmatrix} = 2_2 . 3_3 . 3_1 . 1_2 \{1_1 . 2_3 - F_1 . 1_\xi\}$, by (D_2).

The first term reduces by applying formula (A) to $1_2 . 2_3$. The second term contains $F_1 . 1_2$. Using (E_2), this

$$\equiv \begin{pmatrix} 1 & 3 & \omega \\ \xi & 1 & 3 \end{pmatrix} \begin{pmatrix} 2 & \\ & 2 \end{pmatrix}^2 - \begin{pmatrix} 1 & 3 & 2 & \omega \\ \xi & 3 & 1 & 2 \end{pmatrix}.$$

Hence forms (a), (d) are all equivalent. We retain one of them.

33. II $\begin{pmatrix} i & j & k & i \\ i_1 & j_1 & k_1 & \end{pmatrix}$.—Without loss of generality we may consider only

$$\begin{pmatrix} 1 & 2 & 3 & 1 \\ i & j & k & \end{pmatrix}.$$

(a) If $j = 2$, then i cannot be 1, by formula (A); nor can $i = 3$, for then $k = 1$, and the form contains $1_1 . 3_2 . 2_3$, which reduces by (H).

(b) If $i = 2$, then $j \neq 3$ by (A); nor can $j = 1$, by (G).

(c) If $k = 2$, then $j \neq 3$ by (H); nor can $i = 3$, by (A).

So all the forms reduce.

34. Summing up, Grade 6 has three forms

$$\begin{pmatrix} 1 & 2 & 3 & \omega \\ \xi & 3 & 2 & 1 \end{pmatrix}, \quad \begin{pmatrix} 3 & 2 & 1 & \omega \\ \xi & 1 & 2 & 3 \end{pmatrix},$$

and *one* of

$$\begin{pmatrix} 2 & i & j & \omega \\ \xi & 2 & j & i \end{pmatrix}.$$

There are no irreducible forms of higher grade, as all the upper or lower symbols must differ, so that three is the maximum number of each.

The System $K^{(2)}$.

35. This consists of chains involving $2'_2$.

By formulæ (A), (B), the combination $2'_2 . i_j$ is reducible if i, j denote 1, 1 or 1, 3 or 3, 1 or 3, 3; for the formulæ express these factors in terms of forms not involving $2'_2$ at all.

It follows that the only possible chains involve not more than two symbols i, j.

Again, the chain $\begin{pmatrix} i & 2' & \\ & 2 & j \end{pmatrix}$ is reducible by (A) if $i \neq j$. Hence we retain the following nine forms :—

$$\binom{2'\ \omega}{\xi\ 2},\quad \binom{2'}{2}^2,\quad \binom{2'\ 2}{2},\quad \binom{2'\ i}{\xi\ 2\ \xi},\quad \binom{\omega\ 2'\ \omega}{2\ i},\quad \binom{i\ 2'\ \omega}{\xi\ 2\ i},$$

$$i = 1, 3.$$

The System $K^{(3)}$.

36. This system, involving factors F_1 and F_2, consists of two parallel sets: $J^{(4)}$ involving F_1 alone and $J^{(5)}$ involving F_2 alone. For, by formula (J), the product $F_1 F_2$ is reducible.

We consider then forms $P = F'_1 M$, where M consists only of chains and tags.

Formulæ (C) to (K) shew that we need not consider terms with the factors

$$3_\xi,\ 1_\xi,\ 1_2,\ 3_2,\ 2'_2. \tag{10}$$

There are three cases, besides the isolated form F_1^2, viz.

$$\text{I}\quad F_1\binom{}{1}\binom{}{3}^{(2)},\qquad \text{II}\quad F_1\binom{}{1,\ 3}^{(2)},\qquad \text{III}\quad F_1\binom{\ \ 2}{i,}\binom{}{j},$$

where i, j are 1, 3.

