SOME NON-PRIMARY PERPETUANT SYZYGIES OF THE SECOND KIND

By H. T. H. PIAGGIO.

[Received February 11th, 1913.—Read February 13th, 1913.]

CONTENTS.

IN a previous paper* syzygies of the n-th "kind" were shewn to exist among perpetuant types for all degrees greater than $(n+1)$. It was also shewn that for degrees less than $(2n+2)$, these syzygies were all formed by one simple method, namely, symbolical multiplication of the single syzygy of the $(n+2)$-th degree by perpetuant products. We called these syzygies primary.

It was proved that syzygies not so formed (*non-primary*) certainly do arise for degree $(2n+2)$. For the first "kind" $(n = 1)$, we have

$$2n+2 = 4;$$

thus non-primary syzygies first occur for degree 4. These are the well-known Stroh syzygies. For the second kind, $2n+2 = 6$, so non-primary syzygies first occur for degree 6. The present paper deals with these.

* "Perpetuant Syzygies of the n-th Kind" (*Proc. London Math. Soc.*, Ser. 2, Vol. 8), which see for definitions and a general account of the subject.

It may be pointed out that one object of this paper and of similar work is to find some general method of forming syzygies of all "kinds" that will apply to all possible cases. In the previous paper it was shewn that for the n-th kind we can obtain by "symbolical multiplication" all those of the first n degrees, as well as many of higher degrees. A paper by Young and Wood ("Perpetuant Syzygies," *Proc. London Math. Soc.*, Ser. 2, Vol. 2) may be summed up as establishing that for the first kind the syzygies of all degrees up to eight are derived by symbolical multiplication from either the Jacobian identity or the Stroh syzygies.

It is found that the non-primary syzygies of the second kind of degree 6 consist of three different types. One of these is formed according to the general method explained in the last section of the previous paper. The second* is a generalisation of the Stroh syzygies. The third is rather more complex.

It is proved finally that these three types of non-primary syzygies, together with certain specified primary ones, are such that all others (of the second kind of degree 6) can be expressed as linear functions of them.

I. *Notation and Preliminary Details.*

Notation.—The Stroh syzygy

$$\sum_0^w \binom{w}{r} \left[(a_1 a_2)^{w-r}(a_4 a_3)^r - (a_1 a_3)^{w-r}(a_4 a_2)^r\right] \equiv 0,$$

we shall denote by $\{a_1 a_2 a_4 a_3\}_w$ or even by $\{1243\}_w$, writing only the suffixes.

The result of multiplying symbolically $\{a_1 a_2 a_4 a_3\}_w$ by the perpetuant product $(a_4 a_5)^u (a_3 a_6)^v$ will be denoted by

$$\{(a_1 a_2 a_4 a_3)_w (a_4 a_5)_u (a_3 a_6)_v\},$$

or by

$$\{(1243)_w (45)_u (36)_v\}.$$

The result of multiplying the Jacobian identity

$$(a_1 a_2 a_3) \equiv A_1(a_2 a_3) - A_2(a_1 a_3) + A_3(a_1 a_2) \equiv 0,$$

by the product $(a_1 a_4)^u (a_2 a_5)^v$ will be written

$$J\left[(a_1 a_4)^u (a_2 a_5)^v (a_1 a_2 a_3)\right],$$

or

$$J\left[(14)^u (25)^v (123)\right],$$

but, for the sake of brevity, $J\left[(14)^{w-1}(123)\right]$ will be replaced by $J_w(1423)$.

If $w \geqslant 3$, from four given symbols six essentially distinct J_w's can be formed, since

$$J_w(abcd) = -J_w(abdc) = (-1)^{w-1} J_w(bacd) = (-1)^w J_w(badc).$$

We shall define $J_2(abcd)$ as the form into which $(ab)(cd) - (ab)(ad) + (ab)(ac)$ degenerates, *i.e.*, as

$$(ab)(cd) - \tfrac{1}{2}(ad)^2 + \tfrac{1}{2}(db)^2 + \tfrac{1}{2}(ac)^2 - \tfrac{1}{2}(cb)^2 = \tfrac{1}{4}\left[\{abcd\}_2 + \{acdb\}_2 - \{adbc\}_2\right].$$

* This second type was made known to me by Dr. A. Young.

From four given symbols only three J_2's can be formed, as in addition to the relations given above

$$J_2(abcd) = J_2(cdab).$$

For reasons connected with the "resolution" of compound syzygies, it is convenient to use J_2's instead of Stroh syzygies of weight 2. The three J_2's formed from four given symbols are expressible linearly in terms of the three Stroh syzygies, and conversely.

Arrangement Scheme for Compound Syzygies of Degree 6.

(1) The order of precedence is

(i) C_3S_3, (ii) C_2S_4, (iii) C_1S_5, (iv) $C_1C_2S_3$, (v) $C_1^2S_4$, (vi) $C_1^3S_3$.

