ON INVARIANTS AND COVARIANTS OF LINEAR HOMOGENEOUS DIFFERENTIAL EQUATIONS

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IT is known* that the most general transformations which convert the system of linear homogeneous differential equations

$$(1) \qquad y_i'' + \sum_{k=1}^{n} (2p_{ik}y_k' + q_{ik}y_k) = 0 \quad (i = 1, 2, \ldots, n),$$

where p_{ik} and q_{ik} are functions of the independent variable x, into another of the same form, are given by the equations

$$(2) \qquad y_\kappa = \sum_{\lambda=1}^{n} a_{\kappa\lambda}(x)\, \eta_\lambda \quad (\kappa = 1, 2, \ldots, n),$$

$$(3) \qquad \xi = \xi(x),$$

where $a_{\kappa\lambda}$ and ξ are arbitrary functions of x, for which the determinant

$$|a_{\kappa\lambda}| \quad (\kappa, \lambda = 1, 2, \ldots, n)$$

does not vanish identically. A function of the coefficients of (1) and their derivatives, and of the dependent variables and their derivatives, which has the same value for (1) as for any system derived from (1) by the transformations (2), is called a *semi-covariant*; if the function keeps the same value when (1) is transformed by (3) also, it is called a *covariant*. A semi-covariant or a covariant is called a *seminvariant* or an *invariant*, respectively, if the dependent variables and their derivatives do not occur in the function.

In a preceding paper† the writer has obtained a complete system of seminvariants for the system (1). It is the object of the present paper to complete the problem by the calculation of complete systems of invariants,

* Wilczynski, *Projective Differential Geometry of Curves and Ruled Surfaces*, Leipzig, 1906, Chap. I.

† "On Seminvariants of Linear Homogeneous Differential Equations", 1916, *Proc. London Math. Soc.*, Ser. 2, Vol. 15, p. 217.

semi-covariants, and covariants. The methods used largely avoid the solution of the complicated systems of partial differential equations which arise by the Lie theory.

By means of the system (1) and its invariants and covariants we can study the projective differential properties of any spread (except for special cases) generated by ∞^1 flats in space of any number of dimensions. It is thus unnecessary to employ systems of higher order. This geometry will be discussed in papers to follow.

1. *The Invariants.*

As invariants are also seminvariants, the calculation of invariants merely requires the discovery of functions of the seminvariants which keep the same value after the system (1) is transformed in accordance with equation (3).

In the calculation of the seminvariants, it was found convenient to introduce auxiliary functions of the coefficients of (1) and of their derivatives, denoted by

$$u_{ik},\ v_{ik},\ w_{ik}\quad (i, k = 1, 2, \ldots, n),$$

and defined by the equations

$$(4)\qquad \left\{\begin{aligned} u_{ik} &= p'_{ik}-q_{ik}+\sum_{j=1}^{n} p_{ij}p_{jk} \\ v_{ik} &= u'_{ik}+\sum_{j=1}^{n}(p_{ij}u_{jk}-p_{jk}u_{ij}) \\ w_{ik} &= v'_{ik}+\sum_{j=1}^{n}(p_{ij}v_{jk}-p_{jk}v_{ij}) \end{aligned}\right\}\quad (i, k = 1, 2, \ldots, n).$$

If I is a seminvariant involving u_{ik}, r successive applications of the operator

$$D_u = \sum_{i=1}^{n}\frac{\partial}{\partial u_{ii}}$$

upon I are indicated by

$$I^{(r)} = D_u^r I = \frac{1}{r!}\sum_{i=1}^{n}\left(\frac{\partial}{\partial u_{ii}}\right)^r I.$$

Similarly, successive applications of the operators

$$D_{uv} = \sum_{i=1}^{n}\sum_{k=1}^{n} v_{ik}\frac{\partial}{\partial u_{ik}},$$

$$D_{vw} = \sum_{i=1}^{n}\sum_{k=1}^{n} w_{ik}\frac{\partial}{\partial v_{ik}}$$

are indicated by

$$I^{(rs)} = D_{uv}^{s} I^{(r)} = \frac{1}{s!} \sum_{i=1}^{n} \sum_{k=1}^{n} \left(v_{ik} \frac{\partial}{\partial u_{ik}}\right)^{s} I^{(r)},$$

$$I^{(rst)} = D_{vw}^{t} I^{(rs)} = \frac{1}{t!} \sum_{i=1}^{n} \sum_{k=1}^{n} \left(w_{ik} \frac{\partial}{\partial v_{ik}}\right)^{t} I^{(rs)},$$

respectively.

For the purpose of the calculation of the invariants it is convenient to replace the complete system of seminvariants* consisting of

$$(5) \quad \begin{cases} I^{(rs)}\,(r = 0, 1, \ldots, n-1\,;\ s = 0, 1, 2, \ldots\,;\ r+s \leqslant n),\ I_1^{(rs)}\,(s > 2), \\ I^{(rs1)}\,(s > 0), \quad D_{uv}^{t} J\ (t = 0, 1, \ldots, n-2), \end{cases}$$

and certain derivatives, by the system consisting of

$$(6) \quad \begin{cases} I^{(r00)} = I^{(r)} \quad (r = 0, 1, \ldots, n-1), \\ I^{(rs0)} = I^{(rs)}\,(r = 0, 1, \ldots, n-1\,;\ s = 1, 2, \ldots\,;\ r+s \leqslant n), \\ I^{(rst)}\ (r = 0, 1, \ldots, n-1\,;\ s = 1, 2, \ldots\,;\ r+s \leqslant n\,;\ t = 1, 2, 3\,;\ t \leqslant s), \end{cases}$$

and certain derivatives.

