THE INVARIANT THEORY OF THREE QUADRICS

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Introduction.

The following pages give in outline a complete system of concomitants of three quadrics. In §§ 20–22 the invariants are dealt with, and a complete list of these is given in § 23. In § 5, the *prepared system* of bracket types is explained, and in § 14 tabulated.

A geometrical discussion of these results is deferred.

I. *Notation.*

1. In symbolic form let the point, plane, and line equations of the quadrics be

$$\left.\begin{aligned} f = a_x^2 = a_x'^2 = \ldots, \qquad \phi = u_\alpha^2 = u_{\alpha'}^2 = \ldots, \\ f_1 = b_x^2 = b_x'^2 = \ldots, \qquad \phi_1 = u_\beta^2 = \ldots, \\ f_2 = c_x^2 = c_x'^2 = \ldots, \qquad \phi_2 = u_\gamma^2 = \ldots, \\ \text{and} \qquad \pi = (Ap)^2 = \ldots, \\ \pi_1 = (Bp)^2 = \ldots, \\ \pi_2 = (Cp)^2 = \ldots. \end{aligned}\right\} \qquad (1)$$

These symbols refer to quaternary forms wherein

$$a_x = a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4,$$

$$A = (aa') \text{ a second degree element,}$$

$$\alpha = (aa'a'') \text{ a third degree element,}$$

$$u_x = 0,$$

$$p = (uv),$$

$$x = (uvw).$$

Any single term concomitant of f, f_1, f_2 is denoted by P. The word *member* will be used to signify a concomitant.

The symbols a, A, α, u, p, x are called *elements* of various *degrees* one, two, or three; and these three degrees are distinguished respectively by (1) small italic letters, (2) capital italic letters, and (3) Greek letters, together with x.

Reducibility.

2. Following Gordan* in his theory of two quadrics we introduce the symbols c_i, $c_{\nu\mu}$ to denote the character of a form P. Let c_1, c_2, c_3 denote the degree of P in the coefficients of f, f_1, f_2 respectively. Let $c_{1\mu}$, $c_{2\mu}$, $c_{3\mu}$ refer to f, f_1, f_2 respectively: and in $c_{1\mu}$ let μ denote the number of brackets in P, each of which contains μ symbols a, a', ... or the equivalent of μ symbols in the higher currencies A, α. Then μ may not exceed 4.

Then a form P_1 is held to be simpler than P_2 if one of c_1, c_2, c_3 in P_1 is less than the corresponding degree in P_2, while the other two are not greater. In this sense, forms are considered in ascending degree.

To distinguish forms of the same degree, P_1 is simpler than P_2 if in P_1 one of c_{14}, c_{24}, c_{34} is greater than in P_2, the other two being not less. If this test fails, then c_{i3}, c_{i2} are examined in succession.†

If $c_{\nu 4} > 0$, P_1 *is reducible.*

3. As before, the symbol a_α implies the factor a_α^2.

Equivalent Forms.

4. P_1, P_2 are equivalent if $P_1 - P_2$ is reducible. This is symbolised by

$$P_1 - P_2 \equiv 0 \text{ mod } R,$$

or

$$P_1 - P_2 \equiv 0,$$

or

$$P_1 \equiv P_2.$$

Prepared Forms.

5. To begin with, P consists of four types of bracket factor: $(dd_1 d_2 d_3)$,

* *Math. Ann.*, Bd. 56.

† Cf. Turnbull, "System of Two Quadratics," *Proc. London Math. Soc.*, Ser. 2, Vol. 18, p. 74.

(dd_1d_2u), (dd_1p), d_x, where d denotes a, b, or c. Wherever in a factor two or three d's refer to one quadric they are replaced by D or δ respectively. Now every symbol d must occur twice in P. But if, say, dd_1 stand for aa_1 in one bracket, it does not follow that the complementary a, a_1 will be found to be also convolved in another bracket. Yet, by a proper introduction of new bracket types, we arrive at an alternative form of P in which *every* symbol d, D, or δ is explicitly paired. This is called the prepared form of P, and must now be investigated.

II. *The Prepared System.*

6. A bracket of P may have four or less a's: *i.e.* it may contain a_a, a, A, or a, or no reference to the quadric f. The first of these implies the invariant a_a^2, so we pass on to the second case, where $\alpha = (aa'a'')$ occurs in a bracket. By the use of new brackets

$$(\alpha\beta p), \quad (\alpha\gamma p), \quad (\alpha\beta\gamma x),$$

we may collect the complements of $aa'a''$ which occur bracketed once.

The proof is the same as for two quadrics* with the additional case of

$$(aa'a''i)\, a_\delta a'_{\delta'} a''_x.$$

This is seen, by interchanging the a's in every way, to be

$$= \tfrac{1}{6}(aa'a''i)(aa'a''.\,\delta\delta' x)$$

$$= \tfrac{1}{6} i_\alpha (\alpha\delta\delta' x).$$

The bracket $(\alpha\delta\delta' x)$ is $(\alpha\beta\gamma x)$ or else is zero.

The bracket $(\alpha\beta\gamma x)$.

7. This bracket is the reciprocal or dual of $(abcu)$ and does not appear for less than three quadrics. It obeys the same rule of interchange as its dual, and, expressed in the original form, is a six-term series

$$\dot{a}_\beta \dot{a}'_\gamma \dot{a}''_x \quad (aa'a'' = \alpha),$$

where the dots indicate a determinantal permutation.

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* Cf. Turnbull, *ibid.*, p. 75, § 10.

Interchangeability of α.

8. Since $e_\alpha e'_{\alpha'} - e'_\alpha e_{\alpha'} \equiv 0 \mod c_{14}$, any two single α's in P may be interchanged. Nor would this reduction break down for an α contained in the new brackets $(\alpha\delta p)$, $(\alpha\beta\gamma x)$. We may then suppress any distinguishing marks between the α's; so also for β, γ. A form P will now contain an even number, or none, of each of α, β, γ. Moreover this is true for n quadrics if we add the new bracket $(\alpha\beta\gamma\delta)$.

The Element A.

9. The next step is more complicated: we must consider the pairing of A. Let a^i, a^j denote any two of a, a', a'', a'''. As in the case of two quadrics, if P contain brackets $(aa'kl)(a^i a^j mn)$, then we may express this in terms of $(aa'kl)(aa'mn)$, and terms with more than two symbols a in the second bracket. It is important to notice that the other symbols kl, mn of the original brackets are undisturbed in the equivalent brackets.

As P will originally contain either an even or an odd number of brackets (c_{12}), each with two symbols like a, a', we may thus pair off all such to become pairs of A's except possibly one odd pair. This gives two cases:—

$$\text{(i)}\quad P = \{\Pi(Aij)(Akl)\}\, M,$$

$$\text{(ii)}\quad P = \{\Pi(Aij)(Akl)\}\, (aa'mn)\, a_\rho a'_\sigma M,$$

where both ρ, σ involve b, c, u, p, x, but no reference to the quadric f.

The same applies to B and C. Hence P has at most one of each sort (aa'), (bb'), (cc') unpaired, which leads to three cases:—

Case I.—One, (aa') say, occurs, but all symbols b, b' are in separate factors: as also c, c'.

Case II.—Two are unpaired, (aa'), (bb') say.

Case III.—Three are unpaired, (aa'), (bb'), (cc').

