A new system of simple groups.

By

Leonard Eugene Dickson of Chicago.

Introduction.

One of the five isolated simple continuous groups not occurring in Lie's four systems is the group of 14 parameters studied by Killing, Cartan, and Engel. This group is a special case of a linear group on 7 variables with coefficients in an arbitrary field or domain of rationality. The structure of the latter has been determined*) by the writer for fields not having modulus 2. The problem for modulus 2, which requires a different analysis, is solved in the present paper. For $q > 1$, we obtain a simple group of order $2^{6q}(2^{6q}-1)(2^{2q}-1)$. For $q = 1$, the group has a simple subgroup of index 2 and order 6048. The latter is shown to be holoedrically isomorphic with the simple group**) of all ternary hyperorthogonal substitutions of determinant unity in the Galois Field of order 3^2. The generational relations of the isomorphic abstract group are determined and a transitive representation on 36 letters exhibited.

For $q = 1$, the group of order 12096 is shown to be simply isomorphic with a subgroup of index 120 of the senary Abelian group modulus 2, of order $2^9 \cdot 3^4 \cdot 5 \cdot 7$. The latter is known***) to be simply isomorphic with the group of the equation for the 28 bitangents to a quartic curve without double points. It therefore has resolvents of degrees $63 = 2^6 - 1$ and 120, the latter not hitherto noticed.

Definition of the group G_q.

Consider the linear homogeneous transformations S on 7 variables with coefficients in the Galois Field of order 2^q which leave invariant

$$(1) \qquad \xi_0^2 + \xi_1\eta_1 + \xi_2\eta_2 + \xi_3\eta_3 .$$

*) *Transactions Amer. Math. Soc.*, vol. 2 (1901), pp. 383—391.

**) *Annalen*, Bd. 52, pp. 561—581.

***) Jordan, Traité, pp. 229—242; a simpler proof by the writer, *Transactions*, vol. 3, pp. 377—382.

We study the group G_q of those of the transformations S which, when operating cogrediently upon the two sets of variables

$$\xi_0,\ \xi_1,\ \eta_1,\ \xi_2,\ \eta_2,\ \xi_3,\ \eta_3; \qquad \bar{\xi}_0,\ \bar{\xi}_1,\ \bar{\eta}_1,\ \bar{\xi}_2,\ \bar{\eta}_2,\ \bar{\xi}_3,\ \bar{\eta}_3,$$

leave invariant the system of 6 equations

$$(2) \qquad X_l + Y_{mn} = 0, \qquad Y_l + X_{mn} = 0,$$

where l, m, n form any cyclic permutation of 1, 2, 3, and

$$X_i = \begin{vmatrix} \xi_0 & \xi_i \\ \bar{\xi}_0 & \bar{\xi}_i \end{vmatrix}, \quad Y_i = \begin{vmatrix} \xi_0 & \eta_i \\ \bar{\xi}_0 & \bar{\eta}_i \end{vmatrix}, \quad X_{ij} = \begin{vmatrix} \xi_i & \xi_j \\ \bar{\xi}_i & \bar{\xi}_j \end{vmatrix}, \quad Y_{ij} = \begin{vmatrix} \eta_i & \eta_j \\ \bar{\eta}_i & \bar{\eta}_j \end{vmatrix}, \quad Z_{ij} = \begin{vmatrix} \xi_i & \eta_j \\ \bar{\xi}_i & \bar{\eta}_j \end{vmatrix}.$$

A very simple discussion*) shows that, for modulus 2, a transformation S which leaves (1) absolutely invariant must have the form

$$(3) \begin{cases} \xi_i' = \sum_{j=1}^{3} (\alpha_{ij}\xi_j + y_{ij}\eta_j), \qquad \eta_i' = \sum_{j=1}^{3} (\beta_{ij}\xi_j + \delta_{ij}\eta_j) \qquad (i = 1, 2, 3), \\ \xi_0' = \xi_0 + \sum_{j=1}^{3} (\alpha_{0j}\xi_j + y_{0j}\eta_j), \end{cases}$$

where

$$(4) \quad \alpha_{0j}^2 = \alpha_{1j}\beta_{1j} + \alpha_{2j}\beta_{2j} + \alpha_{3j}\beta_{3j}, \quad y_{0j}^2 = y_{1j}\delta_{1j} + y_{2j}\delta_{2j} + y_{3j}\delta_{3j},$$

$$(5) \quad \sum_{i=1}^{3} (\alpha_{ij}\beta_{ik} + \alpha_{ik}\beta_{ij}) = 0, \quad \sum_{i=1}^{3} (y_{ij}\delta_{ik} + y_{ik}\delta_{ij}) = 0$$

$$(j, k = 1,2,3; j \neq k).$$

$$(6) \quad \sum_{i=1}^{3} (\alpha_{ij}\delta_{ik} + \beta_{ij}y_{ik}) = 0 \quad \sum_{i=1}^{3} (\alpha_{ij}\delta_{ij} + \beta_{ij}y_{ij}) = 1$$

For modulus 2, (5) and (6) are precicely the conditions that the partial transformation (3) on ξ_i, η_i $(i = 1, 2, 3)$ shall leave absolutely invariant**) $Z_{11} + Z_{22} + Z_{33}$, so that it belongs to the senary special Abelian group. *Hence G_q is simply isomorphic with a subgroup of the senary special Abelian group in the* $GF[2^q]$.

The conditions obtained in *Transactions*, p. 385, for the invariance of equations (2) now simplify considerably, since we have $\alpha_{i0} = \beta_{i0} = 0$ $(i = 1, 2, 3)$, $\alpha_{00} = 1$. We obtain

$$(7) \qquad \alpha_{li} = \begin{vmatrix} y_{0j} & y_{0k} \\ y_{lj} & y_{lk} \end{vmatrix} + \begin{vmatrix} \delta_{mj} & \delta_{mk} \\ \delta_{nj} & \delta_{nk} \end{vmatrix}, \quad y_{li} = \begin{vmatrix} \alpha_{0j} & \alpha_{0k} \\ \alpha_{lj} & \alpha_{lk} \end{vmatrix} + \begin{vmatrix} \beta_{mj} & \beta_{mk} \\ \beta_{nj} & \beta_{nk} \end{vmatrix},$$

$$(8) \qquad \beta_{li} = \begin{vmatrix} y_{0j} & y_{0k} \\ \delta_{lj} & \delta_{lk} \end{vmatrix} + \begin{vmatrix} y_{mj} & y_{mk} \\ y_{nj} & y_{nk} \end{vmatrix}, \quad \delta_{li} = \begin{vmatrix} \alpha_{0j} & \alpha_{0k} \\ \beta_{lj} & \beta_{lk} \end{vmatrix} + \begin{vmatrix} \alpha_{mj} & \alpha_{mk} \\ \alpha_{nj} & \alpha_{nk} \end{vmatrix},$$

---

*) Dickson, *Linear Groups* (Leipzig, 1901), p. 200; *American Journal*, vol. 21, p. 244.

