

The Problem of the Resisting Medium.

By *Selig Brodetsky.*

In A. N. nos. 4308, 4334, 4341-43, Dr. T. J. J. See has assumed that when a particle moves in the manner defined by the restricted problem of three bodies, but in a resisting medium, the Jacobian integral corresponding to this case can be derived from the Jacobian integral as obtained in the supposed absence of such a medium, by the addition to the constant C of a secular term αt . This assumption is not a priori justifiable, for the effects of the resistance must be considered in the original equations of motion. Dr. See also assumed that α is on the whole positive, but there is nothing to indicate that this should necessarily be so. It will appear in the course of this paper that α can be considered as being negative at any time, and then Dr. See's argument breaks down. Let us then put in the resistance in the equations of motion and see how it will affect the solution.

The problem will be affected by the manner in which the medium is related to the Sun-Jove system. The medium may be absolutely stationary as a whole, so that as many particles of this medium are moving in any one direction with any velocity, as in any other direction with any other velocity. Or perhaps the medium may be stationary as a whole with respect to the rotating axes used in the problem; or the medium might be affected by some other such motion as a whole, which would have a profound influence on the solution. In any case two assumptions seem to be necessary. Firstly the medium must be too thin to affect appreciably the motions of the Sun and Jove, but dense enough to affect the »small particle« moving under the Sun-Jove system. Secondly, that this so called »small particle« is big enough to enable us to take the effect of the resistance to be the same as if the medium had no differential motions distributed among its parts, so that the resistance is in the nature of a frictional retardation. Both these assumptions are reasonable enough when dealing with actual problems in nature.

In any case the constitution and mode of resistance of the medium imports much difficulty into the consideration

of this problem. Let us take as wide an hypothesis as possible. The medium can be supposed to vary in density with the time — as it must under such a powerful influence as the solar attraction; and it may differ in density in different places. It seems clear that it would in fact become more and more attenuated as we recede from the Sun: but in order to avoid making the assumption that it is arranged in spherical strata round the Sun, we shall merely say that its average effective density at any point is a function of the time and the coordinates. By average effective density at any point I mean the average density of a portion of the medium enclosed by a sphere whose centre is at this point, and whose radius is the same as that of one »small particle«. The second assumption made above enables us to say that this effective density does not depend on the size of this sphere, assuming it is small enough. Let the axes be the rotating axes taken in Dr. See's papers. Let x, y be the coordinates of any point referred to these axes with the Sun as origin; r, θ the corresponding polar coordinates. Let X, Y, R, Θ refer to parallel axes, but the centre of mass of the Sun and Jove as origin. Then the density will be a function of t, x, y ; or t, r, θ ; or t, X, Y ; or t, R, Θ : any of these forms may be used according to convenience. The important thing to notice about this function is that it always remains positive, as long as we are considering a possible case in nature.

Again, we have to define the nature of the resistance. The effect can be taken to be a retardation acting on the particle, along its line of instantaneous motion relatively to the medium: the retardation will be a function of the velocity relatively to the medium. Here too it is important to notice that this function is always positive. Now we are in a position to put down the modified equations of motion. As a first case let us consider the medium as a whole to follow the axes: i. e. it is of the nature of a solar envelope, accompanying the Sun in the same way as Jove. Relatively to the moving axes defined above we get

$$\begin{aligned} \frac{d}{dt} \left(\frac{dX}{dt} - nY \right) - n \left(\frac{dY}{dt} + nX \right) &= \frac{\partial U}{\partial X} - f \cdot \varphi(V) \frac{1}{V} \frac{dX}{dt} \\ \frac{d}{dt} \left(\frac{dY}{dt} + nX \right) + n \left(\frac{dX}{dt} - nY \right) &= \frac{\partial U}{\partial Y} - f \cdot \varphi(V) \frac{1}{V} \frac{dY}{dt} \end{aligned}$$

in which n is the angular velocity of the axes, i. e. of Jove round the Sun,
 V is the velocity relatively to the moving axes,
 U is the potential function,

f is the resistance factor, a positive function of time and position,
 and $\varphi(V)$ is a function of the velocity, defining the retardation.