Case I.—P contains $(aa'mn)\, a_\rho a'_\sigma$. Here we may write

$$2(aa'mn)\, a_\rho a'_\sigma = (aa'mn)(a_\rho a'_\sigma - a'_\rho a_\sigma) = (aa'mn)(aa'\rho\sigma),$$

introducing the new bracket $(aa'\rho\sigma)$, which is unnecessary if ρ or σ may be broken up, *i.e.* if ρ or $\sigma = (bcu)$. Besides this, the bracket $(aa'\rho\sigma)$ resolves itself into two simpler ones, or to zero, if $\rho = \sigma$, or if ρ, σ both contain B, C, or p. The cases wherein there is no reduction are given in the following table:—

	(1)		(2)	(4)′			(3)	(4)	(5)	(6)	(7)					(8)	(9)	
$\rho =$	x	x	x	x	x	x	β	β	β	β	β	γ	γ	γ	γ	Bu	Bc	Cb
$\sigma =$	β	γ	Bu	Bc	Cu	Cb	γ	Cu	cp	Cb	bp	Bu	bp	cp	Bc	Cu	cp	bp

In these tabulated cases, any attempt to bracket aa' in one or other factor a_ρ or a'_σ fails to simplify.

10. *Case* II.—This may be dealt with as Case I, unless the odd symbols a, a', b, b' are convolved at least once. P therefore may contain (aa'), (bb'), together with

either $$(abQ)(a'b'R), \tag{1}$$

or $$(abQ)\, a'_\rho b'_\sigma; \tag{2}$$

where Q, R, containing neither a nor b, can only be C, p, or cu: the last of which at once reduces. Since $Q \neq R$ only one possibility is left, $Q = C$, $R = p$. Hence the bracket pair (1) is $(abC)(a'b'p)$, which is conveniently written as

$$(ABCp). \tag{3}$$

Again, in form (2), if $Q = C$, the form may be written

$$(\dot{a}\dot{b}C)\, \dot{a}'_\rho \dot{b}'_\sigma,$$

since the complementary elements aa', and bb', are convolved. This form is symmetrical in A, B, C as regards its first bracket. For either A or B may be explicitly bracketed by breaking C up. This shows that ρ, σ must be independent of a, b and c. So they are both equal to x. This gives one new bracket type $(\dot{a}\dot{b}C)\, \dot{a}'_x \dot{b}_x$ which may be written

$$(ABCxx). \tag{4}$$

Exactly the same argument shows that if $Q = p$, then ρ, σ can only be γ, γ: leading to $(AB\gamma\gamma p)$. Similarly for

$$(BCaap), \quad (CA\beta\beta p). \tag{5}$$

11. *Case* III.—Here the symbols a, a', b, b', c, c' are left over after pairing existing sets A, B, C: and unless they are all convolved they may

be treated as in Cases I and II. This leaves only the following to be considered :—

(i) $(abcu)\,a'b'c'$, which reduces by bracketing aa' in $(abcu)$;

(ii) $(\dot{a}\dot{b}Q)(\dot{a}'\dot{c}R)\,\dot{b}'_{\rho}\dot{c}'_{\sigma}$;

(iii) $(\dot{a}\dot{b}Q)(\dot{a}'\dot{c}R)(\dot{b}'\dot{c}'S)$.

Here the symbols Q, R, S can only be A, B, C, or p [else the bracket at once reduces as in (i)]: and no two of Q, R, S are equal; so that at least one of them is A, B, or C. By convolving one or other of aa', bb', cc' into the bracket not containing p, we effect a reduction. So no new form of bracket is needed.

12. New types of bracket are indicated by the table of § 9, and by (3), (4), (5) of § 10. By means of these new types we have now explicitly paired off all the A, B, C symbols of P, and further have proved that among the symbols A, any two may be interchanged indifferently. Such a member P is now *prepared.*

In the prepared form P, the first degree symbols belonging to one form f, say, may be interchanged. For let $I(a, a')P$ denote the effect on P of interchanging *one* a with *one* a'. Then

$$P - I(a, a')P \equiv 0 \mod c_{12},$$

for $\overline{aa'}$ will be bracketed and give rise to an increase in c_{12}, provided that neither the a nor the a' occur in the new types of bracket given in Case I of § 9. Yet even in this case, the pair aa' may be bracketed for the same reasons as those considered in Cases II and III.

It follows that for the last two values of ρ, σ in the table of § 9 there is no need to consider the case where two different first degree elements c, c' occur. By using $I(c, c')P$ the difference is eliminated from this bracket.

The Bracket $(ABCp) \equiv 0$.

13. For let (AB, Cp) denote $(A\dot{c}u)(B\dot{c}'v)$ and for brevity let

$$g = (BC, Ap), \quad h = (CA, Bp), \quad k = (AB, Cp).$$

Then clearly k is unaltered if B, A are interchanged. Now if we bracket

C in the first bracket of k, we obtain by the fundamental identity,

$$k = (Acc')(Buv) + (\dot{a}c'c\dot{u})(B\dot{a}'v);$$

thus $$k = 2(AC)(Bp) - (C\dot{a}\dot{u})(B\dot{a}'\dot{v});$$

transposing, this is $$k+g = 2(AC)(Bp).$$

Similarly for $g+h$, $h+k$: hence

$$k = (AC)(Bp)+(BC)(Ap)-(AB)(Cp),$$

which reduces k at once.

Statement of the Prepared System.

14. We may now sum up the preceding results and give special notations for the various groups of new brackets introduced. The table of § 9 gives these types :—

(1) $(A\beta x) = a_\beta a'_x - a'_\beta a_x = \dot{a}_\beta \dot{a}'_x$.

(2) $(ABux) = (\dot{a}Bu)\dot{a}'_x = (BAux) = (\dot{b}Au)\dot{b}'_x$,

(3) $(A\beta\gamma) = \dot{a}_\beta \dot{a}'_\gamma$,

(4) $(ACu\beta) = (\dot{a}Cu\dot{a}'_\beta = (CAu\beta) = H_2$, and (4)′ $(ABxc) = \dot{a}_x(\dot{a}'Bc) = h_3$,

(5) $(Apc\beta) = (\dot{a}cp)\dot{a}'_\beta = G_{13}$,

(6) $(ACb\beta) = (\dot{a}Cb)\dot{a}'_\beta = (CAb\beta) = F'_4$.

(7) $(Apb\beta) = (\dot{a}bp)\dot{a}'_\beta = F_{12}$,

(8) $(ABCuu) = (\dot{a}Bu)(\dot{a}'Cu) = k$,

(9) $(ABccp) = (\dot{a}Bc)(\dot{a}'cp)$.

To these must be added the results of § 10,

$$(ABCxx) = (A\dot{b}\dot{c})\dot{b}'_x\dot{c}'_x = (B\dot{a}\dot{c})\dot{a}'_x\dot{c}'_x = k,$$

$$(AB\gamma\gamma p) = (\dot{a}\dot{b}p)\dot{a}'_\gamma\dot{b}'_\gamma.$$

The symbols H, h, G, F, k, etc. are found useful for reference, and in the above list several alternative ways of writing each type of bracket are given. These and all the original brackets may now be classified in four groups F_1, F_2, F_3, F_4; the suffix denoting the number of unpaired symbols explicitly found in the prepared bracket. Thus under F_2 would fall $(ABccp)$ which requires the two symbols A, B to be paired elsewhere in the member. All the brackets, old and new, of the prepared system are given in the following table.

The Prepared System.