These seminvariants are proved to be independent by methods very similar to those by which the seminvariants (5) were proved to be independent. In fact, it is only necessary to find a non-vanishing determinant of maximum order in their functional matrix. First put

$$u_{ik} = 0 \quad (i \neq k).$$

It is then easy to see that there exists a determinant of maximum order which has for one factor the cube of the non-vanishing Jacobian of the elementary symmetric functions of $u_{11}, u_{22}, \ldots, u_{nn}$. The other factor F is a determinant in which the coefficient of the highest power of u_{nn} is different from zero. In fact, this coefficient is the product of the determinant which corresponds to F, in the case of $n-1$ variables in equations (1), and the functional determinant of $I^{(rst)}$ $(r+s = n,\ t = 0, 1, 2, 3)$ with respect to all variables v_{ik}, w_{ik} for which $i = n$ or $k = n$, except $v_{n,\,n-1}$, v_{nn}, w_{nn}. If we arrange the rows and columns in proper order in this latter determinant, and put w_{in} $(i = 1, 2, \ldots, n-2)$ and all v_{ik} except $v_{i,\,i-1}$ $(i = 2, 3, \ldots, n)$ equal to zero, the determinant has all elements of the principle diagonal different from zero and all elements above the

* Stouffer, *loc. cit.*, p. 226.

principle diagonal equal to zero. Thus this factor does not vanish, and the whole functional determinant is different from zero, provided it is non-vanishing in the case of $n-1$ variables in (1). But it is easily seen to be different from zero in the case of $n = 3$. Hence by induction the seminvariants are proved to be independent.

The system (6) contains $2n^2+1$ seminvariants. But the system (5) also contains $2n^2+1$ independent seminvariants, and these with the successive derivatives of the n^2 seminvariants of (5) which contain w_{ik} form a complete system. Therefore the system (6) and the successive derivatives of any n^2 of the seminvariants $I^{(rst)}$ $(t = 1, 2, 3)$ form a complete system. We shall use the successive derivatives of the n^2 seminvariants for which $t = 1, 2$.

In order to calculate the invariants we shall need to find the effect of the transformation (3) upon the seminvariants. By this transformation equations (1) are converted into

$$\frac{d^2 y_i}{d\xi^2}+\sum_{k=1}^{n}\left(2\bar{p}_{ik}\frac{dy_k}{d\xi}+\bar{q}_{ik}y_k\right)=0 \quad (i = 1, 2, \ldots, n),$$

where

$$(7) \qquad \begin{cases} \bar{p}_{ii} = \dfrac{1}{\xi'}(p_{ii}+\tfrac{1}{2}\eta), \\ \bar{p}_{ik} = \dfrac{1}{\xi'}p_{ik} \quad (i \neq k), \\ \bar{q}_{ik} = \dfrac{1}{(\xi')^2}q_{ik}, \end{cases}$$

with

$$(8) \qquad \eta = \frac{\xi''}{\xi'}.$$

If the transformation (3) is made infinitesimal by putting

$$(9) \qquad \xi(x) = x+\phi(x)\,\delta t,$$

where $\phi(x)$ is an arbitrary function of x and δt an infinitesimal, we have, on neglecting higher powers of δt than the first,

$$(10) \qquad \begin{cases} \delta p_{ii} = \bar{p}_{ii}-p_{ii} = (-\phi' p_{ii}+\tfrac{1}{2}\phi'')\,\delta t, \\ \delta p_{ik} = \bar{p}_{ik}-p_{ik} = -\,\phi' p_{ik}\,\delta t \quad (i \neq k), \\ \delta q_{ik} = \bar{q}_{ik}-q_{ik} = -\,2\phi' q_{ik}\,\delta t\,; \end{cases}$$

whence, by substitution in (4),

$$
(11)\qquad \begin{cases}
\delta u_{ii} = (-2\phi' u_{ii} + \tfrac{1}{2}\phi^{(3)})\delta t, \\
\delta u_{ik} = -2\phi' u_{ik}\delta t \quad (i \neq k), \\
\delta v_{ii} = (-3\phi' v_{ii} - 2\phi'' u_{ii} + \tfrac{1}{2}\phi^{(4)})\delta t, \\
\delta v_{ik} = (-3\phi' v_{ik} - 2\phi'' u_{ik})\delta t \quad (i \neq k), \\
\delta w_{ii} = (-4\phi' w_{ii} - 5\phi'' v_{ii} - 2\phi^{(3)} u_{ii} + \tfrac{1}{2}\phi^{(5)})\,\delta t, \\
\delta w_{ik} = (-4\phi' w_{ik} - 5\phi'' v_{ik} - 2\phi^{(3)} u_{ik})\,\delta t \quad (i \neq k).
\end{cases}
$$