Change coordinates to x, y ; then $X = x - \frac{1}{\nu + 1}$; $Y = y$; $n^2 = \nu + 1$; $V = \nu$; where the radius vector and the mass of Jove are each taken as unity, and that of the Sun as ν .

We get

$$\frac{d^2x}{dt^2} - 2n \frac{dy}{dt} = \frac{\partial \Omega}{\partial x} - f(x, y, t) \frac{\varphi(\nu)}{\nu} \frac{dx}{dt}$$

$$\frac{d^2y}{dt^2} + 2n \frac{dx}{dt} = \frac{\partial \Omega}{\partial y} - f(x, y, t) \frac{\varphi(\nu)}{\nu} \frac{dy}{dt}$$

where

$$2\Omega = \nu \left(r^2 + \frac{2}{r} \right) + \left(\rho^2 + \frac{2}{\rho} \right)$$

in which r is the distance from the Sun, and ρ is the distance from Jove.

We have seen that f and $\varphi(\nu)$ are both positive. Also $\frac{1}{\nu} \frac{dx}{dt}$ and $\frac{1}{\nu} \frac{dy}{dt}$ are really direction cosines, whose signs are defined by $\frac{dx}{dt}$ and $\frac{dy}{dt}$, so that ν must be taken to mean the arithmetical value of the relative velocity. Thus the product $f(x, y, t) \frac{\varphi(\nu)}{\nu}$ is always positive.

The above equations give us

$$\frac{d}{dt} (\nu^2 - 2\Omega) = -2f(x, y, t) \frac{\varphi(\nu)}{\nu} \cdot \nu^2$$

$$\nu^2 = 2\Omega - C - 2 \int_0^t f(x, y, t) \frac{\varphi(\nu)}{\nu} \cdot \nu^2 dt$$

the integrand being always positive;

$$\nu^2 = 2\Omega - C - I$$

where I is a positive and increasing quantity.

Thus we see that in the case of a resisting medium moving round with the S-J-axis, the quantity αt given by Dr. See should really be the complicated integral above; and that it conforms itself to Dr. See's hypothesis by being always positive, and increasing with the time. Nevertheless Dr. See's conclusions hardly appear to be justified as will appear from an examination of the conditions.

Suppose a particle starts off with velocity ν_0 , and on the curve Ω_0 of the curves of zero relative velocity. I starts off at zero and increases steadily. If we can assume that the increase in I is always greater than the possible diminution in ν^2 , then since $2\Omega = \nu^2 + I + C$, Ω will increase steadily. This will be the case if

$$\frac{d\nu^2}{dt} + \frac{dI}{dt} > 0$$

always; i. e.

$$\left(\frac{d\nu}{dt} + f(x, y, t) \varphi(\nu) \right) \nu > 0.$$

ν is merely a number. The condition is that

$$\frac{d\nu}{dt} + f(x, y, t) \varphi(\nu) > 0.$$

This case seems to be one most favourable to Dr. See's arguments. Let us however see what results it gives. Draw Sir G. Darwin's curves of zero relative velocity, ignoring the resisting medium. The curves are $2\Omega = C$ for different values of C . The curves then are also the different Ω curves. Only the critical curves are shown in the figure; the remaining ones can be easily filled in. From $2\Omega = 38.88$, Ω increases outwards from the oval part, and inwards from the hourglass part. If then Ω_0 above is greater than $\frac{1}{2} \times 38.88$, the particle must either move nearer and nearer to S or J if it starts off inside the hourglass $2\Omega = 38.88$, or go away to infinity if it starts outside the oval $2\Omega = 38.88$. If, however, the particle starts off in the space between the two parts of $2\Omega = 38.88$, we can say nothing more definite about its motion than that it must either go away to infinity, or be captured by S or J. But in this reasoning we are at liberty to commence the consideration of the particle when we like. These results can, therefore, be applied to the particle at any stage in its history. Thus if the particle is ever outside the oval $2\Omega = 38.88$, it can never be or have been inside the hour-glass $2\Omega = 38.88$; and vice versa. So that capture is only possible if the whole path of the particle is inside the oval $2\Omega = 38.88$; in other words capture from outside the system is impossible. We see then, that what appeared to be a case most favourable to Dr. See's arguments, turns out quite contrary to his conclusions.