12	F_1	a_x b_x c_x u_α u_β u_γ (Ap) (Bp) (Cp) a_α b_β c_γ
		a_β a_γ (bcp) $(\beta\gamma p)$ (Abu) (Acu) $(A\beta x)$ $(A\gamma x)$ (BC)
		b_γ b_α (cap) $(\gamma\alpha p)$ (Bcu) (Bau) $(B\gamma x)$ $(B\alpha x)$ (CA)
		c_α c_β (abp) $(\alpha\beta p)$ (Cau) (Cbu) $(C\alpha x)$ $(C\beta x)$ (AB)
	F_2	$(BC)' = (BCux)$ $(BC)'' = (BCaap)$ $(BC)''' = (BC\alpha\alpha p)$
		$(CAux)$ $(CAbbp)$ $(CA\beta\beta p)$
36		$(ABux)$ $(ABccp)$ $(AB\gamma\gamma p)$
		$(abcu)$ $(\alpha\beta\gamma x)$
		(Abc) $(A\beta\gamma)$ $(Apb\beta) = F_{12}$ $(Apc\gamma) = F_{13}$ $(Apc\beta) = G_{13}$ $(Apb\gamma) = G_{12}$
		(Bca) $(B\gamma\alpha)$ $(Bpc\gamma) = F_{23}$ $(Bpa\alpha) = F_{21}$ $(Bpc\alpha) = G_{23}$ $(Bpa\gamma) = G_{21}$
	F_3	(Cab) $(C\alpha\beta)$ $(Cpa\alpha) = F_{31}$ $(Cpb\beta) = F_{32}$ $(Cpa\beta) = G_{31}$ $(Cpb\alpha) = G_{32}$
		$(BCu\alpha) = H_1$ $(BCxa) = h_1$ $(ABCuu) = K$
		$(CAu\beta) = H_2$ $(CAxb) = h_2$ $(ABCxx) = k$
28		$(ABu\gamma) = H_3$ $(ABxc) = h_3$
3	F_4	$(BCa\alpha) = F_4$ $(CAb\beta) = F_4'$ $(ABc\gamma) = F_4''$

III. *Generalised Identities.*

15. The prepared system, now tabulated, shows clearly a principle of duality, the algebraic equivalent of reciprocation. For this system is symmetrical in regard to the line coordinate elements p, A, B, C: and to every group involving any of a, b, c, u corresponds a group of α, β, γ, x. Some of the factors, *e.g.* (Ap), $(BC)'$, F_{12}, F_4, are self-reciprocal: others form pairs of reciprocals H_1 with h_1, K with k, and so on.

This duality goes further: it may be affirmed that whatever identity or syzygy exists between symbolic forms, has consequently a dual identity or syzygy. For example, the fundamental identity

$$(abcd)\, c_x - (abce)\, d_x + \ldots = 0 \qquad (1)$$

implies the existence of

$$(\alpha\beta\gamma\delta)\, u_\epsilon - (\alpha\beta\gamma\epsilon)\, u_\delta + \ldots = 0, \tag{2}$$

where each of α, β, γ, δ, ϵ are third degree elements in the coefficients of the quadrics. The second identity is readily established by resolving each 12 degree bracket $(\alpha\beta\gamma\delta)$ into factors $a_\beta a'_\gamma a''_\delta$. The same holds true of

$$(abp)\, c_x + (bcp)\, a_x + (cap)\, b_x = 0 \tag{3}$$

and

$$(\alpha\beta p)\, u_\gamma + (\beta\gamma p)\, u_\alpha + (\gamma\alpha p)\, u_\beta = 0. \tag{4}$$

Again, the identity $a_\pi b_\rho - a_\rho b_\pi = (ab\pi\rho)$ is self-reciprocal; whereas

$$(\dot{a}\dot{b}K)(\dot{c}\dot{d}L) = (abcd)(KL) \tag{5}$$

leads to the dual form

$$(\dot{\alpha}\beta K)(\dot{\gamma}\dot{\delta}L) = (\alpha\beta\gamma\delta)(KL). \tag{6}$$

It is a straightforward matter to write down all the linear types of quaternary identities, and then to copy the dual forms such as (2), (4), (6) above. By resolving the component parts of these latter into their elementary brackets, they can all be proved true. Now whatever process of reduction is used to test the reducibility of a member of a quaternary system, this process must ultimately depend upon two—and only two—things, (1) the fundamental linear identities, and (2) the interchange of equivalent symbols. Since both these principles apply to either type of symbol a or α, it follows that any identity or syzygy whatever may be reciprocated.

Reducibility.

16. The criteria $c_1 \ldots c_{33}$ of § 2 must now be supplemented. When two members P_1 and P_2 have the same characters $c_1 \ldots c_{33}$, let the number of F_i brackets ($i = 1, 2, 3, 4$) be counted, i being the greatest suffix in P_1 or P_2. Then P_1 is simpler than P_2 if its number of F_i brackets is less than that of P_2.

Failing this, let W_3, W_2, W_1 denote the number of brackets in P containing, respectively, three, two, one of the symbols A, B, C. Then P_1 is simpler than P_2, if for P_1, W_3 is less than it is for P_2. Failing this, W_2 is similarly examined.

This gives an order of precedence among the F_3 brackets which require one further discrimination, viz. that the six brackets (Abc), $(A\beta\gamma)$, etc. are the simplest F_3 brackets with *one* symbol A, B or C; next come the six F_{ij}; and next G_{ij}. Other F_3 brackets precede or follow this group because less or more symbols A, B, C occur.

The Reduction System.

17. The prepared system contains 79 elements, but a product of two of these elements is often reducible. Thus the product of two F_3 brackets $(abcu)(a\beta\gamma x)$ is identically equal to $\Sigma \dot{a}_\alpha \dot{b}_\beta \dot{c}_\gamma u_x$, which eliminates the F_3 brackets and therefore reduces the product. It is possible to carry out a systematic examination of every such product, and to construct a table in which any such product of two of these 79 factors is shown to be either (i) reducible, or (ii) irreducible, or (iii) equivalent to another product. This table consists of 79 rows and columns—one row and one column for each different factor, from a_ϵ to F_4''. The following fragment of the complete table should make clear the method of classification :—

	H_1	H_2	H_3	h_1	h_2	h_3
H_1	0					
H_2	.	0				
H_3	.	.	0			
h_1	.	x	x	0		
h_2	x	.	x	.	0	
h_3	x	x	.	.	.	0

x = reducible, 0 = irreducible, · = equivalent to another product.

Here, for example, it is shown that the product $h_1 H_2$ is reducible, that $H_3 H_3$ is irreducible, and $H_1 H_2$ is equivalent to another product. The whole table is a large triangle with an hypotenuse of 79 marks of irreducibility which indicate the squares of 79 factors $a_x \ldots F_4''$. This table is called the Reduction System.

Construction of the Reduction System.

18. This table is constructed by examining a product of factors, for example $(Abu)(p\beta\gamma)$. Here, by permuting bu, p we arrive at the identity

$$(Abu)(p\beta\gamma) \equiv G_{12} u_\beta - F_{12} u_\gamma - (pA)\, b_\gamma u_\beta,$$

suppressing reducible terms involving b_β. In accordance with the conditions of § 16, the reducible mark x is placed opposite G_{12} and u_β in the table, and the mark · is placed twice, to correspond with $(Abu)(p\beta\gamma)$ and with $F_{12}u_\gamma$. The third term $(pA)\,b_\gamma u_3$ has three factors and is analysed independently.