**) The equation $Z_{11} + Z_{22} + Z_{33} = 0$ is a consequence of (2), *Transactions*, p. 384.

$$(9)\qquad C_{11}=C_{22}=C_{33},\ C_{rs}=0;\ d_{11}=d_{22}=d_{33},\ d_{rs}=0 \quad (r,s=1,2,3;\ r\neq s),$$

where l, m, n and i, j, k from any cyclic permutation of 1, 2, 3, and

$$C_{rs}\equiv\begin{vmatrix}\alpha_{0r} & y_{0s}\\ \alpha_{lr} & y_{ls}\end{vmatrix}+\begin{vmatrix}\beta_{mr} & \delta_{ms}\\ \beta_{nr} & \delta_{ns}\end{vmatrix},\quad d_{rs}\equiv\begin{vmatrix}\alpha_{0r} & y_{0s}\\ \beta_{lr} & \delta_{ls}\end{vmatrix}+\begin{vmatrix}\alpha_{mr} & y_{ms}\\ \alpha_{nr} & y_{ns}\end{vmatrix}.$$

We may readily express all the coefficients in terms of the 18 y_{ij}, δ_{ij}, $(i,j=1,2,3)$, using $(7)_1$, $(8)_1$, and (4). The expressions for the α_{0j}^2 are initially very long, but simplify*) greatly. Thus

$$(10)\quad \alpha_{02}^2=\begin{vmatrix}\delta_{23} & \delta_{21}\\ \delta_{33} & \delta_{31}\end{vmatrix}\cdot\begin{vmatrix}y_{23} & y_{21}\\ y_{33} & y_{31}\end{vmatrix}+\begin{vmatrix}\delta_{33} & \delta_{31}\\ \delta_{13} & \delta_{11}\end{vmatrix}\cdot\begin{vmatrix}y_{33} & y_{31}\\ y_{13} & y_{11}\end{vmatrix}+\begin{vmatrix}\delta_{13} & \delta_{11}\\ \delta_{23} & \delta_{21}\end{vmatrix}\cdot\begin{vmatrix}y_{13} & y_{11}\\ y_{23} & y_{21}\end{vmatrix},$$

the expressions for α_{03}^2, α_{01}^2 following by cyclic permutation. To avoid loss of symmetry, we will, however, retain all the α_{ij}, β_{ij}, y_{ij}, δ_{ij}.

Generators and order of G_q.

Theorem: *The group G_q is generated by*

$$M=(\xi_1\eta_1)(\xi_2\eta_2)(\xi_3\eta_3),$$

$$T_{i,\tau}T_{j,\tau^{-1}}:\xi_i'=\tau\xi_i,\ \eta_i'=\tau^{-1}\eta_i,\ \xi_j'=\tau^{-1}\xi_j,\ \eta_j'=\tau\eta_i,$$

$$Q_{i,j,\lambda}:\xi_i'=\xi_i+\lambda\xi_j,\ \eta_j'=\eta_j-\lambda\eta_i,$$

$$X_{i,\lambda}:\xi_0'=\xi_0-\lambda\eta_i,\ \xi_i'=\xi_i-\lambda^2\eta_i,\ \eta_j'=\eta_j+\lambda\xi_k,\ \eta_k'=\eta_k-\lambda\xi_j,$$

for i, j, k any permutation of 1, 2, 3.

These transformations are seen to leave invariant (1) and the system (2), modulo 2. From them we obtain

$$(11)\qquad Q_{j,i,1}Q_{i,j,1}Q_{j,i,1}\equiv P_{ij}=(\xi_i\xi_j)(\eta_i\eta_j),$$

$$(12)\quad MX_{i,\lambda}M\equiv Y_{i,\lambda}:\xi_0'=\xi_0-\lambda\xi_i,\ \eta_i'=\eta_i-\lambda^2\xi_i,\ \xi_j'=\xi_j+\lambda\eta_k,\ \xi_k'=\xi_k-\lambda\eta_j.$$

Let S be any given transformation (3) of G_q. We show that there exists a transformation K derived from the preceding, such that KS is the identity. We may assume that $\alpha_{11}\neq 0$. For, if $\alpha_{1i}\neq 0$, $P_{i1}S$ has $\alpha_{11}\neq 0$; if $y_{1i}\neq 0$, MS has $\alpha_{1i}\neq 0$. Then $S_1=Q_{1,3,\alpha_{13}}Y_{2,y_{13}}T_{1,\alpha_{11}^{-1}}T_{2,\alpha_{11}}S$ replaces ξ_1 by a function of the form $\xi_1+y_{11}\eta_1+\alpha_{12}\xi_2+y_{12}\eta_2$. Then $S_2=Q_{1,2,\alpha_{12}}S_1$ replaces ξ_1 by a function of the form $\xi_1+y_{11}\eta_1+y_{12}\eta_2$. If $y_{11}\neq 0$, $X_{1,y_{11}^{1/2}}Q_{2,1,x}S_2$, where $y_{12}-xy_{11}=0$, leaves ξ_1 unaltered. If $y_{11}=0$, $Y_{3,y_{12}}S_2$ leaves ξ_1 unaltered.

*) To α_{02}^2, given by $(4)_1$, we apply $(7)_1$ and $(8)_1$. Expanding, we obtain 48 terms, including the 12 terms of (10). The coefficients of y_{01} and y_{03} are $\equiv 0$ (mod. 2), while that of $y_{01}y_{03}$ is zero by $(5)_2$ for $j=1$, $k=3$. The remaining terms are $y_{01}^2(y_{13}\delta_{13}+y_{23}\delta_{23}+y_{33}\delta_{33})+y_{03}^2(y_{11}\delta_{11}+y_{21}\delta_{21}+y_{31}\delta_{31})=y_{01}^2y_{03}^2+y_{03}^2y_{01}^2\equiv 0.$

Consider therefore a transformation S' which leaves ξ_1 unaltered. Then $\delta_{11}=1$ by $(6)_2$. Applying to S' in succession the left-hand multipliers $Q_{3,1,\delta_{13}}$, $X_{2,\beta_{13}}$, $Q_{2,1,\delta_{12}}$, we obtain a transformation S'' which replaces ξ_1 by ξ_1, and η_1 by $\beta_{11}\xi_1+\eta_1+\beta_{12}\xi_2$. Then

$$\Sigma \equiv X_{3,\beta_{12}} Q_{3,1,\beta_{11}^{1/2}\beta_{12}} Y_{1,\beta_{11}^{1/2}} S''$$

leaves ξ_1 and η_1 unaltered.

Giving Σ the notation (3) and applying (5) and (6), we have

$$\alpha_{11}=\delta_{11}=1,\ \beta_{11}=y_{11}=0,\ \alpha_{1j}=\alpha_{j1}=y_{1j}=y_{j1}=\beta_{1j}=\beta_{j1}=\delta_{1j}=\delta_{j1}=0 \quad (j=2,3).$$

Then $\alpha_{01}=y_{01}=0$ by (4). By (9), for $(l, r, s)=(2, 2, 1)$, $(2, 3, 1)$, $(3, 2, 1)$, $(3, 3, 1)$, we get $\beta_{32}=0$, $\beta_{33}=0$, $\beta_{22}=0$, $\beta_{23}=0$, respectively. Then $\alpha_{02}=\alpha_{03}=0$ by $(4)_1$. Hence $y_{li}=0$ $(l, i=1, 2, 3)$ by $(7)_2$. Then $y_{02}=y_{03}=0$ by $(4)_2$. By $(8)_2$ we get

$$\delta_{32}=\alpha_{23},\ \delta_{23}=\alpha_{32},\ \delta_{33}=\alpha_{22},\ \delta_{22}=\alpha_{33}.$$

Finally, by $(7)_1$ for $l=i=1$, we get

$$\delta_{22}\delta_{33}-\delta_{23}\delta_{32}=1. \tag{13}$$

Hence Σ is the following transformation of determinant unity:

$$\begin{aligned} \eta_2' &= \delta_{22}\eta_2+\delta_{23}\eta_3,\quad \eta_3'=\delta_{32}\eta_2+\delta_{33}\eta_3,\\ \xi_2' &= \delta_{33}\xi_2+\delta_{32}\xi_3,\quad \xi_3'=\delta_{23}\xi_2+\delta_{22}\xi_3. \end{aligned} \tag{14}$$

If $\delta_{22}=\delta_{33}=0$, $\Sigma=T_{2,\delta_{32}^{-1}}T_{3,\delta_{32}}P_{23}$. If δ_{22} and δ_{33} are not both zero, we may take $\delta_{33}\neq 0$, transforming by P_{23} if necessary. Then

$$\Sigma = Q_{2,3,\delta_{32}\delta_{33}^{-1}} Q_{3,2,\delta_{23}\delta_{33}} T_{2,\delta_{33}} T_{3,\delta_{33}^{-1}}.$$

Corollary. *The order of* G_q *is* $2^{6q}(2^{6q}-1)(2^{2q}-1)$.