From these values for δu_{ik} and the definition of $I^{(r)}$, it follows that

$$
\delta I^{(r)} = \sum_{u_{ik}} \frac{\partial I^{(r)}}{\partial u_{ik}} \delta u_{ik} = \sum_{u_{ik}} \frac{\partial I^{(r)}}{\partial u_{ik}} (-2\phi' u_{ik}\delta t) + \sum_{v_{ii}} \frac{\partial I^{(r)}}{\partial u_{ii}} (\tfrac{1}{2}\phi^{(3)}\delta t),
$$

or

$$
(12)\qquad \delta I^{(r)} = -2(n-r)\,\phi' I^{(r)}\delta t + \tfrac{1}{2}(r+1)\,\phi^{(3)} I^{(r+1)}\delta t.
$$

If f is an invariant, the system of partial differential equations* which it must satisfy is obtained by equating to zero the symbols of the infinitesimal transformations of f. If f depends only on $I^{(0)}, I^{(1)}, \ldots, I^{(n-1)}$, equations (12) show that it must satisfy

$$
(13)\qquad \begin{cases}
\sum_{j=0}^{n} (j+1)\, I^{(j+1)} \dfrac{\partial f}{\partial I^{(j)}} = 0, \\
\sum_{j=0}^{n} (n-j)\, I^{(j)} \dfrac{\partial f}{\partial I^{(j)}} = 0.
\end{cases}
$$

We proceed to find all these invariants. The second equation of (13) merely requires that absolute invariants be isobaric of weight zero, or relative invariants be isobaric. Since there are n independent variables and two independent equations in the system (13), there are $n-2$ absolute invariants or $n-1$ relative invariants here. The method of undetermined coefficients shows that two isobaric functions which satisfy the first equation of (13) are

$$
\Theta_2 = (I^{(n-1)})^2 - \frac{2n}{n-1} I^{(n-2)},
$$

$$
\Theta_3 = (I^{(n-1)})^3 - 3I^{(n-1)}\Theta_2 - \frac{6n^2}{(n-1)(n-2)} I^{(n-3)}.
$$

* Lie, *Mathematische Annalen*, 1884, Vol. 24, p. 537.

It is now easily shown by induction that the first equation of (13) is satisfied by the isobaric functions

$$\Theta_\alpha = (I^{(n-1)})^\alpha - \alpha! \sum_{i=2}^{\alpha-1} \frac{1}{(\alpha-i)!\, i!} (I^{(n-1)})^{\alpha-i}\,\Theta_i$$

$$- \frac{\alpha!\, n^{\alpha-1}}{(n-1)(n-2)\ldots(n-\alpha+1)} I^{(n-\alpha)} \quad (\alpha = 2, 3, \ldots, n).$$

These $n-1$ expressions are relative invariants whose respective weights are twice the subscript of Θ. They are independent since each contains an independent seminvariant not contained in those of lower weight. Since there are only $n-1$ such relative invariants, we have them all.

We next proceed to find the invariants involving $I^{(rs)}$ $(s > 0)$. If $\Theta_{\alpha+\beta}$ denotes such a relative invariant, homogeneous of degree α in u_{ik} and of degree β in v_{ik}, that is, isobaric of weight 2α in μ_{ik} and isobaric of weight 3β in v_{ik}, the transformation (3) converts it into $\bar{\Theta}_{\alpha+\beta}$, where

$$\bar{\Theta}_{\alpha+\beta} = \frac{1}{(\xi')^{2\alpha+3\beta}} \Theta_{\alpha+\beta},$$

and the infinitesimal transformation (9) gives to $\Theta_{\alpha+\beta}$ the increment

$$\delta\Theta_{\alpha+\beta} = -(2\alpha+3\beta)\,\phi'\Theta_{\alpha+\beta}\,\delta t.$$

It follows that

$$\delta(D_{uv}\Theta_{\alpha+\beta}) = \sum_{i=1}^{n}\sum_{k=1}^{n} v_{ik}\frac{\partial}{\partial u_{ik}}\left[\delta\Theta_{\alpha+\beta} - \delta u_{ik}\frac{\partial}{\partial u_{ik}}\Theta_{\alpha+\beta}\right] + \sum_{i=1}^{n}\sum_{k=1}^{n} \delta v_{ik}\frac{\partial}{\partial u_{ik}}\Theta_{\alpha+\beta}$$

$$= \{-(2\alpha+3\beta-2)\,\phi' D_{uv}\Theta_{\alpha+\beta}$$

$$+(-3\phi' D_{uv}\Theta_{\alpha+\beta} - 2\alpha\phi''\Theta_{\alpha+\beta} + \tfrac{1}{2}\phi^{(4)} D_u\Theta_{\alpha+\beta})\}\,\delta t,$$

$$\text{(14)} \quad \frac{\delta(D_{uv}\Theta_{\alpha+\beta})}{\delta t} = -(2\alpha+3\beta+1)\,\phi' D_{uv}\Theta_{\alpha+\beta} - 2\alpha\phi''\Theta_{\alpha+\beta} + \tfrac{1}{2}\phi^{(4)} D_u\Theta_{\alpha+\beta}.$$

Now $$D_u\Theta_2 = 0,$$

whence by induction $D_u\Theta_\alpha = 0$ $(\alpha = 2, 3, \ldots, n)$.