By interchanging symbols a, A, α with b, B, β or c, C, γ this one identity implies five others. By reciprocating these we get six others, as, for example,

$$(A\beta x)(pbc) \equiv G_{13}\,b_x - \dots .$$

And further, by interchanging in a *linear* identity the symbols a, A, α with u, p, x we obtain a new identity, equally valid, since the convolution of two of u, p, x is reducible, and also since the symbols u, p, x behave analytically in the same way as a, A, α. For example, by interchanging b, B, β with u, p, x in the above identity we may forecast the new relation

$$(Abu)(B\gamma x) \equiv H_3\,b_x - (AB)'\,b_\gamma - (AB)u_\gamma b_x.$$

Thus from one product $(Abu)(p\beta\gamma)$ a large number of other products may be dealt with at considerable economy of labour.

Below is subjoined the table of the reduction system, broken up for convenience into three parts: these deal respectively with (i) F_1F_2 brackets, (ii) one F_1 or F_2 with one F_3 or F_4, and (iii) F_3F_4 brackets. The detailed proofs are not given, for they are tedious but all of the same kind: and it is easy to test any assertion made in the table by applying one or other linear identity.

I.

	a_x b_x c_x	u_α u_β u_γ	a_β a_γ b_γ	b_α c_α c_β	bcp cap abp	$\beta\gamma p$ $\gamma\alpha p$ $\alpha\beta p$	Ab Ac Bc Ba Ca Cb	$A\beta$ $A\gamma$ $B\gamma$ $B\alpha$ $C\alpha$ $C\beta$	BC' CA' AB'	BC'' CA'' AB''	
g*	0	0	0	0							
bcp cap abp	x 0 0 0 . 0 0 0 .	0	0		0						
$\beta\gamma p$ $\gamma\alpha p$ $\alpha\beta p$	0	x 0 0 0 . 0 0 0 .	0		0	0					
Abu Acu Bcu	0	0	0 0 0 0 0 0 . 0 0	0 . 0 . 0 0 0 0 0	0 . 0 0 0 . 0 0 .	. . . .	0				
Bau Cau Cbu			0 0 0 0 0 . 0 . 0	0 0 . 0 0 0 0 0 0	. 0 0 . 0 0 0 . 0	. . . .	0				
$A\beta x$ $A\gamma x$ $B\gamma x$	0	0	0 . 0 . 0 0 0 0 0	0 0 0 0 0 0 . 0 0	. . . .	0 . 0 0 0 . 0 0 .	0 0 . . . . 0 0 . . . . . . 0 0 . .	0			
$B\alpha x$ $C\alpha x$ $C\beta x$			0 0 . 0 0 0 0 0 0	0 0 0 0 0 . 0 . 0	. . . .	. 0 0 . 0 0 0 . 0	. . 0 0 . . . . . . 0 0 . . . . 0 0				Ap Bp Cp
$BCux$ $CAux$ $ABux$	0	0	. . x . x . x . .	. . x . x . x . .	x	x	. . 0 0 0 0 0 0 . . 0 0 0 0 0 0 . .	. . 0 0 0 0 0 0 . . 0 0 0 0 0 0 . .	0		x 0 0 0 . 0 0 0 .
$BCaap$ $CAbbp$ $ABccp$	0 . . . 0 . . . 0	. . . .	0 0 . . . 0 . . .	. . . 0 . . . 0 0	x 0 0 0 x 0 0 0 x	. . . .	. . x 0 0 x 0 x . . x 0 x 0 0 x . .	x x . 0 0 . 0 . x x . 0 . 0 0 . x x	x . . . x . . . x	0 . 0 . . 0	BC''' CA''' AB'''
$BC\alpha\alpha p$ $CA\beta\beta p$ $AB\gamma\gamma p$	. . . .	0 . . . 0 . . . 0	. . . 0 . . . 0 0	0 0 . . . 0 . . .	. . . .	x 0 0 0 x 0 0 0 x	x x . 0 0 . 0 . x x . 0 . 0 0 . x x	. . x 0 0 x 0 x . . x 0 x 0 0 x . .	x . . . x . . . x	. x x x . x x x .	0 . 0 . . 0

* The factors (BC), (CA), (AB) do not reduce with any of the above F'_1 and F'_2 factors. Also g in the above denotes any of the twelve factors a_x, b_x, …, c_α, c_β.

II.

	$a_x\ b_x\ c_x$	$u_\alpha\ u_\beta\ u_\gamma$	$Ap\ Bp\ Cp$	$a_\beta\ a_\gamma\ b_\gamma\ b_\alpha\ c_\alpha\ c_\beta$	$bcp\ cap\ abp$	$\beta\gamma p\ \gamma\alpha p\ \alpha\beta p$	$Ab\ Ac\ Bc\ Ba\ Ca\ Cb$	$A\beta\ A\gamma\ B\gamma\ B\alpha\ C\alpha\ C\beta$	$BC\ CA\ AB$	$BC'\ CA'\ AB'$	$BC''\ CA''\ AB''$	$BC'''\ CA'''\ AB'''$
$abcu$ $\alpha\beta\gamma x$	0	0	x	0	0 x	x 0	0 x	x 0	x	x	x	x
Abc Bca Cab	0	x 0 0 0 x 0 0 0 x	0	0	0	. . . .	0 0 0 . . 0 . 0 0 0 0 . 0 . . 0 0 0	0 0 . . . . . . 0 0 . . . . . . 0 0	0	. . . .	. 0 0 0 . 0 0 0 .	x . . . x . . . x
$A\beta\gamma$ $B\gamma\alpha$ $C\alpha\beta$	x 0 0 0 x 0 0 0 x	0	0	0	. . . .	0	0 0 . . . . . . 0 0 . . . . . . 0 0	0 0 0 . . 0 . 0 0 0 0 . 0 . . 0 0 0	0	. . . .	x . . . x . . . x	. 0 0 0 . 0 0 0 .
F_{12} F_{13} F_{23} F_{21} F_{31} F_{32}	x 0 . x . 0 . x 0 0 x . 0 . x . 0 x	x 0 . x . 0 . x 0 0 x . 0 . x . 0 x	0	0 . 0 0 . 0 . 0 0 . 0 0 . 0 0 . 0 0 0 0 . 0 0 . 0 0 . 0 0 . 0 . 0 0 . 0	0 . 0 0 0 . 0 0 . . 0 0 . 0 0 0 . 0	0 . 0 0 0 . 0 0 . . 0 0 . 0 0 0 . 0	0 . x x . . . 0 . . x x . . 0 . x x x x . 0 . . x x . . 0 . . . x x . 0	0 . x x . . . 0 . . x x . . 0 . x x x x . 0 . . x x . . 0 . . . x x . 0	. 0 0 . 0 0 0 . 0 0 . 0 0 0 . 0 0 .	0	. 0 . . . 0 . . 0 0 . . 0 . . . 0 .	. 0 . . . 0 . . 0 0 . . 0 . . . 0 .
G_{12} G_{13} G_{23} G_{21} G_{31} G_{32}	x 0 x x x 0 x x 0 0 x x 0 x x x 0 x	x x 0 x 0 x 0 x x x x 0 x 0 x 0 x x	0	x 0 0 0 x x 0 x . x 0 0 x x x 0 0 0 0 0 0 x . x 0 0 x x x 0 . x 0 0 0 x	0 x 0 0 0 x 0 0 x x 0 0 x 0 0 0 x 0	0 0 x 0 x 0 x 0 0 0 0 x 0 x 0 x 0 0	0 x x x x x x 0 x x . x x x 0 x x x . x x 0 x x x x x x 0 x x x . x x 0	x 0 x x . x 0 x x . x x . x x 0 x x x x 0 x x . x x . x x 0 x . x x 0 x	0	. x x . x x x . x x . x x x . x x .	x 0 . x . 0 . x 0 0 x . 0 . x . 0 x	x . 0 x 0 . 0 x . . x 0 . 0 x 0 . x
H_1 H_2 H_3	. x x x . x x x .	0	0	. . x 0 0 x 0 x . . x 0 x 0 0 x . .	x	x	. . 0 0 0 0 0 0 . . 0 0 0 0 0 0 . .	x x x 0 0 x 0 x x x x 0 x 0 0 x x x	0 . . . 0 . . . 0	0 . . . 0 . . . 0	0 x x x 0 x x x 0	0 . . . 0 . . . 0
h_1 h_2 h_3	0	x x x x x x x x x	0	0 0 x . . x . x 0 0 x . x . . x 0 0	x	x	x x x 0 0 x 0 x x x x 0 x 0 0 x x x	. . 0 0 0 0 0 0 . . 0 0 0 0 0 0 . .	0 . . . 0 . . . 0	0 . . . 0 . . . 0	0 . . . 0 . . . 0	0 x x x 0 x x x 0
$ABCuu$ $ABCxx$	x x x 0 0 0	0 0 0 x x x	0	x	x	x	0 0 0 0 0 0 x x x x x x	x x x x x x 0 0 0 0 0 0	0	0	x	x
F_4 F'_4 F''_4	0 x x x 0 x x x 0	0 x x x 0 x x x 0	x 0 0 0 x 0 0 0 x	0 0 x 0 0 x 0 x 0 0 x 0 x 0 0 x 0 0	x	x	x x x 0 0 x 0 x x x x 0 x 0 0 x x x	x x x 0 0 x 0 x x x x 0 x 0 0 x x x	0	x x x x x x x x x	0 x x x 0 x x x 0	0 x x x 0 x x x 0