(S_p is used to denote a syzygy of degree p).

(2) If we have two compounds C_2S_4 and C_2S_4', the first precedes the second if (α) S_4 is a Stroh syzygy of weight $\geqslant 4$ and S_4' is a J_w; or if (β) S_4 is a J_w ($w \geqslant 3$) and S_4' is a J_2.

(3) (α) $(a_p a_q)^\lambda S_4$ precedes $(a_r a_s)^\mu S_4'$, where the above is not sufficient to establish precedence, if the first of the differences $p-r$, $q-s$ that does not vanish is *negative*. Here p is supposed to be less than q, and r less than s.

(β) The order of the compounds $(a_p a_q)^\lambda S_4$, where S_4 is a Stroh syzygy, is given later (in Section VI). It should be added that $(a_p a_q)^\lambda S_4$ precedes $(a_p a_q)^\mu S_4'$ if both S_4 and S_4' are Stroh syzygies and $\lambda < \mu$.

The above is merely the application to syzygies of the first kind of degree 6 of the more detailed scheme for those of the n-th kind given in the previous paper, with some variations in the arrangement of Stroh syzygies. As Stroh syzygies were not used in the last paper (except incidentally in the last section), these variations do not affect the results given there.

Use of Uncanonical Forms.

Theoretically the irreducible syzygies of the first kind whose compounds appear in the syzygies of the second kind should all be *canonical,* that is, members of a more or less arbitrarily chosen set in terms of which all others can be expressed linearly. But this would destroy the symmetry of the work. We therefore keep the uncanonical forms, with the understanding that they are to be regarded *merely as a short way of writing their canonical equivalents*. The scheme of arrangement has been chosen in such a way that this does not affect the reasoning in the following work. The compound syzygies resolved are always such as to have their irreducible parts canonical. The apparently eccentric arrangement of Stroh syzygies in Section VI is due to the necessity of satisfying this condition.

Similar remarks apply to the use of such reducible forms as $(a_1 a_4)^3 (a_1 a_3)$, which is to be understood as being merely an abbreviation for the equivalent sum of irreducible perpetuants.*

II. *Syzygies of the Second Kind of Degrees less than* 6.

These are needed to obtain the compound syzygies of the second kind of degree 6.

Putting $n = 2$ in the results of the previous paper, we see that :—

(i) None exist for degrees less than 4.

(ii) For degree 4 there is only one, namely,

$$(a_1 a_2 a_3 a_4) \equiv A_1 (a_2 a_3 a_4) - A_2 (a_1 a_3 a_4) + A_3 (a_1 a_2 a_4) - A_4 (a_1 a_2 a_3).$$

(iii) For degree 5 they are all given by the generating function

$$4x + 10x^2/(1-x).$$

The irreducible ones are all of the form

$$\begin{aligned} J^{(2)}[(a_p a_t)^\lambda (a_p a_q a_r a_s)] \equiv (a_p a_t)^\lambda (a_q a_r a_s) &- A_q J[(a_p a_t)^\lambda (a_p a_r a_s)] \\ &+ A_r J[(a_p a_t)^\lambda (a_p a_q a_s)] \\ &- A_s J[(a_p a_t)^\lambda (a_p a_q a_r)], \end{aligned}$$

where p, q, r, s, t are the five suffixes 1, 2, 3, 4, 5 in any order. These correspond to the part $10x^2/(1-x)$. The part $4x$ corresponds to the *compound* ones of form $A_p (a_q a_r a_s a_t)$, of which 4 (out of 5) are linearly independent.

---

* We are using Grace's form for a perpetuant, viz.,

$$(a_1 a_2)^\lambda (a_1 a_3)^\mu \dots (a_1 a_p)^\omega,$$

where $\lambda \geqslant 2^{p-2}, \quad \mu \geqslant 2^{p-3}, \quad \dots, \quad \omega \geqslant 1.$

III. *Generating Functions for Linearly Independent Syzygies of the Second Kind of Degree* 6.

The method of procedure is that of the previous paper. If from the *total* number of compound syzygies of a given weight and degree we subtract the number of those that are linearly independent, we are left with the number of linearly independent relations. But a linear relation between compound syzygies is a syzygy of the second kind. Hence we get the number of linearly independent syzygies of the second kind of that weight and degree.

Now each compound syzygy resolves* a product of at least *three* perpetuant factors, so we shall start by writing down all the possible forms of these products of total degree 6. They are

$$\text{(i) } C_2^3, \quad \text{(ii) } C_1C_2C_3, \quad \text{(iii) } C_1^2C_4, \quad \text{(iv) } C_1^2C_2^2, \quad \text{(v) } C_1^3C_3,$$
$$\text{(vi) } C_1^4C_2, \quad \text{(vii) } C_1^6,$$

where C_p denotes a perpetuant of degree p.

The generating functions for the syzygies of the first kind resolving these may be deduced from Young and Wood's paper. They give there the generating functions for *irresolvable* products of various forms. Subtracting these from the corresponding functions† for *all* products of these forms, we get the functions for the syzygies resolving them.