But suppose we may not assume that

$$\frac{d\nu}{dt} + f(x, y, t) \varphi(\nu) > 0$$

i. e. that $\nu^2 + I$ keeps on increasing. Let us consider the moment when the particle was or will be at relative momentary rest, i. e. $\nu_0 = 0$. (This is not to be confused with the equilibrium configurations, which we shall discuss later.) If Ω_0 is the value of Ω at this time and place, then since I increases with the time, and ν^2 is always either greater than or equal to zero, we see that Ω must always be greater than Ω_0 . Thus the position of momentary rest will tell us whether the particle will be captured or not. This position cannot be found without actually integrating the equations of motion: so that this criterion cannot be used practically. At any rate the probability that the position of momentary rest is inside the oval $2\Omega = 38.88$, is too small to make any great accumulation of matter from outside possible.

This is not all the argument against the capture theory in this rather more general case. The particle will not have only one position of momentary rest. If there were no resisting medium, the positions of momentary rest would all be on a given Ω curve, defined by the quantity C entering into the equation $\nu^2 = 2\Omega - C$. But in the presence of a resisting medium, the additional term I will cause the orbit to lose this property, and the various positions of momentary rest that the particle has had in the past and will have in the future are quite distinct. We can extend this consideration as far back in the past as we please. In other words, the particle cannot be captured unless all its positions of momentary rest in the past have been inside the oval $2\Omega = 38.88$. Such capture one can scarcely call capture from outside. The particle must already be closely associated

with Jove, before the process of capture can commence. Incidentally we see that although the curves of zero relative velocity lose their properties as such and as barrier curves,

when a resisting medium is introduced; yet in a modified form the second property still persists, through not in such a way as to make it of much use in practical applications.

Curves of Zero Relative Velocity. (Darwin).

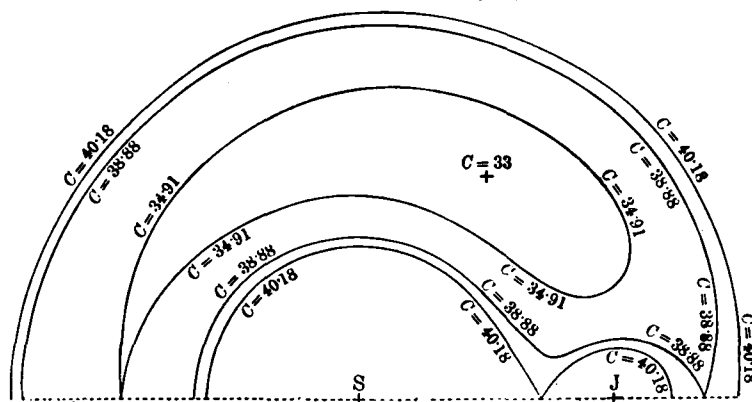


Fig.1

It is interesting to examine the stability of the equilibrium configurations in this case of the restricted problem of three bodies, with a resisting medium introduced. These configurations are indicated in the figure. $C = 33$ gives us the two equilateral-triangle positions: at these points Ω is a minimum. $C = 34.91$, $C = 38.88$, and $C = 40.18$ give us the three straightline arrangements. The way in which these positions are critical is at once understood from the figure. When there is no resisting medium we know that the first two are stable or unstable according to the value of ν , and the last three are always unstable. Let us examine the problem in which we postulate the resisting medium already defined. The particle is at relative rest at one of these positions when it is disturbed slightly. The disturbance can be defined by giving the particle a small but finite velocity v_0 say. Let Ω_0 be the initial value of Ω . Then since $v_0^2 = 2\Omega_0 - C$, and $\nu^2 = 2\Omega - C - I$ at any time, I being zero initially, we get that $2(\Omega - \Omega_0) = I + \nu^2 - v_0^2$. Now I is an increasing positive quantity, and v_0^2 is very small. Hence $I - v_0^2$ must after a time become positive, even if the velocity begins to decrease as the particle begins to move. Therefore after a time Ω must again be equal to Ω_0 if it was ever less than Ω_0 , and after then will always be greater than Ω_0 , also since I is positive, ν is not equal to v_0 when $\Omega = \Omega_0$.