III.

	abc	$\alpha\beta\gamma$	Abc	Bca	Cab	$A\beta\gamma$	$B\gamma\alpha$	$C\alpha\beta$	F_{12}	F_{13}	F_{23}	F_{21}	F_{31}	F_{32}	G_{12}	G_{13}	G_{23}	G_{21}	G_{31}	G_{32}	H_1	H_2	H_3	h_1	h_2	h_3	K	k	F_4	F_4'	F_4''
abcu	0																														
$\alpha\beta\gamma x$	x	0																													
Abc																															
Bca	0	x	0																												
Cab																															
$A\beta\gamma$			0	.	.																										
$B\gamma\alpha$	x	0	.	0	.	0																									
$C\alpha\beta$			.	.	0																										
F_{12}			0	x	0	0	x	0	0																						
F_{13}			0	0	x	0	0	x	.	0																					
F_{23}	x		0	0	x	0	0	x	x	0	0																				
F_{21}			x	0	0	x	0	0	x	x	.	0																			
F_{31}			x	0	0	x	0	0	x	x	x	0	0																		
F_{32}			0	x	0	0	x	0	0	x	x	x	.	0																	
G_{12}			0	.	x	0	x	.	0	0	x	x	x	x	0																
G_{13}			0	x	.	0	.	x	0	0	x	x	x	x	x	0															
G_{23}	x		x	0	.	.	0	x	x	x	0	0	x	x	x	x	0														
G_{21}			.	0	x	x	0	.	x	x	0	0	x	x	x	x	x	0													
G_{31}			.	x	0	x	.	0	x	x	x	x	0	0	x	x	x	x	0												
G_{32}			x	.	0	.	x	0	x	x	x	x	0	0	x	x	x	x	x	0											
H_1			x	.	.	.	0	0	x	x	x	.	.	x							0										
H_2	x		.	x	.	0	x	0	.	x	x	x	x	.	x						.	0									
H_3			.	.	x	0	0	x	x	.	.	x	x	x							.	.	0								
h_1			.	0	0	x	.	.	x	x	x	.	.	x							.	x	x	0							
h_2	x		0	x	0	.	x	.	.	x	x	x	x	.	x						x	.	x	.	0						
h_3			0	0	x	.	.	x	x	.	.	x	x	x							x	x	.	.	.	0					
K	x		x			x			x						x						0	0	0	x	x	x	0				
k																					x	x	x	0	0	0	x	0			
F_4			x	0	0	x	0	0	x	x	x	0	0	x							0	x	x	0	x	x	x	x	0		
F_4'	x		0	x	0	0	x	0	0	x	x	x	x	0	x						x	0	x	x	0	x	x	x	x	0	
F_4''			0	0	x	0	0	x	x	0	0	x	x	x							x	x	0	x	x	0	x	x	x	x	0

IV. *The Complete System.*

19. From the prepared system of § 14 we may in theory proceed to the complete system for three quadrics. This may be sub-divided into four groups K_1, K_2, K_3, K_4 say, corresponding to the four kinds of factors F_1, F_2, F_3, F_4 of the Prepared System. Each K group is defined as a group containing no factor F_i if i is greater than the suffix of K, while at least one factor with the suffix of K is present in the form.

It appears that the groups K_1, K_4 are small, whereas K_2 and K_3 are unwieldy. No effort will be made to count the members of K_2 and K_3, but it will be shown that they are strictly finite.

As for special types of members, all the *invariants* will be found.

The K_1 Group.

This consists of 12 forms made by squaring the 12 factors of the prepared system F_1 (§ 14).

The K_2 Group.

This consists of the forms made by squaring the 36 F_2 brackets (§ 14), together with all possible chains (i, i) where $i = a, b, c, \alpha, \beta, \gamma, A, B, C$; and also chains whose end elements are either x, p or u. A chain* has much the same significance as in the case of ternary forms, being a convenient abbreviation of a lengthy product. An example should make this clear:—

$\begin{pmatrix} a & b & c & \alpha & \gamma \\ x & C & A & \beta & B & u \end{pmatrix}$ is a chain of grade 9, representing

$$a_x(aCu)(Cbu)(bAu)(Acu)(c_\beta)(\alpha\beta p)(\alpha Bx)(B\gamma x)u_\gamma.$$

The grade is the number of different elements not reckoning x, p, u. Each element a, etc. may stand in the upper or lower line. Manifestly all the elements of a chain must differ except possibly the end elements. The grade of a chain may be anything between two and nine inclusive. Theoretically then the K_2 system can be written out: it is finite but

* Cf. Turnbull, "Ternary Quadratic Types," *Proc. London Math. Soc.*, Ser. 2, Vol. 9, p. 83, and Vol. 18, p. 79.

numerous. It is indeed limited further, since no pair of the three elements a, A, α may be adjacent, the same applying to b, B, β and c, C, γ. On the other hand the juxtaposition of BC would indicate four possible factors (BC), $(BCux)$, $(BCaap)$, $(BC\alpha ap)$.

This procedure does not guarantee that all the remaining members of K_2 are irreducible. A detailed application of the fundamental identities would eliminate a considerable number more. One useful step further may be taken by seeking the invariants of the group.