Therefore, if we put $\beta = 0$ in (14), we have

$$\frac{\delta(D_{uv}\Theta_\alpha)}{\delta t} = -(2\alpha+1)\,\phi' D_{uv}\Theta_\alpha - 2\alpha\phi''\Theta_\alpha.$$

Consequently, the isobaric expressions

$$(15) \qquad \Theta_{a,1} = 2\Theta_2 D_{uv}\Theta_a - a\Theta_a D_{uv}\Theta_2 \quad (a = 3, 4, \ldots, n),$$

are relative invariants of weight $2a+2$ in u_{ik} and 3 in v_{ik}.

A more general form of (15) is obtained if we notice that, according to (14),

$$(16) \qquad \Theta_{a+\beta,1} = 2\Theta_2 D_{uv}\Theta_{a+\beta} - a\Theta_{a+\beta} D_{uv}\Theta_2$$

is a relative invariant provided that

$$D_u\Theta_{a+\beta} = 0.$$

Since

$$(17) \qquad D_u(D^r_{uv}\Theta_a) = D^r_{uv}(D_u\Theta_a) = 0,$$

we have
$$D_u\Theta_{a,1} = 0.$$

Therefore

$$\Theta_{a,2} = 2\Theta_2 D_{uv}\Theta_{a,1} - (a+1)\Theta_{a,1} D_{uv}\Theta_2 \quad (a = 3, 4, \ldots, n)$$

are invariants. Again, by (17),

$$D_u\Theta_{a,2} = 0,$$

whence
$$\Theta_{a,3} = 2\Theta_2 D_{uv}\Theta_{a,2} - (a+2)\Theta_{a,2} D_{uv}\Theta_2 \quad (a = 3, 4, \ldots, n)$$

are invariants. In fact a continuation of this reasoning gives the $\frac{n^2+n-6}{2}$ relative invariants

$$\Theta_{a,\beta} = 2\Theta_2 D_{uv}\Theta_{a,\beta-1} - (a+\beta-1)\Theta_{a,\beta-1} D_{uv}\Theta_2$$
$$(a = 3, 4, \ldots, n;\ \beta = 1, 2, \ldots, a),$$

whose respective weights are $2a+2\beta$ in u_{ik} and 3β in v_{ik}.

The invariants Θ_a, $\Theta_{a,\beta}$ are $\frac{n^2+3n-8}{2}$ in number. Moreover they are independent, since each contains a seminvariant not in those which precede when they are arranged in order of ascending values of a and β. The system of partial differential equations which these invariants must satisfy contains 4 equations and $\frac{n^2+3n}{2}$ variables $I^{(rs)}$. There are, therefore,

$$\frac{n^2+3n}{2} - 4 + 1 = \frac{n^2+3n-6}{2}$$

independent relative invariants here. We have all but one. It is easily found by the method of undetermined coefficients that

$$\vartheta_{0,2} = 4\Theta_2(I^{(n-1,1)})^2 - \frac{8n}{n-1} I^{(n-2,2)} \Theta_2 - (\Theta_2')^2$$

is a relative invariant of weight 10. That it is independent of the other invariants is seen by arranging them in the order Θ_a, $\vartheta_{0,2}$, $\Theta_{a,\beta}$, and in order of ascending values of a and β.

We continue the process of finding the invariants by seeking those which involve $I^{(rst)}$ $(t = 1, 2, 3)$. It may be verified easily by direct substitution that the expressions

$$\vartheta_{2a+1} = 4a\Theta_a''\Theta_a - (4a+1)(\Theta_a')^2 + \frac{16a^2}{n} I^{(n-1)}\Theta_a^2,$$

called the quadriderivatives* of Θ_a, are relative invariants. The quadriderivatives of Θ_2 and Θ_3

$$\vartheta_5 = 8\Theta_2''\Theta_2 - 9(\Theta_2')^2 + \frac{64}{n} I^{(n-1)}\Theta_2^2,$$

$$\vartheta_7 = 12\Theta_3''\Theta_3 - 13(\Theta_3')^2 + \frac{144}{n} I^{(n-1)}\Theta_3^2,$$

contain $I^{(n-1,1,1)}$, $I^{(n-2,1,1)}$, and $I^{(n-3,1,1)}$, but no other seminvariants $I^{(rst)}$ $(t > 0)$. For Θ_2 and Θ_3 contain only $I^{(n-1)}$, $I^{(n-2)}$, and I^{n-3}, whose second derivatives expressed in terms of seminvariants are respectively

$$I^{(n-1,1,1)}, \quad 2I^{(n-2,2)} + I^{(n-2,1,1)}, \quad 2I^{(n-3,2)} + I^{(n-3,1,1)}.$$

It is not difficult to verify the extended form of equation (14),

$$(18) \qquad \frac{\delta(D_{vw}\Theta_{a,\beta})}{\delta t} = -\phi'(2a+5\beta+1)\,D_{vw}\Theta_{a,\beta} - 5\beta\phi''\Theta_{a,\beta} - 2\phi^{(3)}D_{vu}\Theta_{a,\beta} + \tfrac{1}{2}\phi^{(5)}D_v\Theta_{a,\beta}.$$

Moreover, by induction, it is easily shown that

$$D_{vu}\Theta_{a,\beta} = 0, \qquad D_v\Theta_{a,\beta} = 0.$$

Consequently (18) becomes

$$(19) \qquad \frac{\delta(D_{vw}\Theta_{a,\beta})}{\delta t} = -\phi'(2a+5\beta+1)D_{vw}\Theta_{a,\beta} - 5\beta\phi''\Theta_{a,\beta}.$$

* Wilczynski, *loc. cit.*, p. 112.