We shall use $[C_lC_m]$ to denote the syzygy resolving the product C_lC_m and $C_k[C_lC_m]$ for the compound syzygy formed by multiplying the previous one by C_k. This compound syzygy will clearly resolve a product $C_kC_lC_m$.

(i) C_2^3. The generating function for all syzygies of form $C_2[C_2^2]$

$$= \binom{6}{2} \times \text{function for } C_2 \times \text{function for } [C_2^2]$$

$$= \binom{6}{2}\, x/(1-x) \times \{\text{function for } \textit{all} \text{ products of form } C_2^2 \text{ less function for } \textit{irresolvable} \text{ products of this form}\}$$

$$= \binom{6}{2} \frac{x}{1-x} \left\{ \frac{3x^2}{(1-x)^2} - \frac{x^4-x^7}{(1-x)^3} \right\} = \frac{15x}{1-x}\, \frac{3x^2+3x^3+2x^4+x^5}{1-x}.$$

* For the definition of "resolution," see Section I of the previous paper.

† Which can be easily written down from the principles of permutations and Grace's Perpetuant Type Theorem.

The function for *all* products of form C_2^3 is $15x^3/(1-x)^3$, and that for *irresolvable* products of this form is $(x^6-x^9-x^{11}+x^{14})/(1-x)^5$; therefore that for linearly independent syzygies resolving products of form C_2^3 is the difference of these two expressions. Now from Young and Wood's paper, we can see that all these syzygies are compound. Hence the function for linearly independent syzygies of the second kind resolving such compound syzygies is

$$\frac{15x}{1-x}\,\frac{3x^2+3x^3+2x^4+x^5}{1-x}-\frac{15x^3}{(1-x)^3}+\frac{x^6-x^9-x^{11}+x^{14}}{(1-x)^5}$$

$$=30x^3+90x^4+165x^5+241x^6+305x^7+360x^8+409x^9+\frac{455x^{10}}{1-x}+\frac{45x^{11}}{(1-x)^2}.$$

(ii) Similarly, starting with the product $C_1C_2C_3$, we can obtain the generating function $20x^4/(1-x)^2$ for the associated syzygies of the second kind.

(iii) $C_1^2C_4$. All products of this form are irresolvable, so no syzygies arise. Similarly for (v) $C_1^3C_3$ and (vii) C_1^6.

(iv) $C_1^2C_2^2$. The function is $60x^2/(1-x)$.

(vi) $C_1^4C_2$. The function is $10x$.

Adding the results for these seven cases, we see that the generating function for all the linearly independent syzygies of the second kind of degree 6 is

$$10x+60x^2/(1-x)+20x^4/(1-x)^2+30x^3+90x^4+165x^5$$
$$+241x^6+305x^7+360x^8+409x^9+455x^{10}/(1-x)+45x^{11}/(1-x)^2.$$

IV. *Formation of Primary Syzygies.*

Primary syzygies are those formed by symbolical multiplication of the single syzygy of the $(n+2)$-th degree by perpetuant products.

(A) Multiply symbolically the simplest syzygy of the second kind, *i.e.*,

$$(1234)\equiv A_1(234)-A_2(134)+A_3(124)-A_4(123) \text{ by } (15)^\lambda(16)^\mu,$$

where $$\lambda\geqslant 2,\quad \mu\geqslant 1.$$

We get a new irreducible syzygy of the second kind, of degree 6:—

$$(15)^\lambda(16)^\mu(234)-A_2J[(15)^\lambda(16)^\mu(134)]+A_3J[(15)^\lambda(16)^\mu(124)]$$
$$-A_4J[(15)^\lambda(16)^\mu(123)],$$

which we call $$J^{(2)}[(15)^{\lambda}(16)^{\mu}(1234)].$$

It resolves $(15)^{\lambda}(16)^{\mu}(234)$, of form C_3S_3, in terms of compound syzygies of form C_1S_5.

(B) (*a*) Multiply symbolically (1234) by $(16)^{\lambda}(25)^{\mu}$, where λ and μ are both equal or greater than 2. We get a new irreducible syzygy of the second kind, of degree 6 :—

$$(16)^{\lambda}J_{\mu+1}(2534)-(25)^{\mu}J_{\lambda+1}(1634)+A_3J[(16)^{\lambda}(25)^{\mu}(124)]$$
$$-A_4J[(16)^{\lambda}(25)^{\mu}(123)],$$

which we call $$J^{(2)}[(16)^{\lambda}(25)^{\mu}(1234)].$$

It resolves $(16)^{\lambda}J_{\mu+1}(2534)$, of form C_2S_4, in terms of $(25)^{\mu}J_{\lambda+1}(1634)$ and compound syzygies of form C_1S_5.