Simplicity of the group G_q, for $q>1$.

Suppose that G_q has a self-conjugate subgroup J which contains a transformation S, not the identity, of the form (3).

Lemma I: *If* $q>1$, *the group* J *contains a transformation which multiplies* ξ *by a constant and differs from the identity.*

a) Let first $y_{11}\neq 0$. From what precedes, G_q contains a transformation R which leaves ξ_1 fixed and replaces η_1 by

$$\beta_{11}\xi_1+\eta_1+\beta_{12}\xi_2+\delta_{12}\eta_2+\beta_{13}\xi_3+\delta_{13}\eta_3 \qquad (\beta_{1i},\ \delta_{1i}\ \text{arbitrary}).$$

By suitable choice of the β_{1i}, δ_{1i}, the product $P=T_{1,y_{11}^{-1}}T_{2,y_{11}}R$ will replace ξ_1 by $y_{11}^{-1}\xi_1$, and η_1 by the same function as that by which S replaces ξ_1. Hence J contains $S_1=P^{-1}SP$, which replaces ξ_1 by $y_{11}^{-1}\eta_1$. The demonstration is completed as in *Transactions*, p. 389.

b) For $y_{11}=0$, but α_{12} and α_{13} not both zero, we readily make $\alpha_{12}=1$. The transform of S by $Y_{1,y_{13}}Q_{2,3,\alpha_{13}}$ replaces ξ_1 by $\alpha_{11}\xi_1+\xi_2+y_{12}\eta_2$. We make $\alpha_{11}=0$ by transforming by $Q_{2,1,\alpha_{11}}$. Transforming by $X_{2,y_{12}^{1/2}}$, we obtain in J a transformation S_1 which replaces ξ_1 by ξ_2. Then J contains

$$S_1^{-1}\cdot T_{2,\lambda}T_{3,\lambda^{-1}}S_1T_{2,\lambda^{-1}}T_{3,\lambda} \qquad (\lambda \neq 0, 1),$$

which replaces ξ_1 by $\lambda\xi_1$.

c) For $y_{11}=\alpha_{12}=\alpha_{13}=0$, either S replaces ξ_1 by $\alpha_{11}\xi_1$ or is conjugate with S' which replaces ξ_1 by $\alpha_{11}\xi_1+\eta_2+y_{13}\eta_3$. Then $Q_{3,2,y_{13}}X_{3,\alpha_{11}}$ transforms S' into S_2 which replaces ξ_1 by η_2. Hence J contains

$$S_2^{-1}Q_{3,1,1}^{-1}S_2Q_{3,1,1}$$

which leaves ξ_1 unaltered and is not the identity.

Lemma II: *If $q>1$, the group J contains a transformation which leaves ξ_1 and η_1 unaltered and differs from the identity.*

By Lemma I, J contains a transformation $S\neq 1$ which replaces ξ_1 by $\alpha\xi_1$, and η_1 by $f=\Sigma(\beta_{1j}\xi_j+\delta_{1j}\eta_j)$, where $\delta_{11}=\alpha^{-1}$ by $(6)_2$. We may assume that f has one of the three forms

$$\beta_{11}\xi_1+\alpha^{-1}\eta_1,\quad \beta_{11}\xi_1+\alpha^{-1}\eta_1+\eta_2,\quad \beta_{11}\xi_1+\alpha^{-1}\eta_1+\xi_2+\delta_{12}\eta_2.$$

For if β_{12} and β_{13} are not both zero, we may take $\beta_{12}\neq 0$, transforming by P_{23} if necessary. To make $\beta_{12}=1$, we transform by $T_{2,\lambda}T_{3,\lambda^{-1}}$. Then transforming by $Q_{2,3,\beta_{13}}Y_{1,\delta_{13}}$, we obtain

$$\xi_1'=\alpha\xi_1,\ \eta_1'=\beta_{11}'\xi_1+\alpha^{-1}\eta_1+\xi_2+\delta_{12}'\eta_2.$$

Next, if $\beta_{12}=\beta_{13}=0$, while δ_{12} and δ_{13} are not both zero, we may set $\delta_{12}=1$, $\delta_{13}=0$.

a) Let first $f=\beta_{11}\xi_1+\alpha^{-1}\eta_1$. If $\alpha\neq 1$, the transform S' of S by

$$Y_{1,\lambda},\ \beta_{11}+\lambda^2(\alpha-\alpha^{-1})=0,$$

replaces ξ_1 by $\alpha\xi_1$, η_1 by $\alpha^{-1}\eta_1$. Hence $S'=T_{1,\alpha}T_{2,\alpha^{-1}}S_1$, where S_1 leaves ξ_1 and η_1 unaltered, and hence is of the form (14). If S' is not commutative with E, where E is one of the two transformations P_{23}, $Q_{2,3,1}$, J contains $S'^{-1}E^{-1}S'E$, which leaves ξ_1 and η_1 fixed, without reducing to the identity. If S' is commutative with both P_{23} and $Q_{2,3,1}$, then $\delta_{33}=\alpha\delta_{22}$, $\delta_{23}=\alpha\delta_{32}=0$. Then $\alpha\delta_{22}^2=1$ by (13). Hence $S'=T_{1,\delta^{-2}}T_{2,\delta}T_{3,\delta}$, $\delta\neq 1$. If $\delta^3\neq 1$, $S'^{-1}P_{12}^{-1}S'P_{12}$ leaves ξ_3 and η_3 unaltered and replaces ξ_1 by $\delta^3\xi_1\neq\xi_1$. If $\delta^3=1$, $S'^{-1}Y_{1,\lambda}^{-1}S'Y_{1,\lambda}=Y_{1,\tau}$, where $\tau\equiv\lambda(1+\delta^2)$ may be made unity. Hence J contains every $Y_{i,1}$ and every $X_{i,1}$ and therefore $(X_{3,1}Y_{2,1})^2=Q_{3,2,1}$, which leaves ξ_1 and η_1 unaltered. If $\alpha=1$, the lemma is proved if $\beta_{11}=0$. For $\alpha=1$, $\beta_{11}\neq 0$, we transform by $T_{1,\tau}T_{2,\tau^{-1}}$ and make $\beta_{11}=1$. Then $S=Y_{1,1}S_2$, where S_2 is of the form (14). Now Y_{11} is commutative with P_{23} and $Q_{2,3,1}$.