I

$$F_1\binom{}{1}\binom{}{3}^{(2)}.$$

37. The presence of (2) implies that in the other tags no upper link is 2. Likewise no suffix can be 1 or 3.

Thus $$F_1\binom{i\ \ldots}{1}\binom{j\ \ldots}{3}\binom{2\ \ \ldots}{k}$$

can only be $$F_1 \cdot 1^\omega \cdot 3^\omega \cdot 2_\xi,\quad F_1 \cdot 1^\omega \cdot 3^\omega \binom{2\ \omega}{2},$$

having regard to the list (10) given above.

II

$$F_1\binom{}{1,\ 3}^{(2)}.$$

38. The chain is $\binom{i}{1\ 3}$ at most. For $\binom{i\ k\ \ldots}{1\ j\ \ \ 3}$ implies $j = 2$ and one of $i, k = 1$ or 3, and so is reducible. So we have three cases, $i = 1, 2,$ or 3.

$$\textit{Case 1.}\quad F_1\begin{pmatrix} & 1 & \\ 1 & & 3\end{pmatrix}^{(2)}$$

The tag (2) may be 2_ξ,

$$\begin{pmatrix}2 & \omega\\ i & \end{pmatrix},\quad i = 1 \text{ or } 2 \text{ or } 3;\ \text{or}\ \begin{pmatrix}2 & j & \dots\\ & i & \end{pmatrix}:$$

here $j \neq 2$, so take $j = 1$ or 3.

If $j = 1$, then i cannot be 1 or 3 because of $\begin{pmatrix} & 1 & \\ 1 & & 3\end{pmatrix}$. If $j = 1$, then i cannot be 2, since j_i would reduce by formula (E). Thus j cannot be 2 or 1. If however $j = 3$, then j_ξ is inadmissible. So the tag must contain another link k and be $\begin{pmatrix}2 & 3 & \dots\\ i & k & \end{pmatrix}$. Here $i, k \neq 2$, since 3_2 is a reducing factor. Thus $i, k = 1, 3$; and the form is again reducible, because of the chain $\begin{pmatrix} & \\ 1, & 3\end{pmatrix}$ repeated. We are left with the above two types only. Of these, $F_1\begin{pmatrix} & i & \\ 1 & & 3\end{pmatrix}\begin{pmatrix}2 & \omega\\ j & \end{pmatrix}$ reduces if i, j are 1, 3 in either order. For this contains $i_\xi \,.\, 2_j \,.\, F_1 \equiv F_1^2 \,.\, i_\xi$ and reducible terms, by formulæ (C), (D).

$$\textit{Case 2.}\quad F_1\begin{pmatrix} & 2 & \\ 1 & & 3\end{pmatrix}^{(2)}.$$

The tag (2) may be 2_ξ,

$2_2\,\omega$,

$$\begin{pmatrix}2 & j & \dots\\ & i & \end{pmatrix};$$

here i can only be 2, because of the factors 2_1, 2_3 already in $\begin{pmatrix} & 2 & \\ 1 & & 3\end{pmatrix}$. Hence j cannot be 1 or 3, because j_i would then reduce the form. Thus no form exists.

Case 3.

This is parallel with Case 1.

III.

$$F_1\binom{\cdots 2}{i}\binom{\cdot}{j} \quad (i, j \text{ being } 1, 3).$$

39. If the upper link 2 appears in $\binom{\cdot}{i}$, this becomes a form already discussed, having the chain $\binom{\cdot \quad \cdot}{i, \, j}$. Hence the upper links of $\binom{\cdot}{j}$ are i, j. It follows that 2 cannot be a suffix, as i_2, j_2 are reducing factors. Nor can i, j be suffixes.

This leaves j^{ω}, $j^k\xi$ ($k = 1$ or 3) as possible. Of these the latter reduces, because of k_ξ. Hence j^{ω} alone remains.

The possible forms are $\qquad F_1 \, . \, i^2 \, . \, j^{\omega}$,

$$F_1\binom{q \;\; \cdots \; 2}{i \;\; r \quad}j^{\omega};$$

here $r \neq i, j$, Case II. But if $r = 2$, then $q \neq i, j$. Nor can $q = 2$. Hence the form always reduces.