Therefore, the expressions

(20) $$\Theta_{\alpha,\beta,1} = \Theta_{3,1} D_{vw} \Theta_{\alpha,\beta} - \beta\Theta_{\alpha,\beta} D_{vw} \Theta_{3,1}$$

$$(\alpha = 3, 4, \ldots, n;\ \beta = 1, 2, \ldots, \alpha),$$

are relative invariants of weights $2\alpha+5\beta+12$. Of course $\Theta_{3,1,1}$ vanishes identically. Moreover, since

$$D_{vw}(D^l_{vw}\Theta_{\alpha,\beta}) = 0, \qquad D_v(D^l_{vw}\Theta_{\alpha,\beta}) = 0,$$

we have $$\frac{\delta(D_{vw}\Theta_{\alpha,\beta,1})}{\delta t} = -\phi'(2\alpha+5\beta+13)D_{vw}\Theta_{\alpha,\beta,1} - 5\beta\phi''\Theta_{\alpha,\beta,1},$$

whence it follows that

$$\Theta_{\alpha,\beta,2} = \Theta_{3,1} D_{vw}\Theta_{\alpha,\beta,1} - \beta\Theta_{\alpha,\beta,1} D_{vw}\Theta_{3,1}$$

is an invariant of weight $2\alpha+5\beta+24$. A continuation of this process shows that

$$\Theta_{\alpha,\beta,\gamma} = \Theta_{3,1} D_{vw}\Theta_{\alpha,\beta,\gamma-1} - \beta\Theta_{\alpha,\beta,\gamma-1} D_{vw}\Theta_{3,1}$$

$$(\alpha = 3, 4, \ldots, n;\ \beta = 1, 2, \ldots, \alpha;\ \gamma = 1, 2, \ldots, \beta),$$

are invariants of weight $2\alpha+5\beta+12\gamma$, respectively. If only the seminvariants (6) are to be included, γ must be limited, to the values 1, 2, 3.

Again, $$\frac{\delta(D_{vw}\vartheta_{0,2})}{\delta t} = -11\phi' D_{vw}\vartheta_{0,2} - 10\phi''\vartheta_{0,2},$$

which shows that

$$\vartheta_{0,3} = 2\Theta_2 D_{vw}\vartheta_{0,2} - 5\Theta_2'\vartheta_{0,2},$$

$$\vartheta_{0,4} = 4\Theta_2(2\Theta_2 D^2_{vw}\vartheta_{0,2} - 5\Theta_2' D_{vw}\vartheta_{0,2}) + 25(\Theta_2')^2\vartheta_{0,2}$$

are relative invariants.

If the invariants

(21) $$\Theta_\alpha,\ \vartheta_{0,2},\ \Theta_{\alpha,\beta},\ \vartheta_5,\ \vartheta_7,\ \vartheta_{0,3},\ \vartheta_{0,4},\ \Theta_{\alpha,\beta,\gamma}$$

$$(\alpha = 2, 3, 4, \ldots, n;\ \beta = 1, 2, \ldots, \alpha;\ \gamma = 1, 2, 3;\ \gamma \leqslant \beta)$$

are arranged in this order and in order of ascending values of α, β, γ, they are seen to be independent since each contains a seminvariant $J^{(rst)}$ not in those which precede.

In each invariant the weight is twice the first subscript plus five times the second, if it occurs, plus twelve times the third, if it occurs.

The system of partial differential equations which the invariants depending on the seminvariants (6) must satisfy contains five independent equations and $2n^2+1$ variables, and has, therefore, $2n^2-4$ solutions. Thus there are $2n^2-3$ such relative invariants. The set (21) contains the required number of independent relative invariants.

The complete system of seminvariants involves, in addition to the seminvariants (6), the successive derivatives of n^2 of the seminvariants $I^{(rst)}$, say those for which $t=1, 2$. The first derivative thus introduces n^2 new seminvariants, and the system of partial differential equations for the invariants has one more equation than before. Thus there are n^2-1 new relative invariants.

To obtain these let us notice that, from two relative invariants Θ_μ, Θ_ν of weights μ and ν, respectively, we can obtain a relative invariant

$$\mu\Theta_\mu\Theta'_\nu-\nu\Theta_\nu\Theta'_\mu,$$

the so called Jacobian* of Θ_μ and Θ_ν. Thus the combination by the Jacobian process of some invariant, say Θ_2, with the n^2-1 invariants ϑ_5, ϑ_7, $\vartheta_{0,3}$, $\vartheta_{0,4}$, $\Theta_{\alpha,\beta,1}$, $\Theta_{\alpha,\beta,2}$ gives n^2-1 new relative invariants involving first derivatives of $I^{(rs1)}$ and $I^{(rs2)}$. Moreover they must be independent of each other and of the invariants (21). For, inasmuch as these new invariants are merely combinations of the invariants (21) and their derivatives, a relation involving them would imply a relation between the invariants (21).