If S_2 is not commutative with both, the lemma follows. In the contrary case, $\delta_{32}=\delta_{23}=0$, $\delta_{22}=\delta_{33}$, whence $\delta_{22}\delta_{33}=1$ by (13). Then

$$S=Y_{1,1}T_{2,\delta^{-1}}T_{3,\delta}.$$

Its transform by $T_{1,\mu^{-1}}T_{2,\mu}$ is $S''=Y_{1,\mu}T_{2,\delta^{-1}}T_{3,\delta}$. Hence J contains $S''S^{-1}=Y_{1,\mu+1}$. It is transformed into $Y_{1,\tau(\mu+1)}$ by $T_{1,\tau^{-1}}T_{2,\tau}$. Hence J contains $Y_{1,1}$, so that the lemma follows as above.

b) Let next $f=\beta_{11}\xi_1+\alpha^{-1}\eta_1+\eta_2$. If $\alpha\neq 1$, we make $\beta_{11}=0$ as in a). Then $S=T_{1,\alpha}T_{3,\alpha^{-1}}Q_{2,1,1}K$, where K is of the form (14). Then $S^{-1}Q_{2,3,1}^{-1}SQ_{2,3,1}$ leaves ξ_1 and η_1 unaltered. If it is the identity, $\delta_{23}=0$, $\delta_{22}=\alpha\delta_{33}$. Let $\delta_{33}=\delta$. Then $\alpha=\delta^{-2}$ by (13). Hence

$$S=T_{1,\delta^{-2}}T_{3,\delta^2}Q_{2,1,1}T_{2,\delta}T_{3,\delta^{-1}}=T_{1\delta^{-2}}T_{2,\delta}T_{3,\delta}Q_{2,1,\delta}.$$

Then J contains $S^{-1}(T_{1,\tau^{-1}}T_{3,\tau})^{-1}ST_{1,\tau^{-1}}T_{3,\tau}=Q_{2,1,\delta(\tau+1)}$. Its transform by P_{13} leaves ξ_1 and η_1 unaltered. If $\alpha=1$, we transform by $T_{1,\mu}T_{3,\mu^{-1}}$ and make $\beta_{11}=1$ or 0. Then $S=Y_{1,\beta}Q_{2,1,1}K$, K of the form (14) and $\beta=0$ or 1. Then $S^{-1}Q_{2,3,1}^{-1}SQ_{2,3,1}$ leaves ξ_1 and η_1 unaltered. If it is the identity, $\delta_{23}=0$, $\delta_{33}=\delta_{22}$ in K, whence $\delta_{22}=1$ by (13). Then $K=Q_{2,3,\delta}$, $\delta\equiv\delta_{32}$. Then $P_{23}M$ transforms S into $X_{1,\beta}Q_{1,3,1}Q_{2,3,\delta}$. Hence J contains $X_{1,\beta}Q_{1,3,1}Q_{2,1,1}Y_{1,\beta}$. According as $\beta=0$ or 1, its square or cube is $Q_{2,3,1}$.

c) The third case may be treated by the same method.

For $q>1$ the group J therefore contains a transformation K which alters neither ξ_1 nor η_1 and differs from the identity. Hence K is of the form (14). But the transformations (14) evidently form a group holoedrically isomorphic with the simple binary group in the $GF[2^q]$, $q>1$. Hence J contains every transformation (14) and therefore every $Q_{i,j,\tau}$, $P_{i,j}$, $T_{i,\tau}T_{j,\tau^{-1}}$, and

$$X_{i,\lambda}^{-1}(T_{i,\tau}T_{j,\tau^{-1}})^{-1}X_{i,\lambda}(T_{i,\tau}T_{j,\tau^{-1}})\equiv X_{i,\sigma},\qquad \sigma=\lambda(\tau-1).$$

Since $q>1$, we may take $\tau\neq 0,1$ and choose λ to make σ assume any value in the field. Hence $J\equiv G_q$, which is therefore simple.

Factors of composition of G_1.

For $q=1$, an analysis analogous to the preceding leads to the result that a self conjugate subgroup J of G_1 must contain the $P_{i,j}$, $Q_{i,j,1}$ and the products two at a time of the transformations $X_{i,1}$, $Y_{i,1}$, M, each of period 2; also that the order of J is either equal to or one-half of the order 12096 of G_1. Such a troublesome alternative has presented itself elsewhere in the theory of linear groups.*) The question is here decided by means of a rectangular table of the transformations of J.

*) Compare the discriminanting invariant, *Linear Groups*, § 205, p. 206.

Independent of what precedes, we make a direct study of the group generated by P_{12} and MX_{11}. It contains

$$P_{23}=(MX_{11})^3,\quad P_{13}=P_{23}P_{12}P_{23},\quad P_{1i}MX_{11}P_{1i}=MX_{i1}=Y_{i1}M,$$
$$X_{j1}X_{i1}=(MX_{j1})^{-1}(MX_{i1}),\quad Y_{j1}X_{j1},\quad Q_{3,2,1}=(X_{31}Y_{21})^2.$$

Hence it is identical with the group J just mentioned. Since the group Γ of order 168 of all ternary linear transformations modulo 2 is generated by binary transformations, and since J contains every P_{ij} and $Q_{i,j,1}$, it follows that J contains a senary group simply isomorphic with Γ, the correspondence of operators being obtained by taking the ternary partial transformation on ξ_1, ξ_2, ξ_3.

In view of a later application, we study the abstract groups H and G simply isomorphic with J and Γ, respectively. By *Linear Groups*, p. 303, G is generated by two operators S and T such that

(15) $$T^2=1,\ S^7=1,\ (ST)^3=1,\ (S^4T)^4=1,$$

while the linear group Γ is obtained by setting

(16) $$T=Q_{3,2,1},\quad S=P_{12}Q_{3,2,1}P_{23}Q_{1,2,1}.$$

The abstract group H is generated by P_{12} and X subject to the generational relations (19), (20), (21), in which occur our old symbols with a new meaning defined as follows:

(17) $$P_{23}=X^3,\ P_{13}=P_{23}P_{12}P_{23},\ Q_{2,1,1}=(XP_{12})^4,$$
$$Q_{i,j,1}=P_{1j}P_{2i}Q_{2,1,1}P_{2i}P_{1j}.$$

Eliminating T and S from (15) and (16) we obtain four relations (15′). From these must follow every true relation holding for the linear transformations P_{ij}, $Q_{i,j,1}$, in particular (11) and

(18) $$\begin{cases}P_{ij}^2=1,\ Q_{i,j,1}^2=1,\ Q_{i,j,1}Q_{i,k,1}=Q_{i,k,1}Q_{i,j,1},\ Q_{j,i,1}Q_{k,i,1}=Q_{k,i,1}Q_{j,i,1},\\ Q_{i,j,1}Q_{k,i,1}Q_{i,j,1}=Q_{k,j,1}Q_{k,i,1}\ (i,\,j,\,k \text{ a permutation of } 1,\,2,\,3).\end{cases}$$

Between the linear transformations P_{ij}, $Q_{i,j,1}$ and $X=X_{11}$ hold the relations (17) and the following:

(19) $$(XP_{12}X^{-1}P_{12})^2=Q_{3,2,1}Q_{3,1,1},\quad XQ_{2,3,1}=Q_{3,2,2}X,$$

(20) $$XQ_{3,1,1}P_{12}XP_{12}XP_{13}X^{-1}=P_{13}Q_{2,1,1}Q_{3,1,1}$$

(21) $$X^{-1}P_{13}X^{-1}P_{12}X^{-1}Q_{1,3,1}X=P_{23}P_{13}Q_{3,1,1}Q_{1,2,1}.$$

From (17) and $(19)_3$ follow readily

(22) $$XQ_{1,2,1}=Q_{2,1,1}X,\ XQ_{1,3,1}=Q_{3,1,1}X,\ XQ_{3,2,1,}=Q_{2,3,1}X,$$
$$Q_{1,2,1}=(P_{12}X)^4.$$

We proceed to show that the order ω of H is 6048. We exhibit 36×168 operators (not initially known to be distinct) in a rectangular table $R_1,\cdots,R_{36}$ with the operators of G_{168} in the first row. By showing that these rows are merely permuted upon applying P_{12} and X as right-hand multipliers, and hence by applying an arbitrary operator of H as multiplier, it follows, since R_1 contains the identity, that every operator of H lies in the table, whence $\omega \overline{\gtrless}$ 6048. From the isomorphism of H with J, it follows that $\omega \overline{\gtrless}$ 6048.