Invariants of the K_2 Group.

The six factors a_β, a_γ, ... together with (BC), (CA), (AB), alone lead to invariants. There are only two invariants properly belonging to *three* quadrics:

$$(BC)(CA)(AB) \text{ denoted by } \Phi_{123},$$

and

$$\begin{pmatrix} a & c & b & a \\ & \beta & \alpha & \gamma & \end{pmatrix} \quad \text{,,} \quad \Omega.$$

The latter may be written as $\frac{1}{6}\,(\overline{abc}\,.\,\alpha\beta\gamma)^2$.

Before proceeding with the remaining K_3 and K_4 groups, the invariants of the whole system will be calculated.

The Invariants.

20. These forms are composed of the six factors a_β, a_γ, ..., three factors (BC), (CA), (AB), the six F_3 factors (Abc), $(A\beta\gamma)$, ..., and the three F_4 factors $(BCa\alpha)$, etc.

In the reduction system the following relations are relevant :—

$$\left.\begin{aligned} F_4(Abc) &\equiv (Bac)(AC)\,b_\alpha + (Cab)(AB)\,c_\alpha \\ F_4'(Bca) &\equiv (Cab)(AB)\,c_\beta + (Abc)(BC)\,a_\beta \\ F_4''(Cab) &\equiv (Abc)(BC)a_\gamma + (Bac)(CA)\,b_\gamma \end{aligned}\right\} \qquad \text{(I)}$$

Reciprocally

$$F_4(A\beta\gamma) \equiv (B\alpha\gamma)(AC)\,a_\beta + (C\alpha\beta)(AB)\,a_\gamma \text{ and two others.} \qquad \text{(II)}$$

Again $\quad F_4 c_\beta \equiv (Bac)(C\alpha\beta) - (BC)\,a_\beta c_\alpha$ and five others, $\qquad$ (III)

including $\quad F_4 b_\gamma \equiv (B\alpha\gamma)(Cab) - (BC)\,a_\gamma b_\alpha$. $\qquad$ (IV)

Multiplying (I) by (Abc) and dropping reducible terms,

$$(Bac)(Abc)(AC)b_a+(Cab)(Abc)(AB)\,c_a \equiv 0 \text{ and reciprocally.} \qquad \text{(V)}$$

Likewise from (III) there follows

$$(Bac)(Ca\beta)\,c_\beta \equiv 0; \qquad \text{(VI)}$$

and from (IV) there follows, since $F_4(Abc)$ is reducible in (I),

$$(Ba\gamma)(Cab)(Abc) \equiv 0. \qquad \text{(VII)}$$

Finally the product F_4F_4' is reducible thus:—

$F_4F_4' = (BCaa)(CAb\beta) = (B\dot{c}a)\,\dot{c}_a'(A\dot{c}b)\,\dot{c}_\beta'$: and now by bracketing A in the first bracket this simplifies.* (VIII)

The invariants are found in the K_2, K_3, K_4 groups. Those in the K_2 group have already been discussed.

As for the other invariants, they may be written as a product MN, where M consists of F_3 and F_4 factors, while N has only F_2 factors. A reference to the possible F_2 factors shows that N may consist of chains of the following types—A, B of course standing for any two of the three quadrics—

$$(A,\, B), \quad (a,\, \beta), \quad (a,\, b), \quad (a,\, \beta), \quad (a,\, a).$$

Moreover these chains can only be each of two sorts,

$$\begin{cases}(AB), \\ (AC)(CB),\end{cases} \quad \begin{cases}a_\beta, \\ \begin{pmatrix} a & b & c & \\ & \gamma & a & \beta\end{pmatrix},\end{cases} \quad \begin{cases}\begin{pmatrix} a & b \\ & \gamma\end{pmatrix}, \\ \begin{pmatrix} a & c & b \\ & \beta & a\end{pmatrix},\end{cases} \quad \begin{cases}\begin{pmatrix} a & \beta \\ & c\end{pmatrix}, \\ \begin{pmatrix} a & \gamma & \beta \\ & b & a\end{pmatrix},\end{cases} \quad \begin{cases}\begin{pmatrix} a & c & \\ & \beta & a\end{pmatrix}, \\ \begin{pmatrix} a & b & \\ & \gamma & a\end{pmatrix};\end{cases}$$

any others being immediately reducible.

21. Again, since N consists of chains, there are in N an even number of unpaired symbols standing as end links of these chains. Hence M also must have an even number of unpaired symbols; whence it follows that M has an even number of F_3 brackets. Also M may have F_4 brackets or not: suppose in the first case that M consists entirely of F_3 brackets.

* Analytically this is analogous to the formula (J) in reducing two quadrics. Cf. Turnbull, *ibid.*, p. 81.

Excluding the cases reducible by (VII), M may have two or four F_3 brackets, but cannot have six brackets: when the complementary factors of N are inserted this gives the following forms:—

(i) $(Abc)^2$ and its dual $(A\beta\gamma)^2$.

(ii) $(Abc)(Bca)(A, B)(a, b)$ and its dual $(A\beta\gamma)(B\gamma\alpha)(A, B)(\alpha, \beta)$.

(iii) $(Abc)(A\beta\gamma)(b, \gamma)(c, \beta)$, $(Abc)(A\beta\gamma)(b, \beta)(c, \gamma)$ and $(Abc)(A\beta\gamma)(b, c)(\beta, \gamma)$.

(iv) $(Abc)(B\gamma a)(A, B)(b, \gamma)(c, a)$

,, ,, ,, $(b, a)(c, \gamma)$

,, ,, ,, $(b, c)(\gamma, a)$.

(v) $(Abc)(A\beta\gamma)(Bac)(Ba\gamma)\,N$.

Of these, (i) is irreducible; as also is (ii) for the case when (A, B) is (AB). But the other type

$$(Abc)(Bca)(AC)(CB)(a, b)$$

reduces when the final chain is either $\begin{pmatrix} a & b \\ & \gamma \end{pmatrix}$ or $\begin{pmatrix} a & c & b \\ \beta & a & \end{pmatrix}$ by squaring the third of identities (I) or by using (V), respectively.

The next type (iii) gives $(Abc)(A\beta\gamma)\,b_\gamma c_\beta$ and $(Abc)(A\beta\gamma)\begin{pmatrix} b & c \\ & a \end{pmatrix}\begin{pmatrix} & a & \\ \beta & & \gamma \end{pmatrix}$ only: any other possible form of chain at once duplicates a link.

The next type (iv) must not contain the link b_a, owing to identity (VI). This leaves only two forms for the chains

$$(AB)\,b_\gamma c_a \quad \text{and} \quad (AC)(CB)\,b_\gamma c_a,$$

of which the former reduces by squaring an identity of type (IV).

Similarly by forming the product of identities (III) and (IV), type (v) reduces.

22. In the second case, suppose M to contain F_4 brackets. By (VIII) it is seen that only one such bracket, say F_4'' may occur. Excluding pro-

ducts reducible by identities (I)–(IV), the invariant is composed of

$$F_4'', \text{ i.e. } (ABc\gamma) \text{ with } (Abc), (A\beta\gamma), (Bac), (Ba\gamma), a_\gamma, b_\gamma, c_a, c_\beta,$$
$$(BC), (CA), (AB).$$

In no case can an invariant be built of F_4'' followed by a product of these other factors, as is seen by trial. So no more invariants exist, except the squares of F_4'', F_4', and F_4.