40. In Table B overleaf these results are collected. But to make the system more accessible, the 125 forms given below are sorted into their orders in x, p, u, translated back into their original symbolic notation. Reducible terms in the expansion of F_1^2, and the like, are omitted. These sets are given at the beginning in Table A.

VI.

41. TABLE B.

Arranged in Groups to correspond with those of Gordan.

References.

The 21 Forms $J^{(1)}$.

$a_x^2,\ b_x^2,\ (Ap)^2,\ (Bp)^2,\ u_\alpha^2,\ u_\beta^2,\ (abp)^2,\ (aBu)^2,\ (bAu)^2,$
$a_\beta^2,\ b_\alpha^2,\ (AB)^2,\ (AB)(AB)',\ (AB)'^2,\ (A\beta x)^2,\ (B\alpha x)^2,$
$(\alpha\beta p)^2,\ F_1^2,\ F_2^2,\ \alpha_\alpha^2,\ b_\beta^2.$

The 7 Forms $J^{(2)}$.

§ 29. $\binom{i\,2\,i}{2\,i},\ \binom{i\,j\,i}{2\,i},\ \binom{i\,2\,i}{i\,j},\ \binom{1\,3\,1}{1\,3};\quad i, j = 1, 3.$

The 73 Forms $J^{(3)}$.

$\binom{i\ \omega}{\xi j},\ \binom{i\ k}{\xi j\ \xi},\ \binom{\omega\,j\,\omega}{i\,k};\quad i, j, k = 1, 2, 3.$

Grade 4, § 29. $\binom{i\,2\ \omega}{\xi j\ 2},\ \binom{i\,2\,\omega}{\xi\,2\,i},\ \binom{2\,i\,\omega}{\xi\,2j},\ \binom{2\,i\ \omega}{\xi\,i\,2};\quad i, j = 1, 3.$

$\binom{i\,j\,\omega}{\xi\,j\,i},$

§ 29. $\binom{2\,i\,\omega}{\xi\ i\,j},\ \binom{j\,2\,\omega}{\xi\,i\,j},\ \binom{j\,i\ \omega}{\xi\,i\,2},\ \binom{j\ i\,\omega}{\xi\,2\,j}.$

Grade 5, § 31. $\binom{2\ i\,j}{\xi\ 2\,j\,\xi},\ \binom{i\,2\,j}{\xi\,2\ i\,\xi},\ \binom{1\ 2\ 3}{\xi\ 3\ 1\ \xi},\ \binom{2\,i\,j}{\xi\,j\,i\,\xi},$

and four forms similarly for ω.

Grade 6, § 34. $\binom{i\,2\,j\,\omega}{\xi\,j\ 2\,i},\ \binom{2\ j\,i\,\omega}{\xi\ 2\,i\,j}.$

§ 35. $\binom{2'\ \omega}{\xi\ \ 2},\ \binom{2'\ i}{\xi\ \ 2\,\xi},\ \binom{\omega\ 2'\ \omega}{2\ \ i},\ \binom{i\ 2'\,\omega}{\xi\,2\ \ i}.$

The actual numbers of each kind are, respectively,

$(9+9+9)+(4+2+4+2)+4+(2+2+2+2)+2\,(2+2+1+1)$
$+(2+1)+1+2+2+2 = 73.$

The 12 Forms $J^{(4)}$.

§§ 37–39 $F_{1\cdot 1\omega\cdot 3\omega\cdot 2\xi},\ F_{1\cdot 1\omega\cdot 3\omega}\binom{2\ \omega}{2},\ F_1\binom{i}{1\,3}2_\xi,\ F_1\binom{i}{1\,3}\binom{2\ \omega}{i},$

$F\ \binom{i}{1\,3}\binom{2\ \omega}{2},\ F_{.}\binom{2}{1\,3}2_\xi,\ F_1\binom{2}{1\,3}\binom{2\ \omega}{2},\ F_{1\cdot 1}2_{\cdot 3\omega},$

$F_{1\cdot 3}2_{\cdot 1\omega};\quad i = 1 \text{ or } 3.$

The 12 Forms $J^{(5)}$ *are similar.*