We proceed to the computations. The rectangular table is

$$R_1 = G,\ R_2 = GX,\ R_3 = GXP_{12},\ R_4 = GXP_{13},\ R_5 = GXQ_{2,1,1},$$
$$R_6 = GXQ_{3,1,1},\ R_7 = GX^{-1},\ R_8 = GX^{-1}P_{12},\ R_9 = GX^{-1}P_{13},$$
$$R_{10} = GX^{-1}Q_{1,2,1},\ R_{11} = GX^{-1}Q_{1,3,1},\ R_{12} = GXQ_{3,1,1}P_{12},$$
$$R_{13} = GX^{-1}Q_{1,3,1}P_{12},\ R_{14} = GXP_{12}X,\ R_{15} = GXP_{13}X^{-1},$$
$$R_{16} = GXP_{13}X,\ R_{17} = GXP_{12}X^{-1},\ R_{18} = GX^{-1}P_{12}X,$$
$$R_{19} = GX^{-1}P_{13}X,\ R_{20} = GX^{-1}P_{13}X^{-1},\ R_{21} = GX^{-1}P_{12}X^{-1},$$
$$R_{22} = GXP_{13}X^{-1}P_{12},\ R_{23} = GX^{-1}Q_{1,2,1}X,\ R_{24} = GX^{-1}Q_{1,3,1}X,$$
$$R_{25} = GX^{-1}Q_{1,3,1}P_{12}X,\ R_{26} = GX^{-1}Q_{1,2,1}P_{13}X^{-1},$$
$$R_{27} = GXP_{13}X^{-1}P_{12}X,\ R_{28} = GX^{-1}Q_{1,3,1}P_{12}XP_{12},$$
$$R_{29} = GX^{-1}Q_{1,2,1}P_{13}XP_{13},\ R_{30} = GX^{-1}Q_{1,3,1}P_{12}XP_{12}X,$$
$$R_{31} = GX^{-1}Q_{1,2,1}P_{13}XP_{13}X,\ R_{32} = GX^{-1}Q_{1,2,1}P_{13}XP_{13}X^{-1},$$
$$R_{33} = GX^{-1}Q_{1,3,1}P_{12}XP_{12}X^{-1},\ R_{34} = GX^{-1}Q_{1,3,1}P_{12}XP_{12}XP_{12},$$
$$R_{35} = GX^{-1}Q_{1,2,1}Q_{1,3,1},\ R_{36} = GXQ_{3,1,1}Q_{2,1,1}.$$

Applied as a right-hand multiplier, P_{12} gives rise to the permutation

$$(R_2R_3)(R_6R_{12})(R_7R_8)(R_{11}R_{13})(R_{14}R_{21})(R_{15}R_{22})(R_{16}R_{23})$$
$$(R_{19}R_{26})(R_{20}R_{24})(R_{25}R_{28})(R_{30}R_{34})(R_{27}R_{32}),$$

R_1, R_4, R_5, R_9, R_{10}, R_{17}, R_{18}, R_{29}, R_{31}, R_{33}, R_{35}, R_{36} *being unaltered.*

The cases not following by inspection are treated thus:

$$R_4P_{12} \equiv GXP_{13}P_{12} = GXP_{23}P_{13} = GP_{23}XP_{13} = GXP_{13} \equiv R_4.$$
$$R_5P_{12} \equiv GXQ_{2,1,1}P_{12} = GXQ_{1,2,1}Q_{2,1,1} = GQ_{2,1,1}XQ_{2,1,1} \equiv R_5.$$
$$R_{10}P_{12} \equiv GX^{-1}Q_{1,2,1}P_{12} = GX^{-1}Q_{2,1,1}Q_{1,2,1} = GQ_{1,2,1}X^{-1}Q_{1,2,1} \equiv R_{10}.$$
$$R_{14}P_{12} \equiv GXP_{12}XP_{12} = GQ_{2,1,1}(XP_{12})^{-2} = GX^{-1}P_{12}X^{-1} \equiv R_{21}.$$
$$R_{16}P_{12} \equiv GXP_{13}XP_{12} = GP_{13}Q_{2,1,1}X^{-1}P_{12}Q_{3,1,1}Q_{2,1,1}X, \text{ by } P_{23}(20)P_{23}.$$
$$= GX^{-1}Q_{3,2,1}P_{12}Q_{2,1,1}X = GQ_{2,3,1}X^{-1}Q_{2,1,1}Q_{1,2,1}X$$
$$= GQ_{2,3,1}Q_{1,2,1}X^{-1}Q_{1,2,1}X \equiv R_{23}, \text{ by } (22)_1.$$
$$R_{17}P_{12} \equiv GXP_{12}X^{-1}P_{12} = GQ_{3,2,1}Q_{3,1,1}P_{12}XP_{12}X^{-1} \equiv R_{17}, \text{ by } (19)_1.$$
$$R_{18}P_{12} \equiv GX^{-1}P_{12}XP_{12} = GP_{12}X^{-1}Q_{3,2,1}Q_{3,1,1}P_{12}X, \text{ by } (19)_1,$$
$$= GP_{12}Q_{2,3,1}X^{-1}Q_{3,1,1}P_{12}X = GQ_{1,3,1}X^{-1}P_{12}X \equiv R_{18}.$$

The condition for $R_{19}P_{12} = R_{26}$ is that G shall contain

$$X^{-1}P_{13}X\cdot P_{12}\cdot XP_{13}Q_{1,2,1}X = P_{23}X^{-1}P_{13}P_{12}XP_{12}XP_{13}Q_{1,2,1}X$$
$$= P_{23}X^{-1}P_{13}\cdot Q_{1,2,1}X^{-1}P_{12}X^{-1}P_{12}\cdot P_{13}Q_{1,2,1}X, \text{ by } (22)_4.$$
$$= P_{23}Q_{2,3,1}\cdot X^{-1}P_{13}X^{-1}P_{12}X^{-1}\cdot P_{13}P_{23}Q_{1,2,1}X$$
$$= P_{23}Q_{2,3,1}\cdot P_{23}P_{13}Q_{3,1,1}Q_{1,2,1}X^{-1}Q_{1,3,1}\cdot P_{13}Q_{1,3,1}XP_{23}, \text{ by } (21),$$
$$= Q_{3,2,1}P_{13}Q_{3,1,1,}Q_{1,2,1}X^{-1}Q_{3,1,1}XP_{23}, \text{ by } (11),$$
$$= Q_{3,2,1}P_{13}Q_{3,1,1}Q_{1,2,1}Q_{1,3,1}P_{23}, \text{ by } (22)_2,$$
$$= Q_{3,2,1}Q_{3,1,1}Q_{1,2,1}P_{23}.$$