(β) Let $\mu=1$ and $\lambda>1$. In accordance with our arrangement scheme we consider that $J^{(2)}[(16)^{\lambda}(25)(1234)]$ resolves $(25)J_{\lambda+1}(1634)$ and not $(16)^{\lambda}J_2(2534)$.

(γ) But, if $\lambda=1$, as well as $\mu=1$, we have two J_2's, and so $J^{(2)}[(16)(25)(1234)]$ resolves $(16)J_2(2534)$ in terms of $(25)J_2(1634)$ and other compound syzygies.

V. *The Three Types of Non-Primary Syzygies.*

(*a*) *First type.**

If A and B are two syzygies of the first kind, involving one symbol in common, and if a and b are the corresponding sums of perpetuant products, then, denoting by aB and bA the results of multiplying symbolically B by a and A by b respectively, we see that $bA-aB$ must be a syzygy of the second kind, for when expanded as a sum of perpetuant products it is $ba-ab$, which vanishes identically.

For example, take $A=\{1243\}_{w-1}$ and $B=(456)$, with the symbol 4 in common. Then

$$a\equiv\sum_{0}^{w-1}\binom{w-1}{r}[(12)^{w-r-1}(43)^{r}-(13)^{w-r-1}(42)^{r}],$$

* This is a case of the general method of Section XII of the previous paper.

and $$b \equiv A_4(56) - A_5(46) + A_6(45),$$

and we get the syzygy of the second kind

$$(56)\{1243\}_{w-1} - A_5\{(1243)_{w-1}(46)\} + A_6\{(1243)_{w-1}(45)\} - A_1 A_2 J_w(4356)$$
$$+A_1 A_3 J_w(4256) - A_3(12)^{w-1}(456) + A_2(13)^{w-1}(456)$$
$$-\sum_1^{w-2}\binom{w-1}{r}\left[(12)^{w-r-1}J_{r+1}(4356) - (13)^{w-r-1}J_{r+1}(4256)\right],$$

which we shall denote by

$$\left\{ \begin{matrix} (1243)_{w-1} \\ (456) \end{matrix} \right\}.$$

It resolves $(56)\{1243\}_{w-1}$ of form $C_2 S_4$.

(*b*) *Second type.**

Consider $$Y_w(12,\ 14,\ 36,\ 56)$$

$$\equiv \sum_0^{w-1}\binom{w}{r}\left[(12)^r\{3654\}_{w-r} + (14)^r\{3256\}_{w-r} + (36)^r\{1452\}_{w-r} + (56)^r\{1234\}_{w-r}\right].$$

Replace the syzygies of the first kind in this expression by their equivalent sums of perpetuant products. We get a set of terms of form

$$\Sigma C_2^3 + \Sigma C_1^2 C_2^2 + \Sigma C_1^4 C_2.$$

The coefficient of

$$(12)^\lambda (36)^\mu (54)^{w-\lambda-\mu} = \binom{w}{\lambda}\binom{w-\lambda}{\mu} - \binom{w}{\mu}\binom{w-\mu}{\lambda} = 0.$$

Similarly all the other terms of form C_2^3 cancel.

The coefficient of

$$(36)^r (54)^{w-r} = \binom{w}{0}\binom{w}{r} - \binom{w}{r}\binom{w}{0} = 0,$$

and the coefficient of $$(36)^w = 1 - 1 = 0.$$

Similarly all the other terms of form $C_1^2 C_2^2$ or $C_1^4 C_2$ cancel.

Thus the expanded form of Y_w (12, 14, 36, 56) is identically zero, so we have a syzygy of the second kind. It resolves a syzygy of the first kind of form $C_2 S_4$.

* Due to Dr. A. Young.

(c) *Third type.*

Let

$$V_w(123,\,456) \equiv \sum_0^{w-1} \left\{2\binom{w-1}{r-1} - \binom{w-1}{r}\right\}$$

$$\times[\ (14)^r\{2536\}_{w-r} + (15)^r\{2634\}_{w-r} + (16)^r\{2435\}_{w-r}$$

$$+(24)^r\{3516\}_{w-r} + (25)^r\{3614\}_{w-r} + (26)^r\{3415\}_{w-r}$$

$$+(34)^r\{1526\}_{w-r} + (35)^r\{1624\}_{w-r} + (36)^r\{1425\}_{w-r}],$$

understanding $\binom{w-1}{-1}$ as 0 and $\binom{w-1}{0}$ as 1.

The coefficient of $\quad (14)^{\lambda}(25)^{\mu}(36)^{w-\lambda-\mu}$

$$= 2\left[\binom{w-1}{\lambda-1}\binom{w-\lambda}{\mu} + \binom{w-1}{\mu-1}\binom{w-\mu}{\lambda} + \binom{w-1}{w-\lambda-\mu-1}\binom{\lambda+\mu}{\lambda}\right]$$

$$-\left[\binom{w-1}{\lambda}\binom{w-\lambda}{\mu} + \binom{w-1}{\mu}\binom{w-\mu}{\lambda} + \binom{w-1}{w-\lambda-\mu}\binom{\lambda+\mu}{\lambda}\right].$$

In this expression λ, μ, and $w-\lambda-\mu$ may have any values from 1 to $w-2$. It can be easily verified that when the binomial coefficients are expressed in terms of factorials, the result reduces to zero. Similarly the coefficients of all other terms of form C_2^3 are zero.