The n^2-1 invariants involving the next higher derivatives of $I^{(rst)}$ $(t=1, 2)$ can be obtained by combining Θ_2 by the Jacobian procsss with the n^2-1 just obtained. A continuation of this process gives the relative invariants involving derivatives to as high an order as desired.

We thus have proved that:

The invariants (21) *and those obtained from them by the Jacobian process form a system of invariants, complete in the sense that all invariants can be expressed in terms of them.*

2. *Semi-Covariants.*

It is not necessary to seek semi-covariants which involve higher derivatives of y_i than the first, for such higher derivatives may be removed by means of equation (1). Since seminvariants are also semi-covariants, the systems of partial differential equations which semi-covariants must satisfy are the same as the systems which the

* Wilczynski, *loc. cit.*, p. 112.

seminvariants must satisfy, with the addition of the terms which arise from the infinitesimal transformation of y_i and y'_i. Thus to the system* which determines the seminvariants involving p_{ik}, p'_{ik}, p''_{ik}, q_{ik}, q'_{ik} is added the $2n$ variables y_i and y'_i but no new equations. However, the one relation between those equations ceases to hold, so that there are only $2n-1$ more solutions, that is, there are $2n-1$ absolute semi-covariants, or $2n$ relative semi-covariants, which are not seminvariants. Moreover, the systems which determine the semi-covariants involving higher derivatives of p_{ik} and q_{ik} show that there are no more independent semi-covariants. We proceed now to find $2n$ independent semi-covariants.

The transformation (2) may be made infinitesimal by putting

$$a_{ii}(x) = 1+\phi_{ii}(x)\,\delta t, \quad a_{ik}(x) = \phi_{ik}(x)\,\delta t \quad (i \neq k\,;\ i, k = 1, 2, \ldots, n),$$

where δt is an infinitesimal and the ϕ_{ik}'s are arbitrary functions of x. The infinitesimal transformations of y_i, p_{ik}, u_{ik}, v_{ik} are then found by direct substitution to be

$$\left.\begin{aligned}
\frac{\delta y_i}{\delta t} &= -\sum_{j=1}^{n} \phi_{ij}\, y_j \\
\frac{\delta p_{ik}}{\delta t} &= \sum_{j=1}^{n} (\phi_{jk}\, p_{ij} - \phi_{ij}\, p_{jk}) + \phi'_{ik} \\
\frac{\delta u_{ik}}{\delta t} &= \sum_{j=1}^{n} (\phi_{jk}\, u_{ij} - \phi_{ij}\, u_{jk}) \\
\frac{\delta v_{ik}}{\delta t} &= \sum_{j=1}^{n} (\phi_{jk}\, v_{ij} - \phi_{ij}\, v_{jk})
\end{aligned}\right\} \quad (i, k = 1, 2, \ldots, n).$$

The expressions u_{ik} and v_{ik} are therefore cogredient.

The quantities $$r_i^{(1)} = \sum_{j=1}^{n} u_{ij}\, y_j \quad (i = 1, 2, \ldots, n)$$

are cogredient with y_i. For

$$\begin{aligned}
\frac{\delta r_i^{(1)}}{\delta t} &= \sum_{j=1}^{n} \left[u_{ij} \sum_{\lambda=1}^{n} (-\phi_{j\lambda}\, y_\lambda) + y_j \sum_{\lambda=1}^{n} (\phi_{\lambda j}\, u_{i\lambda} - \phi_{i\lambda}\, u_{\lambda j}) \right] \\
&= -\sum_{j=1}^{n} \phi_{ij} \sum_{\lambda=1}^{n} u_{j\lambda}\, y_\lambda \\
&= -\sum_{j=1}^{n} \phi_{ij}\, r_j^{(1)}.
\end{aligned}$$

* Stouffer, *loc. cit.*, pp. 221, 222.

It follows that each of the $n-2$ sets of quantities

$$\left.\begin{aligned} r_i^{(2)} &= \sum_{j=1}^{n} u_{ij} r_j^{(1)} \\ r_i^{(3)} &= \sum_{j=1}^{n} u_{ij} r_j^{(2)} \\ &\cdots \quad \cdots \quad \cdots \\ r_i^{(n-1)} &= \sum_{j=1}^{n} u_{ij} r_j^{(n-2)} \end{aligned}\right\} \quad (i = 1, 2, \ldots, n)$$

are also cogredient with y_i. Therefore the determinant

$$R = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ r_1^{(1)} & r_2^{(1)} & \cdots & r_n^{(1)} \\ r_1^{(2)} & r_2^{(2)} & \cdots & r_n^{(2)} \\ \cdots & \cdots & \cdots & \cdots \\ r_1^{(n-1)} & r_2^{(n-2)} & \cdots & r_n^{(n-1)} \end{vmatrix}$$

is a semi-covariant.

Again, since u_{ik} and v_{ik} are cogredient, the quantities

$$s_i = \sum_{j=1}^{n} v_{ij} y_j \quad (i = 1, 2, \ldots, n)$$

are cogredient with y_i. Therefore the $n-1$ determinants

$$S_i = \sum_{j=1}^{n} s_j \frac{\partial R}{\partial r_j^{(i)}} \quad (i = 1, 2, \ldots, n-1)$$

are relative semi-covariants.