From (21), $R_{20}P_{12} = R_{24}$. Next, $R_{27}P_{12} = R_{32}$ if G contains

$$XP_{13}X^{-1}P_{12}X\cdot P_{12}\cdot XP_{13}X^{-1}P_{13}Q_{1,2,1}X$$
$$= XP_{13}X^{-1}\cdot X^{-1}P_{12}X^{-1}P_{12}Q_{1,2,1}\cdot P_{13}X^{-1}Q_{3,2,1}P_{13}X, \text{ by } (22)_4,$$
$$= XP_{13}P_{23}XP_{12}X^{-1}Q_{2,1,1}P_{12}P_{13}Q_{2,3,1}X^{-1}P_{13}X$$
$$= XP_{13}XP_{13}\cdot X^{-1}Q_{3,1,1}P_{12}Q_{2,3,1}X^{-1}P_{13}X$$
$$= Q_{3,1,1}P_{13}X^{-1}P_{13}X^{-1}\cdot X^{-1}Q_{3,1,1}Q_{1,3,1}P_{12}X^{-1}P_{13}X$$
$$= Q_{3,1,1}P_{13}X^{-1}P_{13}XP_{23}\cdot Q_{1,3,1}P_{13}P_{12}X^{-1}P_{13}X$$
$$= Q_{3,1,1}P_{13}X^{-1}\cdot P_{13}XQ_{1,2,1}\cdot P_{13}X^{-1}P_{13}X$$
$$= Q_{3,1,1}P_{13}X^{-1}\cdot Q_{2,3,1}P_{13}X\cdot P_{13}X^{-1}P_{13}X$$
$$= Q_{3,1,1}P_{13}Q_{3,2,1}\cdot X^{-1}P_{13}XP_{13}X^{-1}P_{13}\cdot X$$
$$= Q_{3,1,1}P_{13}Q_{3,2,1}\cdot P_{13}X^{-1}Q_{2,3,1}Q_{2,1,1}\cdot X, \text{ by } P_{23}(19)P_{23}.$$
$$= Q_{3,1,1}Q_{1,2,1}Q_{3,2,1}Q_{1,2,1}, \text{ by } (22)_3 \text{ and } (22)_1.$$

$$R_{29}P_{12} = GX^{-1}Q_{1,2,1}P_{13}XP_{23}P_{13} = GX^{-1}Q_{1,2,1}P_{12}P_{13}XP_{13}$$
$$= R_{10}P_{12}P_{13}XP_{13} = R_{10}P_{13}XP_{13} \equiv R_{29}.$$

The condition for $R_{31}P_{12} = R_{31}$ is that G shall contain

$$X^{-1}Q_{1,2,1}P_{13}XP_{13}X\cdot P_{12}\cdot X^{-1}P_{13}X^{-1}P_{13}Q_{1,2,1}X$$
$$= X^{-1}Q_{1,2,1}\cdot Q_{2,1,1}X^{-1}P_{12}Q_{3,1,1}Q_{2,1,1}\cdot P_{13}X^{-1}P_{13}Q_{1,2,1}X, \text{ by } P_{23}(20)P_{23},$$
$$= X^{-1}Q_{2,1,1}P_{12}X^{-1}P_{23}P_{12}Q_{1,3,1}Q_{2,3,1}X^{-1}Q_{3,2,1}P_{13}X$$
$$= Q_{1,2,1}X^{-1}P_{12}X^{-1}P_{23}P_{12}Q_{1,3,1}X^{-1}P_{13}X$$
$$= Q_{1,2,1}P_{23}\cdot X^{-1}P_{13}X^{-1}P_{12}X^{-1}\cdot Q_{3,1,1}P_{13}X$$
$$= Q_{1,2,1}P_{23}\cdot P_{23}P_{13}Q_{3,1,1}Q_{1,2,1}X^{-1}Q_{1,3,1}\cdot Q_{3,1,1}P_{13}X, \text{ by } (21),$$
$$= Q_{1,2,1}P_{13}Q_{3,1,1}Q_{1,2,1}X^{-1}Q_{3,1,1} = Q_{1,2,1}Q_{3,1,1}Q_{1,2,1}, \text{ by } (22)_2.$$

The condition for $R_{33}P_{12} = R_{33}$ is that G shall contain

$$X^{-1}Q_{1,3,1}P_{12}XP_{12}X^{-1}\cdot P_{12}\cdot XP_{12}X^{-1}P_{12}Q_{1,3,1}X$$
$$= X^{-1}Q_{1,3,1}P_{12}Q_{3,2,1}\cdot Q_{3,1,1}Q_{1,3,1}X, \text{ by } (19),$$
$$= X^{-1}Q_{1,3,1}Q_{3,1,1}P_{12}\cdot P_{13}Q_{3,1,1}X = X^{-1}Q_{3,1,1}P_{13}\cdot P_{12}P_{13}XQ_{1,3,1}$$
$$= Q_{1,3,1}X^{-1}P_{23}XQ_{1,3,1} = Q_{1,3,1}P_{23}Q_{1,3,1}.$$

The condition for $R_{35}P_{12} = R_{35}$ is that G shall contain

$$X^{-1}Q_{1,3,1}Q_{1,2,1}\cdot P_{12}\cdot Q_{1,2,1}Q_{1,3,1}X = X^{-1}Q_{1,3,1}Q_{2,1,1}Q_{1,3,1}X$$
$$= X^{-1}Q_{2,3,1}Q_{2,1,1}X = Q_{3,2,1}Q_{1,2,1}, \text{ by } (18)_4,\ (22)_2,\ (22)_1.$$

The condition for $R_{36}P_{12} = R_{36}$ is that G shall contain

$$XQ_{3,1,1}Q_{2,1,1}P_{12}Q_{2,1,1}Q_{3,1,1}X^{-1} = XQ_{3,1,1}Q_{1,2,1}Q_{3,1,1}X^{-1}$$
$$= XQ_{1,2,1}Q_{3,2,1}X^{-1} = Q_{2,1,1}Q_{2,3,1}.$$

Theorem: *Applied as a right-hand multiplier, X gives rise to the permutation*

$$(R_1R_2R_7)(R_{12}R_{22}R_{27})(R_{13}R_{25}R_{26})(R_{34}R_{36}R_{35})(R_3R_{14}R_{15}R_4R_{16}R_{17})$$
$$(R_5R_{10}R_{23}R_6R_{11}R_{24})(R_9R_{19}R_{21}R_8R_{18}R_{20})(R_{28}R_{30}R_{32}R_{29}R_{31}R_{33}).$$

That $R_{12}X = R_{22}$ follows from (20), $R_{36}X = R_{35}$ from $(22)_1$ and $(22)_2$.

$$R_{34}X = GX^{-1}Q_{1,3,1}(P_{12}X)^3 = GX^{-1}Q_{1,3,1}X^{-1}P_{12}Q_{1,2,1}$$
$$= GX^{-1}X^{-1}Q_{3,1,1,}P_{12}Q_{1,2,1} = GXQ_{3,1,1}Q_{2,1,1}P_{12} \equiv R_{36}P_{12} = R_{36}.$$
$$R_{14}X = GXP_{12}X^2 = GXP_{12}P_{23}X^{-1} = GP_{23}XP_{13}X^{-1} = R_{15}.$$
$$R_5X \equiv GXQ_{2,1,1}X = GX^2Q_{1,2,1} = GP_{23}X^{-1}Q_{1,2,1} = R_{10}.$$
$$R_{23}X \equiv GX^{-1}Q_{1,2,1}X^2 = GX^{-1}Q_{1,3,1}X^{-1} = GX^{-1}X^{-1}Q_{3,1,1}$$
$$= GP_{23}XQ_{3,1,1} = R_6.$$

The remaining cases follow by inspection. We may now state the

Theorem: *The group G_1 of order* 12096 *contains a subgroup J of index* 2, *generated by P_{12} and MX_{11}, simply isomorphic with the abstract group H generated by P_{12} and X subject to* (19), (20), (21), *with the amplification* (17), *together with* (15′), *namely* (15) *for the values* (16). *Moreover, J may be represented as a transitive substitution-group on* 36 *letters.*

The simplicity of J may be established by a direct but long analysis, as stated above. However, an indirect proof follows from the isomorphism next established.