The coefficient of

$$(14)^r(25)^{w-r} = \left[2\binom{w-1}{r-1} - \binom{w-1}{r}\right] + \left[2\binom{w-1}{w-r-1} - \binom{w-1}{w-r}\right]$$

$$+\binom{w}{r}\left[2\binom{w-1}{-1} - \binom{w-1}{0}\right],$$

which also reduces to zero for all values of r from 1 to $w-1$. Similarly for all other terms of form $C_1^2 C_2^2$.

Finally, the coefficient of

$$(14)^w = \left[2\binom{w-1}{-1} - \binom{w-1}{0}\right][1-1-1+1] = 0,$$

and similarly for all other terms of form $C_1^4 C_2$.

Hence in the expanded form all terms vanish, so $V_w(123,\,456)$ is a syzygy of the second kind.

VI. *Resolution of Compound Syzygies by means of Syzygies of the Second Kind* (*being the proof that all the linearly independent syzygies of the second kind of degree* 6 *are of the types obtained above*).

The only possible forms of compound syzygies of degree 6 are :—

(i) $C_3 S_3$, (ii) $C_2 S_4$, (iii) $C_1 S_5$, (iv) $C_1 C_2 S_3$, (v) $C_1^2 S_4$, (vi) $C_1^3 S_3$.

(i) $C_3 S_3$. Every such compound is resolvable by a syzygy of the second kind of the form

$$J^{(2)}\left[(ae)^{\lambda}(af)^{\mu}(abcd)\right],$$

where $$\lambda \geqslant 2, \quad \mu \geqslant 1.$$

The corresponding generating function is

$$\binom{6}{3} x^4/(1-x)^2 = 20x^4/(1-x)^2.$$

(ii) $C_2 S_4$. S_4 can be a Stroh syzygy (of weight $\geqslant 4$), a J_w of weight $\geqslant 3$, or a J_2.

We consider it a resolution to express a compound of a Stroh syzygy in terms of J's or a compound of a $J_w (w \geqslant 3)$ in terms of a J_2.

For the resolution of compounds of Stroh syzygies we use such syzygies of the second kind as

$$\left\{\begin{matrix}(1243)_{w-1} \\ (456)\end{matrix}\right\}, \quad Y_w(12, 14, 36, 56), \quad \text{and} \quad V_w(123, 456).$$

For compounds of the J's we use syzygies of the second kind of form

$$J^{(2)}\left[(af)^{\lambda}(be)^{\mu}(abcd)\right].$$

Lemma.—To prove that

$$\left\{\begin{matrix}(5463)_{w-1} \\ (612)\end{matrix}\right\}, \quad Y_w(12, 13, 54, 64), \quad Y_w(21, 23, 54, 64), \quad V_w(134, 256),$$

and $$V_w(156, 234),$$

together resolve the five compounds $(12)^r \{5463\}_{w-r}$, where r has the values 1, 2, 3, 4, 5. This holds good for $w \geqslant 10$.

Denote $(12)^r \{5463\}_{w-r}$ by $F(r)$.

$F(1)$ is always resolved by a syzygy of the second kind of type

$$\left\{\begin{matrix}(5463)_{w-1} \\ (612)\end{matrix}\right\}.$$

Now

$Y_w(12, 13, 54, 64)$, $= A$ say, contains $\Sigma \binom{w}{r} F(r)$,

$Y_w(21, 23, 54, 64)$, $= B$ say, contains $\Sigma \binom{w}{r} (-1)^r F(r)$,

$V_w(134, 256)$, $= C$ say, contains* $-\Sigma\left[2\binom{w}{r}-3\binom{w-1}{r}\right](-1)^{w-r} F(r)$,

$V_w(156, 234)$, $= D$ say, contains $-\Sigma\left[2\binom{w}{r}-3\binom{w-1}{r}\right] F(r)$;

therefore $\frac{1}{2}(A+B)$ contains $\binom{w}{2} F(2)+\binom{w}{4} F(4)+\binom{w}{6} F(6)+\dots$,

and $\frac{1}{2}(A-B)$ contains $\binom{w}{3} F(3)+\binom{w}{5} F(5)+\dots$.