In order to prove that R and S_i are independent of each other and of the seminvariants, we need only show that the functional determinant

$$\frac{\partial(R, S_1, S_2, \ldots, S_{n-1})}{\partial(y_1, y_2, y_3, \ldots, y_n)} \tag{22}$$

does not vanish identically. If we put $u_{ik} = 0$ $(i \neq k)$, the highest power of u_{nn} in R and in all S_i except S_{n-1} occurs in each semi-covariant in the form $u_{nn}^{n-1} y_n$, multiplied by the corresponding semi-covariant for the case of $n-1$ variables in equations (1). Moreover, u_{nn}^{n-1} does not occur in S_{n-1}. Hence in the functional determinant above the highest power of u_{nn} occurs in the form $(u_{nn}^{n-1} y_n)^{n-1}$, multiplied by the partial derivative of S_{n-1} with respect to y_n, and by the functional determinant corresponding to (22), with $u_{ik}=0$ $(i \neq k)$, in the case of $n-1$ variables in (1). If the latter determinant is different from zero, (22) cannot vanish identically. But (22), with $u_{ik}=0$ $(i\neq k)$, does not vanish identically for $n=2$, as is easily verified, and it follows by induction that (22) does not vanish identically.

Again the quantities $\quad t_i = y'_i + \sum_{j=1}^{n} p_{ij}\, y_j \quad (i = 1, 2, \ldots, n)$

are cogredient with y_i. For

$$\begin{aligned}
\frac{\delta t_i}{\delta t} &= -\sum_{j=1}^{n} (\phi'_{ij} y_j + \phi_{ij} y'_j) \\
&\qquad + \sum_{j=1}^{n} \left[p_{ij} \sum_{\lambda=1}^{n} (-\phi_{j\lambda} y_\lambda) + y_j \sum_{\lambda=1}^{n} (\phi_{\lambda j} p_{i\lambda} - \phi_{i\lambda} p_{\lambda j}) + y_j \phi'_{ij} \right] \\
&= -\sum_{j=1}^{n} \phi_{ij} \left[y'_j + \sum_{\lambda=1}^{n} p_{j\lambda} y_\lambda \right] \\
&= -\sum_{j=1}^{n} \phi_{ij}\, t_j.
\end{aligned}$$

Hence the n determinants

$$T_y = \sum_{j=1}^{n} t_j \frac{\partial R}{\partial y_j},$$

$$T_i = \sum_{j=1}^{n} t_j \frac{\partial R}{\partial r_j^{(i)}} \quad (i = 1, 2, \ldots, n-1),$$

are semi-covariants.

To prove that T_y and T_i are independent of each other and of R, S_i and the seminvariants it is only necessary to show that the determinant

$$\frac{\partial (T_y, T_1, T_2, \ldots, T_{n-1})}{\partial (y'_1, y'_2, y'_3, \ldots, y'_n)}$$

does not vanish identically. If we put $u_{ik}=0$ $(i\neq k)$, this determinant becomes the product of $(y_1 y_2 y_3 \ldots y_n)^{n-1}$ and the reciprocal of the Vandermonde determinant

$$\begin{vmatrix}
1 & 1 & 1 & \ldots & 1 \\
u_{11} & u_{22} & u_{33} & \ldots & u_{nn} \\
u_{11}^2 & u_{22}^2 & u_{33}^2 & \ldots & u_{nn}^2 \\
\ldots & \ldots & \ldots & & \ldots \\
u_{11}^{n-1} & u_{22}^{n-1} & u_{33}^{n-1} & \ldots & u_{nn}^{n-1}
\end{vmatrix}$$

It does not vanish identically, therefore, and the semi-covariants thus far found are independent.

We thus have just the required $2n$ independent relative semi-covariants and need seek no further.

3. *The Covariants.*

The covariants are those functions of the semi-covariants (including seminvariants) which are unchanged in value after (1) is transformed by (3). Since there are just $2n-1$ absolute semi-covariants, there can be only $2n-1$ more absolute covariants than absolute invariants, and $2n$ more relative covariants than relative invariants.

If the transformation (3) is assumed to be in the form (9), equations (11) show that the infinitesimal transformations of $r_i^{(1)}$, $r_i^{(2)}$, ... are

$$\left.\begin{aligned} \frac{\delta r_i^{(1)}}{\delta t} &= -2\phi' r_i^{(1)} + \tfrac{1}{2}\phi^{(3)} y_i \\ \frac{\delta r_i^{(2)}}{\delta t} &= -4\phi' r_i^{(2)} + \phi^{(3)} r_i^{(1)} \\ \frac{\delta r_i^{(3)}}{\delta t} &= -6\phi' r_i^{(3)} + \tfrac{3}{2}\phi^{(3)} r_i^{(2)} \end{aligned}\right\} \quad (i = 1, 2, \ldots, n);$$

whence, by induction,

$$\frac{\delta r_i^{(l)}}{\delta t} = -2l\phi' r_i^{(l)} + \tfrac{1}{2}l\phi^{(3)} r_i^{(l-1)} \tag{23}$$

$$(i = 1, 2, \ldots, n;\ l = 1, 2, \ldots, n-1),$$

where we have put $r_i^{(0)} = y_i$. Also, from (11),

$$\frac{\delta s_i}{\delta t} = -3\phi' s_i - 2\phi' r_i^{(1)} + \tfrac{1}{2}\phi^{(4)} y_i \quad (i = 1, 2, \ldots, n). \tag{24}$$