Holoedric isomorphism of H and the simple ternary hyperorthogonal group O in the $GF[3^2]$.

Knowing that the two groups are simple, of the same order 6048, representable as transitive substitution-groups on 28 letters*), and that the periods of the operators of each are 1, 2, 3, 4, 6, 7, 8, 12, those of

*) For O this is shown in *Annalen*, Bd. 55, p. 532. For H it follows since G_1 is simply isomorphic with a subgroup of the senary Abelian group A (as shown above), which is simply isomorphic with the group of the equation for the 28 bitangents to a quartic.

period 7 falling into 2 sets each of $2^5 \cdot 3^3$ conjugates*) the presumption was in favor of their isomorphism.

We proceed to determine a set of substitutions of O which satisfy all the generational relations for the group H.

Since all the substitutions of period 6 in O are conjugate (*Annalen*, Bd. 55, p. 572), we assume that

$$(23)\qquad \begin{aligned} X &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -i-1 & -i+1 \\ 0 & -i-1 & i-1 \end{pmatrix} \equiv [1, -i-1, -i+1], \\ P_{23} &= X^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \end{aligned}$$

where $i^2 \equiv -1 \pmod{3}$. Since $Q_{3,2,1} = P_{23}^{-1} Q_{2,3,1} P_{23}$, we set

$$Q_{2,3,1} = \begin{pmatrix} \beta_{11} & \beta_{12} & \beta_{13} \\ \beta_{21} & \beta_{22} & \beta_{23} \\ \beta_{31} & \beta_{32} & \beta_{33} \end{pmatrix}, \quad Q_{3,2,1} = \begin{pmatrix} \beta_{11} & -\beta_{12} & -\beta_{13} \\ -\beta_{21} & \beta_{22} & \beta_{23} \\ -\beta_{31} & \beta_{32} & \beta_{33} \end{pmatrix}.$$

Then $(19)_2$: $XQ_{2,3,1} = Q_{3,2,1}X$ holds if and only if

$$\beta_{13} = (i+1)\beta_{12},\ \beta_{31} = (1-i)\beta_{21},\ \beta_{32} = -i\beta_{23},\ \beta_{33} = \beta_{22} + (i-1)\beta_{23}.$$

Now a hyperorthogonal substitution (β_{ij}) is of period 2 if and only if

$$\beta_{ij}^3 = \beta_{ji} \qquad (i, j = 1, 2, 3).$$

From $\beta_{32} = \beta_{23}^3$, $\beta_{32} = -i\beta_{23}$, follows $\beta_{23} = 0$ or $\pm(1+i)$. Hence

$$Q_{2,3,1} = \begin{pmatrix} \beta_{11} & \beta_{12} & (1+i)\beta_{12} \\ \beta_{21} & \beta_{22} & 0 \\ (1-i)\beta_{21} & 0 & \beta_{22} \end{pmatrix} \text{ or } \begin{pmatrix} \beta_{11} & \beta_{12} & (1+i)\beta_{12} \\ \beta_{21} & \beta_{22} & \pm(1+i) \\ (1-i)\beta_{21} & \pm(1-i) & \beta_{22}\pm 1 \end{pmatrix}.$$

In the first case a hyperorthogonal condition gives $\beta_{21} = 0$, whence $\beta_{12} = 0$. Also $\beta_{22}^4 = 1$, $\beta_{22} = \beta_{22}^3$, whence $\beta_{22}^2 = 1$. The determinant being 1, $\beta_{11} = 1$. Then $Q_{2,3,1}$ of period 2 must coincide with P_{23}. Hence the first case is excluded. For the second, the hyperorthogonal conditions reduce to

$$\beta_{11}^2 = 1,\ \beta_{11}\beta_{21} + \beta_{21}\beta_{22} \mp \beta_{21} = 0,\ \beta_{21}^4 \mp \beta_{22} + 1 = 0,\ \beta_{21}^4 + \beta_{22}^2 = -1,$$
$$\beta_{21}^4 - \beta_{22}^2 \pm \beta_{22} = -1,\ \beta_{21}^3 = \beta_{12},\ \beta_{22}^3 = \beta_{22}.$$

Hence $\beta_{22}^2 = \mp\beta_{22}$. For $\beta_{22} = 0$, the determinant equals ± 1; for $\beta_{22} = \mp 1$, the determinant equals ∓ 1. Hence the substitution $Q_{2,3,1}$ is

*) Shown for O in *Annalen*, Bd. 55, p. 572; and for H by means of theorems on A recently presented to the *American Journal*.

$$\begin{pmatrix} 1 & \beta_{12} & (i+1)\beta_{12} \\ \beta_{12}^3 & 0 & i+1 \\ (1-i)\beta_{12}^3 & 1-i & 1 \end{pmatrix}, \quad \beta_{12}^4 = -1;$$

$$\text{or} \begin{pmatrix} 1 & \beta_{12} & (1+i)\beta_{12} \\ \beta_{12}^3 & 1 & -1-i \\ (1-i)\beta_{12}^3 & i-1 & 0 \end{pmatrix}, \quad \beta_{12}^4 = 1.$$

Now the hyperorthogonal substitution $\xi_2' = \xi_3$, $\xi_3' = -\xi_2$ transforms the second into W, where $\overline{W}$ (obtained from W by replacing i by $-i$) is of the first form, and transforms X into $\overline{X}$. Hence we may assume that $Q_{2,3,1}$ is of the second form, say $S_{\beta_{12}}$. Now the hyperorthogonal substitution

$$\xi_1' = \mu^{-2}\xi_1, \quad \xi_2' = \mu\xi_2, \quad \xi_3' = \mu\xi_3, \quad \mu^4 = 1$$

is commutative with X and transforms S_β into $S_{\mu\beta}$. Hence we may take $\beta = 1$. Hence we have

$$(24)\quad Q_{2,3,1} = \begin{pmatrix} 1 & 1 & 1+i \\ 1 & 1 & -1-i \\ 1-i & i-1 & 0 \end{pmatrix}, \quad Q_{3,2,1} = \begin{pmatrix} 1 & -1 & -1-i \\ -1 & 1 & -1-i \\ i-1 & i-1 & 0 \end{pmatrix}.$$

The conditions that $Q_{1,3,1} \equiv (\delta_{ij})$ shall be commutative with $Q_{2,3,1}$ reduce to

$$(25)\quad \begin{cases} \delta_{21} = \delta_{12} + (1-i)\delta_{13} - (1+i)\delta_{31}, & \delta_{32} = \delta_{31} + i\delta_{13} - i\delta_{23}, \\ \delta_{22} = \delta_{11} + (i-1)\delta_{23} - (1+i)\delta_{31}, & \delta_{33} = \delta_{11} - \delta_{12} + (1-i)\delta_{13} + (1-i)\delta_{23}. \end{cases}$$

Since $Q_{1,3,1}^2 = 1$, $\delta_{ji} = \overline{\delta}_{ij}$. Expressing the δ_{ij} in the form $a + bi$, we get*)

$$Q_{1,3,1} = \begin{pmatrix} d_{11} & d_{12} + iD_{12} & d_{13} + iD_{13} \\ d_{12} - iD_{12} & d_{22} & d_{23} + iD_{23} \\ d_{13} - iD_{13} & d_{23} - iD_{23} & d_{33} \end{pmatrix}.$$

The conditions (25) reduce to

$$D_{12} = d_{13} - D_{13}, \qquad d_{33} = d_{11} - d_{12} + d_{13} + d_{23} + D_{13} + D_{23},$$
$$d_{23} = d_{13} - D_{13} + D_{23}, \quad d_{22} = d_{11} - d_{13} - d_{23} - D_{13} - D_{23}.$$