In the first two positions, $C = 33$, the particle may come back to its original position, but in general not with zero velocity or velocity v_0 . In all cases we soon get Ω increasing. Since the Ω curves here are practically concentric circles, the particle will recede further and further away from the original position. Capture is possible though not certain.

The critical position $C = 40.18$ gives us a node, and the neighbouring Ω curves are practically hyperbolas, the real axis along SJ for $\Omega > \Omega_0$, and perpendicular to SJ for $\Omega < \Omega_0$. Since Ω must soon be greater than Ω_0 , the particle will enter one of the loops round S or round J, and become an inferior planet, or be captured as a satellite to Jove.

For $C = 38.88$ we get another node: the argument is exactly similar, and the particle either goes right outside the oval, and is never captured, or enters the hour-glass and has the same fate as in the case just considered. For $C = 34.91$, we again get a node, and the particle is constrained to remain outside the critical horseshoe: but nothing can be said as to capture except that it is possible. But in this case, too, the particle recedes further and further away from its original position.

Defining stability as the possibility for the particle to come back to its original position after any small disturbance, we see that the resisting medium confirms the instability of three positions of relative equilibrium, and makes the two positions of contingent stability necessarily unstable.

A word or two about Dr. See's argument in favour of capture in general. Allowing, what is not necessarily true of the general resisting medium, that the additional term entering the Jacobian integral because of the medium, is positive, it does not even then follow according to his argument that there must be capture. In fact we have already seen that for a body to be captured under such circumstances, it must in its past history have been already intimately associated with the Sun-Jove system. Let us now generalise the motion of our resisting medium. Let it rotate as a whole round the centre of mass of S and J with an angular velocity n' . The equations become

$$\begin{aligned} \frac{d}{dt} \left(\frac{dX}{dt} - nY \right) - n \left(\frac{dY}{dt} + nX \right) &= \frac{\partial U}{\partial X} - f(x, y, t) \varphi(V') \frac{1}{V'} \left(\frac{dX}{dt} - nY + n'Y \right) \\ \frac{d}{dt} \left(\frac{dY}{dt} + nX \right) + n \left(\frac{dX}{dt} - nY \right) &= \frac{\partial U}{\partial Y} - f(x, y, t) \varphi(V') \frac{1}{V'} \left(\frac{dY}{dt} + nX - n'X \right) \end{aligned}$$

where

$$V' = \left[\left(\frac{dX}{dt} - Y(n - n') \right)^2 + \left(\frac{dY}{dt} + X(n - n') \right)^2 \right]^{1/2}$$

i. e. velocity relative to the medium.

We get

$$\begin{aligned} \frac{d}{dt} (v^2 - 2\Omega) &= -2f(x, y, t) \frac{\varphi(V')}{V'} \left[v^2 + (n - n') \left(X \frac{dY}{dt} - Y \frac{dX}{dt} \right) \right] \\ &= -2f(x, y, t) \frac{\varphi(V')}{V'} \left(v^2 + (n - n') R^2 \frac{d\Theta}{dt} \right) \\ v^2 &= 2\Omega - C - 2 \int_0^t f(x, y, t) \frac{\varphi(V')}{V'} \left(v^2 + (n - n') R^2 \frac{d\Theta}{dt} \right) dt = 2\Omega - C - I. \end{aligned}$$

The integrand in I is positive if $v^2 + (n - n') R^2 \frac{d\Theta}{dt}$ is positive

and negative if $v^2 + (n - n') R^2 \frac{d\Theta}{dt}$ is negative.

No definite statement as to capture is at all possible. Only rigorous integration of the equations of motion would suffice to give any such definite information as Dr. See offers; and such integration has not yet been effected, even without a resisting medium.