Finally, from (10), and the fact that

$$\frac{\delta y_i'}{\delta t} = -\phi' y_i, \tag{25}$$

we have

$$\frac{\delta t_i}{\delta t} = -\phi' t_i + \tfrac{1}{2}\phi'' y_i \quad (i = 1, 2, \ldots, n). \tag{26}$$

The results given by equations (23), (24), (25), and (26) make it possible to calculate the infinitesimal transformations of the covariants easily. Thus

$$\frac{\delta R}{\delta t} = \sum_{k=1}^{n} \sum_{l=1}^{n-1} (-2l\phi' r_k^{(l)} + \tfrac{1}{2} l \phi^{(3)} r_k^{(l-1)}) \frac{\partial R}{\partial r_k^{(l)}}$$

$$= -n(n-1)\, \phi' R,$$

and R is therefore a relative covariant. Again,

$$\frac{\delta S_i}{\delta t} = \sum_{k=1}^{n} \sum_{l=1}^{n-1} (-2l\phi' r_k^{(l)} + \tfrac{1}{2} l \phi^{(3)} r_k^{(l-1)}) \frac{\partial S_i}{\partial r_k^{(l)}}$$

$$+ \sum_{k=1}^{n} (-3\phi' s_k - 2\phi'' r_k^{(1)} + \tfrac{1}{2}\phi^{(4)} y_k) \frac{\partial S_i}{\partial s_k} \quad (i = 1, 2, \ldots, n-1);$$

whence

$$\frac{\delta S_1}{\delta t} = -[n(n-1)+1]\, \phi' S_1 - 2\phi'' R - \phi^{(3)} S_2,$$

$$\frac{\delta S_i}{\delta t} = -[n(n-1)-2i+3]\, \phi' S_i - \tfrac{1}{2}(i+1)\, \phi^{(3)} S_{i+1} \qquad (i = 2, 3, \ldots, n-1),$$

where

$$S_n = 0.$$

Therefore S_{n-1} is a relative covariant. Also

$$\frac{\delta T_y}{\delta t} = -[n(n-1)+1]\, \phi' T_y + \tfrac{1}{2}\phi'' R - \tfrac{1}{2}\phi^{(3)} T_1,$$

and

$$\frac{\delta T_i}{\delta t} = -[n(n-1)-2i+1]\, \phi' T_i - \tfrac{1}{2}(i+1)\, \phi^{(3)} T_{i+1} \qquad (i = 1, 2, \ldots, n-1),$$

where

$$T_n = 0.$$

Thus T_{n-1} is another relative covariant.

In forming the relative covariants we shall need before us the infinitesimal transformation of $I^{(r)}$ given by equation (12), viz.

$$\frac{\delta I^{(r)}}{\delta t} = -2(n-r)\, \phi' I^{(r)} + \tfrac{1}{2}(r+1)\, \phi^{(3)} I^{(r+1)}.$$

We can now write down the following series of relative covariants:

$$\Phi_n = R,$$

$$\Phi_{n-1} = S_{n-1},$$

$$\Phi_{n-2} = S_{n-2} + \frac{n-1}{n} I^{(n-1)} \Phi_{n-1},$$

$$\Phi_{n-3} = S_{n-3} + \frac{n-2}{n} I^{n-1} \Phi_{n-2} - \frac{(n-2)(n-1)}{n(n-1)} I^{(n-2)} \Phi_{n-1};$$

whence, by induction,

$$\Phi_{n-a} = S_{n-a} + \sum_{j=1}^{a-1} (-1)^{j+1} \frac{(n-a+j)!\,(n-j)!}{(n-a)!\,n!} I^{(n-j)} \Phi_{n-a+j}$$

$$(a = 1, 2, \ldots, n-2).$$

Likewise

$$\Psi_{n-a} = T_{n-a} + \sum_{j=1}^{a-1} (-1)^{j+1} \frac{(n-a+j)!\,(n-j)!}{(n-a)!\,n!} I^{(n-j)} \Psi_{n-a+j}$$

$$(a = 1, 2, \ldots, n-1).$$

Also $\Phi_1 = 2\Theta_2 \left[S_1 + \sum_{j=1}^{n-2} (-1)^{j+1} \frac{(j+1)!\,(n-j)!}{n!} I^{(n-j)} \Phi_{j+1} \right] - R\Theta_2',$

$$\Psi_0 = 8\Theta_2 \left[T_y + \sum_{j=1}^{n-1} (-1)^{j+1} \frac{j!\,(n-j)!}{n!} I^{(n-j)} \Psi_j \right] + R\Theta_2'.$$

If these $2n$ relative covariants are arranged in the order

$$\Phi_n, \Phi_{n-1}, \ldots, \Phi_1, \quad \Psi_{n-1}, \Psi_{n-2}, \ldots, \Psi_0,$$

they are seen to be independent, since each contains a semi-covariant not in those which precede. We therefore have all the independent covariants.