Then

$$(26)\quad Q_{1,3,1} = \begin{pmatrix} d_{11} & d_{12} + i(d_{13} - D_{13}) & d_{13} + iD_{13} \\ d_{12} - i(d_{13} - D_{13}) & d_{11} + d_{13} + D_{23} & d_{13} - D_{13} + D_{23} - iD_{23} \\ d_{13} - iD_{13} & d_{13} - D_{13} + D_{23} - iD_{23} & d_{11} - d_{12} - d_{13} - D_{23} \end{pmatrix}.$$

The six hyperorthogonal conditions are

$$(27)\quad d_{11}^2 + d_{12}^2 + (d_{13} - D_{13})^2 + d_{13}^2 + D_{13}^2 = 1,$$

$$(28)\quad d_{12}^2 + (d_{13} - D_{13})^2 + (d_{11} + d_{13} + D_{23})^2 + (d_{13} - D_{13} + D_{23})^2 + D_{23}^2 = 1,$$

*) Concerning determinants of such matrices, see *Amer. Math. Monthly*, Dec. 1903.

(29) $d_{13}^2 + D_{13}^2 + D_{23}^2 + (d_{13} - D_{13} + D_{23})^2 + (d_{11} - d_{12} - d_{13} - D_{23})^2 = 1$, together with three conditions involving i which give

$$(30)\qquad (d_{13} - D_{13})(d_{13} + D_{13} - d_{11}) = 0,$$

$$(31)\qquad d_{13}^2 + d_{13}D_{23} - d_{13}D_{13} + D_{13}D_{23} - d_{11}d_{12} + d_{12}d_{13} + d_{12}D_{23} = 0,$$

$$(32)\qquad -d_{13}^2 + d_{13}D_{23} + D_{13}D_{23} + d_{12}D_{23} - d_{12}D_{13} - d_{11}d_{13} = 0,$$

$$(33)\qquad d_{13}^2 + D_{13}^2 + d_{12}D_{23} - d_{11}D_{13} + d_{13}D_{23} + D_{13}D_{23} - d_{12}D_{13} = 0,$$

$$(34)\qquad d_{11}(D_{13} - D_{23} - d_{13}) + d_{12}D_{13} - d_{12}D_{23} + d_{13}D_{13} - D_{13}^2 = 0,$$

$$(35)\qquad -d_{13}^2 + d_{13}D_{13} + d_{12}D_{13} - d_{11}D_{23} - d_{12}D_{23} = 0.$$

For $d_{13} = D_{13}$, (31), (32) or (33), (34) or (35) give respectively

$$(36)\qquad d_{12}D_{23} - d_{13}D_{23} - d_{11}d_{12} + d_{12}d_{13} = 0,$$

$$(37)\qquad d_{13}^2 + d_{13}D_{23} - d_{12}D_{23} + d_{12}d_{13} + d_{11}d_{13} = 0,\quad d_{12}d_{13} - d_{11}D_{23} - d_{12}D_{23} = 0.$$

Combining the third with the preceding two we get

$$(d_{11} + d_{13})(d_{12} + D_{23}) = 0, \quad (d_{11} + d_{13})(d_{13} + D_{23}) = 0.$$

If $d_{11} + d_{13} \neq 0$, then $d_{12} = d_{13} = -D_{23}$, and $(37)_2$ gives $D_{23}(d_{11} + D_{23}) = 0$. If also $D_{23} = 0$, (26), of determinant 1, reduces to the identity since $d_{11}^2 = 1$ by (27). But if $D_{23} \neq 0$, (26) reduces to $(24)_1$, when each element is multiplied by d_{11}. Then $d_{11}^2 = 1$ by (27), $d_{11}^3 = 1$ in view of the determinant. Hence (26) reduces to $Q_{2,3,1}$, so that also this case is excluded. Hence $d_{11} + d_{13} = 0$. Then $d_{12}^2 = 1$ by (27). Set $d_{12} = \pm 1$. Then $(37)_2$ gives

$$(d_{11} \pm 1)(D_{23} \pm 1) \equiv 1, \quad d_{11} \pm 1 \equiv D_{23} \pm 1 \pmod{3}.$$

Hence $d_{11} = D_{23}$, $d_{11} = 0$ or ± 1. In either case, the determinant of (26) equals ± 1, so that the upper signs hold. Hence

$$(38)\qquad Q_{1,3,1} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \text{ or } \begin{pmatrix} 1 & 1 & -1-i \\ 1 & 1 & 1+i \\ i-1 & 1-i & 0 \end{pmatrix},$$

the second being $Q_{2,3,1}V$ where V denotes the first.

For $d_{13} \neq D_{13}$, (30) gives $d_{11} = d_{13} + D_{13}$. Hence

$$D_{13} = d_{13} \pm 1, \quad d_{11} = -d_{13} \pm 1.$$

Then (27) or (28) gives $d_{12}^2 = 1$, while (29), (31)—(35) each reduces to

$$d_{12}D_{23} - d_{13}D_{23} - d_{12}d_{13} \mp d_{13} \pm D_{23} \mp d_{12} = 0.$$

Set $D_{23} = -d_{13} + t$. Completing the square in d_{13}, we get

$$\{d_{13} - (d_{12} \pm 1 - t)\}^2 \equiv t^2 - 1 \pmod{3}.$$

Hence $t \neq 0$, $t^2 \equiv 1$, $d_{13} = d_{12} \pm 1 - t$.

Defining $Q_{1,3,1}$ by $(38)_1$, we get

$$Q_{1,2,1} = P_{23}^{-1} Q_{1,3,1} P_{23} = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

$$Q_{3,1,1} = X Q_{1,3,1} X^{-1} = \begin{pmatrix} 0 & -i-1 & 1-i \\ i-1 & 1 & -i \\ 1+i & i & 1 \end{pmatrix},$$

$$P_{13} = Q_{1,3,1} Q_{3,1,1} Q_{1,3,1} = \begin{pmatrix} 1 & i-1 & i \\ -i-1 & 0 & i-1 \\ -i & -i-1 & 1 \end{pmatrix},$$

$$P_{12} = P_{23}^{-1} P_{13} P_{23} = \begin{pmatrix} 1 & 1-i & -i \\ 1+i & 0 & i-1 \\ i & -i-1 & 1 \end{pmatrix}.$$

Then S and T defined by (16) are seen to satisfy (15) since

$$S = \begin{pmatrix} 1 & -1 & 1+i \\ -1-i & -1-i & 0 \\ i & -i & 1-i \end{pmatrix}, \quad S^2 = \begin{pmatrix} 1-i & 1 & i \\ -1+i & 1 & i \\ 0 & 1-i & -1-i \end{pmatrix},$$

$$S^4 = \begin{pmatrix} -1-i & 0 & i-1 \\ -1 & 1-i & -i \\ -i & -1-i & 1 \end{pmatrix}, \quad S^6 = \begin{pmatrix} 1 & -1+i & -i \\ -1 & -1+i & i \\ 1-i & 0 & 1+i \end{pmatrix},$$

$$TS = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & i \\ i & 0 & 0 \end{pmatrix}, \quad S^4 T = \begin{pmatrix} -1 & -1 & 1+i \\ -1-i & 1+i & 0 \\ -i & -i & 1-i \end{pmatrix}.$$

Further, $(17)_3$ or its equivalent $Q_{1,2,1} = (P_{12} X)^4$ is seen to hold. Likewise, $(19)_1$, (20) and (21). The isomorphism is therefore proved.

Chicago, November 1903.