A medium which yields slightly more tractable results has been suggested to me by Sir G. Darwin. He proposes to consider the different particles of the medium to move in circles round the Sun as centre of force, neglecting their effect on one another. As v is usually very large, this amounts to assigning to them circular orbits round the centre of mass of the Sun and Jove. n' will now vary with R , and $n - n'$ will be positive for points outside the orbit of Jove, and negative for points inside. If $\frac{d\Theta}{dt}$ is of the same sign as $n - n'$, then the integrand in I is positive, and we can apply the results obtained in our first case, viz. the resisting medium rotating as a whole with S-J. We must then make $\frac{d\Theta}{dt}$ positive outside the orbit of Jove, and negative inside. If then a small body moving under the influence of

the Sun-Jove system, has had a position of momentary relative rest outside the oval $C = 38.88$, and moves in a large direct orbit round S-J, it will not be captured. If it has a more or less circular direct orbit round J of moderate size, and has had a position of momentary rest inside the hour-glass $C = 38.88$, and never one outside the oval $C = 38.88$, it will almost certainly be captured. But as already remarked, such capture could hardly be styled capture from outside. Further than this we cannot go.

In conclusion I should like to say that the above work shows us that the capture theory propounded by Dr. See is based on such very uncertain mathematical arguments, as to render the very possibility of capture in actual nature with an assumed resisting medium very uncertain. Need I further point out that to assume that such a considerable satellite as a moon was formed in this way is very improbable? It is significant that whereas the major planets in the Solar System have been able to »capture« several comets, rendering them periodic, the earth has failed to accomplish this in any case that we know.

I cannot but acknowledge my great indebtedness to Sir George Darwin for his kindly criticism and encouragement.

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Cometa 1910 a.

| 1910 | T.m. Arcetri | $\Delta\alpha$ | $\Delta\delta$ | Cf. | α app. | $\log pA$ | δ app. | $\log pA$ | Red. ad l. app. | * |
|----------|--|------------------------------------|----------------|------|---|--------------------|---------------|-----------|--|---|
| Febb. 1 | 6 ^h 24 ^m 32 ^s | -1 ^m 25 ^s 53 | -5' 17".6 | 8,4 | 21 ^h 38 ^m 0 ^s 64 | 9.625 | + 3° 4' 41".2 | 0.778 | -1 ^s 86 - 10 ^s 5 | 1 |
| 5 | 6 21 7 | -0 14.79 | +2 27.5 | 12,8 | 21 46 0.26 | 9.627 | + 5 23 26.2 | 0.775 | -1.84 - 10.7 | 2 |
| 7 | 6 26 49 | +0 2.30 | -5 33.7 | 1,1 | 21 49 28.46 | 9.629 | + 6 20 38.1 | 0.777 | -1.83 - 10.8 | 3 |
| 11 | 6 33 1 | -1 10.75 | +9 23.5 | 8,4 | 21 55 28.72 | 9.632 | + 7 58 40.3 | 0.778 | -1.81 - 11.0 | 4 |
| Marz. 20 | 16 24 0 | +0 52.38 | +3 20.9 | 4,2 | 22 29 40.68 | 9.647 _n | +17 10 29.6 | 0.771 | -1.54 - 13.3 | 5 |

Stelle di confronto.

| * | α 1910.0 | δ 1910.0 | Autorità | * | α 1910.0 | δ 1910.0 | Autorità |
|---|--|-----------------|-------------------------------------|---|--|-----------------|-----------------|
| 1 | 21 ^h 39 ^m 28 ^s 03 | + 3° 10' 9".3 | AG Alb 7589 | 3 | 21 ^h 49 ^m 27 ^s 99 | + 6° 26' 22".6 | AG Lpz II 11000 |
| 2 | 21 46 16.89 | + 5 21 9.4 | 1/2 (AG Alb 7627 + AG Lpz II 10968) | 4 | 21 56 41.28 | + 7 49 27.8 | " 11071 |
| | | | | 5 | 22 28 49.84 | +17 7 22.0 | AG Berl A 9218 |