23. *List of Invariants of Three Quadrics.*

	No. of forms.		Degree.
1	12	Forms Δ, Θ, etc. involving one, or two of the quadrics.	
2	1	$(BC)(CA)(AB) = \Phi_{123}$	(2, 2, 2)
3	1	$\begin{pmatrix} a & c & b & a \\ & \beta & a & \gamma \end{pmatrix} = \Omega = a_\beta c_\beta c_a b_a b_\gamma a_\gamma$	(4, 4, 4)
4	6	$(Abc)^2$ and its dual $(A\beta\gamma)^2$	(2, 1, 1) (2, 3, 3)
5	3	$(BCaa)^2 = F_4^2, F_4'^2, F_4''^2$	(4, 2, 2)
6	6	$(Abc)(Bca)(AB)\, a_\gamma b_\gamma$ and its dual $(A\beta\gamma)(B\gamma a)(AB)\, c_a c_\beta$	(3, 3, 4) (5, 5, 4)
7	6	$(Abc)(Bca)(AB)\begin{pmatrix} a & c & b \\ & \beta & a \end{pmatrix}$ and its dual $(A\beta\gamma)(B\gamma a)(AB)\begin{pmatrix} a & \gamma & \beta \\ & b & a \end{pmatrix}$	(6, 6, 2) (6, 6, 6)
8	3	$(Abc)(A\beta\gamma)\, b_\gamma c_\beta$	(2, 4, 4)
9	6	$(Abc)(B\gamma a)(AC)(CB)\, b_\gamma c_a$	(5, 3, 6)
10	3	$(Abc)(A\beta\gamma)\begin{pmatrix} b & c \\ & a \end{pmatrix}\begin{pmatrix} & a & \\ \beta & & \gamma \end{pmatrix}$	(6, 4, 4)

This gives 47 invariants in all.

The K_3 Group.

24. F_3 brackets are of these types

(i) $(abcu)$, $(\alpha\beta\gamma x)$.

(ii) (Abc), $(A\beta\gamma)$.

(iii) F_{ij} $(ij = 1, 2, 3$ and differ).

(iv) G_{ij}.

(v) H_i, h_i.

(vi) K, k.

In accordance with § 16, these six sets may be considered to be of increasing complexity; and to express members of one set in terms of earlier sets is to reduce them. We shall quote results without detailing every proof, as the work is tedious. Investigation shows that no irreducible member can have more than four F_3 brackets, and the cases when 3 or 4 occur are comparatively rare.

(i) $(abcu)$, $(\alpha\beta\gamma x)$.

25. The (r.s.)* shows that only the F_3 factors (Abc), (Bca), (Cab) can exist along with $(abcu)$. The complete system is

$$(abcu)(Abc)(Bca)(A_eB)c_x, \quad (abcu)(Abc)(A, a),$$

$(abcu)\,N$, where N consists of F_2 or F_1 brackets and (A, a) likewise; and where a, b, c may be rearranged. That all three F_3 factors cannot appear simultaneously is proved in § 26.

There is a dual set for $(\alpha\beta\gamma x)$.

(ii) (Abc), $(A\beta\gamma)$.

26. The (r.s.) rules out type $(abcu)$, so this group consists of members involving the six brackets $(Abc) \ldots (A\beta\gamma)$ with F_2 or F_1 brackets. Leaving

* A convenient abbreviation for *reduction system*.

out cases reduced in (VII), § 20, and (v), § 21, there may be the following general types :—

$$(B) \begin{cases} (Bac)N, \qquad (Ba\gamma)N, \\ (Bac)(Ca\beta)N, \\ (Bac)(Bu\gamma)N, \\ (Bca)^2 \text{ and } (B\gamma a)^2, \\ (Bac)(Cab)N \qquad \text{and its dual,} \\ (Bac)(Cab)(Abc)N \qquad ,, \\ (Bca)(Abc)(A\beta\gamma)N \qquad ,, \end{cases}$$

where N consists of F_2 or F_1 brackets. The two latter forms reduce, leaving in this group the forms containing at most two F_3 brackets. Further reduction is not obvious. Herewith is a proof that $(Bca)(Cab)(Abc)$ reduces, which is typical of subsequent reductions, and shorter than that for $(Bca)(Abc)(A\beta\gamma)$.

$$(Bca)(Cab)(Abc)N \equiv 0.$$

From the (r.s.) we select these identities

(1) $(Abc)(Bau) \equiv -(Bca)(Abu)+(abcu)(BA)$,

(2) $(Abc)(Bax) \equiv h_3 b_a$ and also $h_3(Cab) \equiv 0 \bmod (Bac)$,

(3) $(Bca)(Cab)Ap \equiv (Abc)(BC)''+\text{etc.} \equiv 0$, § 16.

If N contains (Bau), the form reduces by multiplying (1) by (Bac). Hence by symmetry N cannot have any of the six (Bau), (Bcu), ...

If N contains (Bax), identity (2) applies. This rules out six more factors.

Since (3) rules out Ap, Bp, Cp it follows that the only factors in N involving A, B, C are $(BC)^i$, $(CA)^i$, $(AB)^i$, which cannot possibly be paired with the odd A, B, C of the first three brackets. Hence $N \equiv 0$.

(iii) F_{ij}, where $F_{12} = (Apb\beta)$.

27. The six F_{ij} factors reduce in product except for types F_{12}^2, $F_{12}F_{13}$, $F_{12}F_{32}$, which lead to four cases,

$$(\alpha)\ F_{ij}^2, \quad (\beta)\ F_{12}F_{13}M_1, \quad (\gamma)\ F_{12}F_{32}M_2, \quad (\delta)\ F_{12}M_3.$$

Now, by (r.s.),

M_1 may contain (Abc), $(A\beta\gamma)$, and F_2, F_1 factors,

M_2 ,, (Abc), (Cab), $(A\beta\gamma)$, $(C\alpha\beta)$,,

M_3 ,, ,, ,, ,,

Further investigation admits only the following to be retained :—

(C)

- F_{ij}^2 and $F_{12}F_{13}(Abc)(A\beta\gamma)$ and the like, all quadratic complexes,
- *$F_{12}F_{13}(Abc)(A\gamma x)[\beta]$ where $[\beta] = u_\beta$ or $\begin{pmatrix}\beta & x \\ & a\end{pmatrix}$,
- $F_{12}F_{13}(bcp)(\beta\gamma p)$ and $F_{12}F_{13}\, b_\gamma c_\beta$,
- $F_{12}F_{32}(A, C)$ where $(A, C) = \begin{pmatrix}A & C \\ & p\end{pmatrix}$ or $(AC)^i$,
- *$F_{12}(Cab)N$,
- $F_{12}(Abc)(A\beta\gamma)N$,
- *$F_{12}(Abc)N$,
- $F_{12}N$, where N has F_2 or F_1 factors.

(iv) G_{ij}, where $G_{12} = (Apb\gamma)$.

28. A form containing G_{ij} is reduced when expressed in terms of preceding factors. Taking G_{12} as typical, the (r.s.) admits of

$$F_{12},\ F_{13},\ (Abc),\ (Bca),\ (A\beta\gamma),\ (C\alpha\beta),\ N.$$

But $G_{12}G_{13} \equiv F_{12}F_{13}$; so that $G_{12}F_{12}F_{13} \equiv 0$. Accordingly we need

* These have dual forms.

only consider the types

(1) $G_{12}F_{12}M$,

(2) $G_{12}F_{13}M$,

(3) $G_{12}M$, where M contains neither G_{ij} nor F_{ij}.