Also $\frac{1}{3}\{2B+(-1)^w C\}$, $= C'$ say, contains

$$\binom{w-1}{2} F(2)-\binom{w-1}{3} F(3)+\binom{w-1}{4} F(4)-\dots,$$

and $\frac{1}{3}\{2A+D\}$, $= D'$ say, contains

$$\binom{w-1}{2} F(2)+\binom{w-1}{3} F(3)+\binom{w-1}{4} F(4)+\dots.$$

Therefore $\frac{1}{2}(C'+D')$ contains $\binom{w-1}{2} F(2)+\binom{w-1}{4} F(4)+\dots$,

and $\frac{1}{2}(D'-C')$ contains $\binom{w-1}{3} F(3)+\binom{w-1}{5} F(5)+\dots$;

therefore $\frac{1}{2}\binom{w-1}{2}(A+B)-\frac{1}{2}\binom{w}{2}(C'+D')$

contains $\left[\binom{w}{4}\binom{w-1}{2}-\binom{w}{2}\binom{w-1}{4}\right] F(4)$,

and $\frac{1}{2}\binom{w-1}{3}(A-B)-\frac{1}{2}\binom{w}{3}(D'-C')$

contains $\left[\binom{w}{5}\binom{w-1}{3}-\binom{w}{3}\binom{w-1}{5}\right] F(5)$.

---

* Since $V_w(134, 256)$ contains $\Sigma\left\{2\binom{w-1}{r-1}-\binom{w-1}{r}\right\}(12)^r\{3546\}_{w-r}$.

The coefficient $=\left\{2\binom{w}{r}-3\binom{w-1}{r}\right\}$,

and $\{3546\}_{w-r}=(-1)^{w-r}\{5364\}_{w-r}=-(-1)^{w-r}\{5463\}_{w-r}$.

Now $\left[\binom{w}{4}\binom{w-1}{2}-\binom{w}{2}\binom{w-1}{4}\right]$

$$= \frac{1}{24} w(w-1)^2(w-2)(w-3) \neq 0, \text{ if } w \geqslant 4;$$

and $\left[\binom{w}{5}\binom{w-1}{3}-\binom{w}{3}\binom{w-1}{5}\right]$

$$= \frac{2}{3!\,5!} w(w-1)^2(w-2)^2(w-3)(w-4) \neq 0, \text{ if } w \geqslant 5.$$

The 3 Stroh syzygies formed from 4 given symbols are all canonical for weights $\geqslant 5$. For weight 4, only 2 out of the 3 are canonical, and for lower weights none are so.

Therefore, if $w \geqslant 10$, $w-5 \geqslant 5$, and none of the Stroh syzygies in $F(1) \ldots F(5)$ are uncanonical, so we get 5 resolutions.

If $w = 9$, we get all 5 resolutions with 2 of the 3 Stroh syzygies and only 4 with the 3rd. We discard here the resolution of

$$F(5) = (12)^5 \{5463\}_{w-5},$$

as there are only two canonical Stroh syzygies of weight $w-5$ $(=4)$.

If $w = 8$, we get, in 2 cases out of 3, 4 resolutions, and only 3 in the 3rd, discarding also $F(4)$, as there are no canonical Stroh syzygies of weight $w-5$ $(=3)$, and only 2 of weight $w-4$ $(=4)$.

Similarly, if $w = 7$, we get 3 resolutions in 2 cases and 2 in the 3rd; if $w = 6$, we get 2 resolutions in 2 cases and 1 in the 3rd; if $w = 5$, we get 1 resolution in 2 cases and none in the 3rd; if $w < 5$, there are no resolutions at all.

We resume the consideration of the compound syzygies of form $C_2 S_4$.

Case (a).—S_4 a Stroh syzygy. We shall arrange the compound syzygies in the following order:—

(i) $(12)^\lambda S_4$, (ii) $(23)^\lambda S_4$, (iii) $(34)^\lambda S_4$, (iv) $(45)^\lambda S_4$, (v) $(56)^\lambda S_4$,

(vi) $(13)^\lambda S_4$, (vii) $(24)^\lambda S_4$, (viii) $(35)^\lambda S_4$, (ix) $(46)^\lambda S_4$, (x) $(14)^\lambda S_4$,

(xi) $(25)^\lambda S_4$, (xii) $(36)^\lambda S_4$, (xiii) $(15)^\lambda S_4$, (xiv) $(26)^\lambda S_4$, (xv) $(16)^\lambda S_4$.

It is a resolution to express any one of these in terms of forms further on in the list, or in terms of compound syzygies of any of the types

(i) $C_2 S_4$ (where S_4 is a J), (ii) $C_1 S_5$, (iii) $C_1 C_2 S_3$, (iv) $C_1^2 S_4$,

(v) $C_1^3 S_3$.

If $\lambda = 1$, we have 45 forms.* All are resolved by a syzygy of the second kind of the type of $\left\{\begin{matrix}(1243)_{w-1}\\(456)\end{matrix}\right\}$, which resolves $(56)\,\{1243\}_{w-1}$.

For values of $\lambda > 1$, we have to examine each form separately. The result is shown in the following table.