Since G_{13} is dual of G_{12} and F_{13} is its own dual, then types (1) and (2) are dual. So (1) and (3) need only be considered. Ultimately we are left with

(D)

- $*G_{12}F_{12}(A\beta\gamma)(A)$ where $(A) = (Ap)$, $\begin{pmatrix} A & & x \\ & b & \end{pmatrix}$ or $\begin{pmatrix} A & & p \\ & B & \end{pmatrix}$,
- $G_{12}(A\beta\gamma)(Abc)\,c_\beta\,(A)$,
- $G_{12}(Abc)\begin{pmatrix} c & & \gamma \\ & \beta & \end{pmatrix}$, where no independent dual exists, and there are only three of this type for all G_{ij}.
- $G_{12}\begin{pmatrix} b & \beta & \\ & c & \gamma \end{pmatrix}(A)$,
- $G_{12}N$, where N consists of F_2, F_1 factors but contains neither c nor β.
- G_{12}^2.

The brevity of the above list is largely due to identities of the type

$$G_{12}i_\beta j_c \equiv G_{13}i_\gamma j_b,$$

where i, j are any two different symbols u, a, A, α.

(v) H_i, h_i: where $H_1 = (BCua)$.

29. The factor H_1 reduces with any F_3 bracket except

$$F_{21},\ F_{31},\ H_2,\ H_3,\ h_1,\ (A\beta\gamma),\ (B\gamma a),\ (Ca\beta),\ (Bca),\ (Cab).$$

But owing to relations such as

$$H_1F_{21} \equiv F_4(Bp)u_a, \quad H_1(Bca) \equiv F_4(Bcu),$$

$$h_1F_{21} \equiv F_4(Bp)a_x, \quad H_1h_1 \equiv F_4(BC)',$$

$$H_1H_2 \equiv K(Ca\beta),$$

$$H_1(A\beta\gamma) \equiv H_2(B\gamma a) \equiv H_3(Ca\beta),$$

the system may be reduced to the following types:—

(E)
$$\begin{cases} *H_1F_{21}(C, a) \text{ where } (C, a) \text{ is } (Cp)a_x, \begin{pmatrix} C & & B & \\ & A & & x \end{pmatrix} \text{ or } \begin{pmatrix} C & & a \\ & B & \end{pmatrix}, \\ *H_1H_2H_3u_\alpha u_\beta u_\gamma, \\ *H_1H_2u_\alpha u_\beta(A, B), \\ H_1h_1\begin{pmatrix} a & & a \\ & B & \end{pmatrix}, \\ *H_1^2, \\ *H_1(Bca)(Cab)c_\alpha b_\alpha u_\alpha, \\ *H_1(B\gamma a)(Ca\beta)u_\alpha u_\beta u_\gamma, \\ *H_1(Bca)N, \\ *H_1(B\gamma a)N, \\ *H_1(A\beta\gamma)N, \\ *H_1N, \text{ where } N \text{ consists of } F_1, F_2 \text{ factors.} \end{cases}$$

This list includes a sextic covariant of degree 3 in the coefficients of each of f, f_1, and f_2, viz.:—

$$h_1h_2h_3a_xb_xc_x = (BCax)(CAbx)(ABcx)a_xb_xc_x.$$

$$\text{(vi)} \quad K = (ABuCu).$$

30. The (r.s.) allows the factors H_1, H_2, H_3 and the types

$$u_\alpha \dots (Ap) \dots (Abu) \dots (BC) \dots (BC)' \dots .$$

The system then is

(F)
$$\begin{cases} *K^2, \\ *KH_1u_\alpha(A) \text{ where } (A) = (Ap), \begin{pmatrix} A & & p \\ & C & \end{pmatrix}, \begin{pmatrix} A' & & p \\ & C & \end{pmatrix}, \begin{pmatrix} A & & C & \\ & b & & p \end{pmatrix}, \\ *K(BC)^i(CA)^j(Cp) \text{ where } (BC)^i = (BC) \text{ or } (BC)' \text{ or } \begin{pmatrix} B & & C \\ & a & \end{pmatrix}, \\ *K(BC)^i(Ap), \\ *K(Ap)(Bp)(Cp). \end{cases}$$

The product KH_1H_2 is reducible.

The K_4 Group.

31. F_4 factors are of one type of which $(ABc\gamma)$ is representative. The (r.s.) shows that forms to be retained are, besides the three F_4^2, $F_4'^2$, $F_4''^2$,

$$F_4''MN \quad \text{and} \quad F_4''M,$$

where M is a product of F_3 factors, and N of F_2, F_1 factors. Further the (r.s.) admits the symbol a only twice, viz. in the factors (Bca) and a_γ. Similarly for b, α, β. Introducing four new symbols, let

$$F_{23}' = (Bca)a_\gamma, \quad F_{23}'' = (B\gamma\alpha)c_\alpha, \quad F_{13}' = (Abc)b_\gamma, \quad F_{13}'' = (A\beta\gamma)c_\beta;$$

and regarding these as new F_3 brackets, we may then express a member P, containing F_4'', as a product of factors selected from

$$c_x,\ u_\gamma,\ (Ap),\ (Bp),\ (Acu),\ (Bcu),\ (A\gamma x),\ (B\gamma x),\ (BC),\ (CA),\ (AB),\ (AB)'',$$

$(AB)'''$ and F_3 brackets, viz. F_{13}, F_{13}', F_{13}'', F_{23}, F_{23}', F_{23}'', H_3, h_3.

Identities show that H_3h_3, H_3F_{13}, $H_3(Bac)$, H_3c_β can each be expressed in terms involving F_4'' or reducible terms. Hence if H_3 occurs in P, no other F_3 factor is present. Similarly for h_3.

There are similar reductions for $F_{13}(Bac)$, $F_{13}(B\alpha\gamma)$; which imply that $F_{13}F_{23}'$, $F_{13}F_{23}''$ are here reducible. Clearly $F_{13}F_{13}'$ is reducible: and further, $F_{13}(BC)$, $F_{13}(B\gamma x)$, $F_{13}(Bcu)$ can all be expressed in terms involving either F_{23} or F_4''. If then *both* F_{13}, F_{23} occur in P, the only other factors involving c, γ are $c_x u_\gamma$, and the form is

$$F_4''F_{13}F_{23}c_x u_\gamma;$$

otherwise a form containing F_{13} has besides only tags and chains.

Again, by (III), (VIII) of § 20, we reduce $F_4''(Bac)(A\beta\gamma)$, so that the only remaining type with *two* F_3 brackets is $F_4''(Bac)(Abc)$ and its dual.

The K_4 group is represented then as follows:—

$$(G)\quad \begin{cases} F_4''F_{13}F_{23}c_x u_\gamma, & F_4''(Bac)a_\gamma(Abc)b_\gamma[c,\gamma], \\ F_4''F_{13}[B], & F_4''(Bac)a_\gamma[B], \quad F_4''H_3[c], \\ F_4''[A, B, c, \gamma], & F_4''^2, \end{cases}$$

where the second, fourth, and fifth have dual forms, and the square brackets indicate chains and tags as discussed in the K_2 system.

This exhausts all cases, and the Complete System is contained in the K_1 and K_2 groups of § 19, together with the sets denoted by (A) to (G) in §§ 25–31.