				Least value of w for the Stroh syzygies involved to be canonical.
$(12)^\lambda$	$Y_w(12, 13, 54, 64)$	resolves	$(12)^2\,\{5463\}_{w-2}$	6
	$Y_w(12, 13, 45, 65)$	,,	$(12)^2\,\{4563\}_{w-2}$	6
	$Y_w(12, 13, 46, 56)$	,,	$(12)^2\,\{4653\}_{w-2}$	7
	$Y_w(21, 23, 54, 64)$	,,	$(21)^3\,\{5463\}_{w-3}$	7
	$Y_w(21, 23, 45, 65)$	,,	$(21)^3\,\{4563\}_{w-3}$	7
	$Y_w(21, 23, 46, 56)$	,,	$(21)^3\,\{4653\}_{w-3}$	8

Similarly for $(23)^\lambda$, $(34)^\lambda$, and $(45)^\lambda$.

$(56)^\lambda$	$Y_w(56, 52, 31, 41)$	resolves	$(56)^2\,\{3142\}_{w-2}$	6
	$Y_w(65, 62, 31, 41)$	,,	$(65)^3\,\{3142\}_{w-3}$	7

We do not get the set of six here. For example,

$$Y_w(56,\ 52,\ 13,\ 43)$$

cannot be used to resolve $(56)^\lambda$ as it involves $(34)^\lambda$, which precedes $(56)^\lambda$.

$(13)^\lambda$ Full set of six as for $(12)^\lambda$.

$(24)^\lambda$	$Y_w(24, 26, 15, 35)$	resolves	$(24)^2\,\{1536\}_{w-2}$	6
	$Y_w(42, 46, 15, 35)$	,,	$(42)^3\,\{1536\}_{w-3}$	7
$(35)^\lambda$	$Y_w(53, 52, 41, 61)$	,,	$(53)^2\,\{4162\}_{w-2}$	6
	$Y_w(53, 52, 14, 64)$	,,	$(53)^2\,\{1462\}_{w-2}$	6
	$Y_w(53, 52, 16, 46)$	,,	$(53)^2\,\{1642\}_{w-2}$	7
	$Y_w(53, 51, 26, 46)$	,,	$(53)^3\,\{2641\}_{w-3}$	7
$(46)^\lambda$	$Y_w(64, 63, 15, 25)$	,,	$(64)^2\,\{1523\}_{w-2}$	6
$(14)^\lambda$	$Y_w(14, 15, 26, 36)$	,,	$(14)^2\,\{2635\}_{w-2}$	6

* $45 = 15 \times 3$; 3 Stroh syzygies can be formed from the same 4 symbols, for all weights $\geqslant 5$.

These resolutions are obtained by using the $\left\{\begin{matrix}(1243)_{w-1}\\(456)\end{matrix}\right\}$ set, and the Y_w set.

Using the V_w set in conjunction with the above, we get (from the Lemma)

$(12)^\lambda$	$V_w(134, 256)$	resolves	$(12)^4\{5463\}_{w-4}$	8
	$V_w(135, 246)$	,,	$(12)^4\{4563\}_{w-4}$	8
	$V_w(136, 245)$	,,	$(12)^4\{4653\}_{w-4}$	9
	$V_w(156, 234)$	,,	$(12)^5\{3645\}_{w-5}$	9
	$V_w(146, 235)$	,,	$(12)^5\{3654\}_{w-5}$	9
	$V_w(145, 236)$	,,	$(12)^5\{3564\}_{w-5}$	10
$(23)^\lambda$	$V_w(126, 345)$	,,	$(23)^4\{6415\}_{w-4}$	8
	$V_w(125, 346)$	,,	$(23)^4\{5416\}_{w-4}$	8
	$V_w(124, 356)$	,,	$(23)^4\{4516\}_{w-4}$	9
$(34)^\lambda$	$V_w(123, 456)$	,,	$(34)^4\{1526\}_{w-4}$	8

All these resolutions are possible if $w \geqslant 10$, giving $45+40+10 = 95$ linearly independent syzygies of the second kind of each weight. The corresponding generating function is $95x^{10}/(1-x)$.

If $w = 9$, one resolution of a Stroh syzygy of weight 4 is lost, so the generating function for this weight is $94x^9$.

If $w = 8$, 4 more resolutions are lost, and the function is $90x^8$.

If $w = 7$, 10 ,, ,, ,, $80x^7$.

If $w = 6$, 19 ,, ,, ,, $61x^6$.

If $w = 5$, *all* those in the above table are lost. The resolutions by such syzygies of the second kind as $\left\{\begin{matrix}(1243)_{w-1}\\(456)\end{matrix}\right\}$ remain, but there will be only two-thirds of the former number, *i.e.*, two-thirds of $45 = 30$, so the function is $30x^5$.

If $w < 5$, there are no resolutions at all.

Hence the total generating function for Case (α) is

$$30x^5+61x^6+80x^7+90x^8+94x^9+95x^{10}/(1-x).$$

Case (β).—S_4 a J_w, with $w \geqslant 3$.

As shewn in Section IV, $J^{(2)}[(16)^\lambda(25)^\mu(1234)]$ resolves $(16)^\lambda J_{\mu+1}(2534)$ in terms of $(25)^\mu J_{\lambda+1}(1634)$ and syzygies of form C_1S_5.

Thus it follows from the order of arrangement that $(12)^\lambda J_{\mu+1}(abcd)$, where a, b, c, d are 3, 4, 5, 6 in any order, is always resolvable. There are 6 in this set, since from 4 symbols 6 essentially different J's of the same weight can be formed.

Similarly for $(13)^\lambda$, $(14)^\lambda$, $(15)^\lambda$, and $(16)^\lambda$.

But from our arrangement scheme, there are only 3 resolutions for $(23)^\lambda J_{\mu+1}$, and not the full set of 6. Similarly for $(24)^\lambda$, $(25)^\lambda$, and $(26)^\lambda$. For $(34)^\lambda$, $(35)^\lambda$, and $(36)^\lambda$, there is only one resolution apiece.

To sum up, this gives 45 resolutions for each weight $\geqslant 5$ (as $\lambda \geqslant 2$ and $\mu \geqslant 2$), the corresponding generating function being $45x^5/(1-x)^2$.

As shewn also in Section IV, $J^{(2)}[(16)^\lambda(25)(1234)]$ resolves $(25)J_{\lambda+1}(1634)$ in terms of $(16)^\lambda J_2(2534)$ and syzygies of form C_1S_5.

Every form $(ab)J_{\lambda+1}$ is thus resolvable when $\lambda \geqslant 2$.

We therefore get an extra generating function $\binom{6}{2} \times 6x^4/(1-x)$ making a total for Case (β) of

$$90x^4/(1-x)+45x^5/(1-x)^2.$$

Case (γ).—S_4 a J_2.

As shewn in Section IV, $J^{(2)}[(16)(25)(1234)]$ resolves $(16)J_2(2534)$ in terms of $(25)J_2(1634)$ and other compound syzygies. This case differs from (β) in that from 4 symbols only 3 essentially different J_2's can be formed, as $J_2(abcd) = J_2(cdab)$.

We get similar resolutions for every other syzygy of form $(ab)J_2(cdef)$, if (ab) is (12), (13), (14), (15), (16), (23), (24), (25) or (26). This gives $9\times 3 = 27$ resolutions.

Also $(34)J_2(5612)$ can be resolved by $J^{(2)}[(34)(56)(3512)]$, and similarly for $(35)J_2(4612)$ and $(36)J_2(4512)$. There are no other resolutions.

Hence the generating function for Case (γ) is $30x^3$.

Resolution of the other Forms of Compound Syzygies of Degree 6.

(iii) C_1S_5 *and* (v) $C_1^2S_4$. All such compounds are irresolvable, so there are no corresponding syzygies of the second kind.

(iv) $C_1C_2S_3$. All these are resolvable by a form of the type of $C_1J^{(2)}[(ab)^\lambda(adef)]$, where $\lambda \geqslant 1$.

The corresponding generating function is

$$6\times\binom{5}{2}\frac{x}{1-x}x = \frac{60x^2}{1-x}.$$

(vi) $C_1^3 S_3$. These are resolvable (if, and only if, an A_1 is among the three C_1's) by such a compound syzygy of the second kind as $C_1^2(pqrs)$, where p is 1. The q, r, and s can be chosen in $\binom{5}{3} = 10$ ways, so the corresponding generating function is $10x$.

Adding together the generating functions for all the forms discussed in this section, we obtain a total of

$$\frac{20x^4}{(1-x)^2}+30x^5+61x^6+80x^7+90x^8+94x^9+\frac{95x^{10}}{1-x}+\frac{90x^4}{1-x}+\frac{45x^5}{(1-x)^2}+30x^3$$
$$+\frac{60x^2}{1-x}+10x.$$

Now this can be easily shown to be equal to

$$10x+\frac{60x^2}{1-x}+\frac{20x^4}{(1-x)^2}+30x^3+90x^4+165x^5+241x^6+305x^7+360x^8$$
$$+409x^9+\frac{455x^{10}}{1-x}+\frac{45x^{11}}{(1-x)^2},$$

which is the theoretical generating function found in Section III.

Hence we have proved that all the linearly independent syzygies of the second kind of degree 6 *have been obtained.*

To sum up, the irreducible syzygies found are :—

(A) The *primary* syzygies

(α) $J^{(2)}[(ae)^\lambda (af)^\mu (abcd)]$, and ($\beta$) $J^{(2)}[(ab)^\lambda (cd)^\mu (acef)]$.

(B) The non-primary syzygies

(α) $\begin{Bmatrix} (abcd)_{w-1} \\ (def) \end{Bmatrix}$, ($\beta$) $Y_w(ab,\ ac,\ de,\ fe)$,

and (γ) $V_w(abc,\ def)$.