THE RELATION BETWEEN THE CONVERGENCE OF SERIES AND OF INTEGRALS

By T. J. I'A. BROMWICH.

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IT has long been known that, when $f(x)$ is a positive function which steadily decreases as x tends to infinity, the series $\overset{\infty}{\Sigma} f(n)$ and the integral $\int^{\infty} f(x)\,dx$ converge or diverge together: and in case of divergence, the difference

$$\int_{\mu}^{\nu} f(x)\,dx - \sum_{\mu}^{\nu} f(n)$$

tends to a definite limit as ν tends to infinity.*

But, when the series contains terms of both signs (although steadily decreasing in numerical value), it is not easy to make a corresponding statement with reference to the relation between the convergence of the series and of the integral.†

It is known, of course, that, when $f(x)$ decreases steadily to zero as a limit, the two series and the two integrals

$$\overset{\infty}{\Sigma} f(n) \sin na, \quad \int^{\infty} f(x) \sin ax\,dx, \quad \overset{\infty}{\Sigma} f(n) \cos na, \quad \int^{\infty} f(x) \cos ax\,dx$$

are all convergent.‡ And these results can be easily extended to cases in which the periodic factors are of the forms

$$\sin ax\, P(\sin^2 ax), \qquad \cos ax\, P(\sin^2 ax),$$

where P is a *polynomial*. But, if the polynomial P is replaced by

* The main part of the theorem goes back to Maclaurin: for a proof of the latter part, see my *Infinite Series*, Art. 11. When the monotonic condition is removed from $f(x)$, the theorem is no longer true; for examples, see p. 423 of my book.

† Of course, in a large number of interesting cases, the terms decrease fast enough to ensure *absolute* convergence. This case is covered, from the practical point of view, by Maclaurin's rule, and we shall suppose that absolute convergence is excluded from the cases discussed here.

‡ See, for example, *Infinite Series*, Arts. 19, 20, 169.

a general continuous function, the cosine-series is known to diverge for certain values of a, although the integral is always convergent; however, with the same values of a the sine-series is convergent.*

If the periodic factor in the series is of the form

$$\sin\phi(x) \quad \text{or} \quad \cos\phi(x),$$

where $\phi(x)$ tends to infinity *more rapidly than* x, it is practically certain that the convergence of the integral gives no information with regard to the nature of the series.

Thus, for instance, the integrals

$$\int^{\infty} f(x)\sin(ax^p)\,dx, \qquad \int^{\infty} f(x)\cos(ax^p)\,dx,$$

where p is positive, will converge if $f(x)$ steadily decreases to any finite limit (not necessarily zero)†: but, on the other hand, the two series

$$\Sigma f(n)\sin(an^p), \qquad \Sigma f(n)\cos(an^p)$$

have only been considered for rational values of a/π and integral values of p; they are then known to diverge [even if $f(x)$ tends to zero] unless a certain condition is satisfied; and this condition is certainly broken even in the simplest case (when p is 2) except for special values of a.‡

The object of the following note is to prove that [with certain restrictions on the functions, stated in (a)–(δ) on p. 329], *when* $\phi(x)$ *tends steadily to infinity, but* MORE SLOWLY THAN x, *the behaviour of the integrals*

$$\int^{\infty} f(x)\sin\phi(x)\,dx, \qquad \int^{\infty} f(x)\cos\phi(x)\,dx$$

entirely settles the character of the series

$$\overset{\infty}{\Sigma} f(n)\sin\phi(n), \qquad \overset{\infty}{\Sigma} f(n)\cos\phi(n).$$

This theorem is then applied, in § 2, to extend (and simplify the proofs of) certain known theorems, the simplest of which is that if

$$A_\nu + iB_\nu = \sum_1^\nu \frac{1}{n^{1+ai}} \quad (a \text{ real}),$$

then $A_\nu + iB_\nu - i/a\nu^{ai}$ tends to a definite limit, so that both A_ν and B_ν have a range of oscillation $2/a$ as ν tends to infinity.

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* See Bromwich and Hardy, *Quarterly Journal*, Vol. XXXIX., May, 1908, pp. 232, 236, 240, and also below, p. 338.

† See my *Infinite Series*, Art. 169 and Ex. 8, p. 468.

‡ See below, § 3, p. 338; and Genocchi, *Atti di Torino*, t. x., 1875, p. 991.

1. *Proof of the Theorem.*

Let us write $F(x) = f(x) \sin \phi(x)$, where we suppose that

$$\left.\begin{array}{ll} (\alpha) & f(x) \text{ tends steadily to zero}^{*} \\ (\beta) & \phi(x) \text{ tends steadily to infinity} \\ (\gamma) & \phi'(x) \text{ tends steadily to zero} \end{array}\right\} \text{ as } x \to \infty.$$

From these conditions it follows that

(1) $$f'(x) \leqslant 0, \qquad \phi'(x) \geqslant 0.$$

Now, if we write $$X_n = \int_n^{n+1} F(x)\,dx - F(n),$$
we have the equations

(2) $$X_n = \int_n^{n+1} \{F(x) - F(n)\}\,dx = \int_0^1 \{F(n+t) - F(n)\}\,dt,$$

(3) $$F(n+t) - F(n) = \int_0^t F'(n+v)\,dv.$$

But $$F'(x) = f'(x) \sin \phi(x) + f(x)\,\phi'(x) \cos \phi(x);$$
and so $$|F'(n+v)| \leqslant |f'(n+v)| + |f(n+v)| \,.\, |\phi'(n+v)|.$$
Now, from conditions (α), (β), (γ) and from the inequalities (1), we see that
$$|f'(n+v)| = -f'(n+v),$$
and $$|f(n+v)| \,.\, |\phi'(n+v)| \leqslant f(n)\,\phi'(n).$$
It follows that $$|F'(n+v)| \leqslant f(n)\,\phi'(n) - f'(n+v).$$
Thus, making use of the last inequality in the equation (3), we find that

(3′) $$\begin{aligned} |F(n+t) - F(n)| &\leqslant t f(n)\,\phi'(n) + f(n) - f(n+t) \\ &\leqslant f(n)\,\phi'(n) + f(n) - f(n+1), \end{aligned}$$

provided that t belongs to the interval (0, 1). Hence, combining (3′) and (2), we see that

(4) $$|X_n| \leqslant f(n)\,\phi'(n) + f(n) - f(n+1).$$

We now introduce the condition that

(δ) *the integral* $\int^{\infty} f(x)\,\phi'(x)\,dx$ *is convergent.*

* As remarked above (foot-note, p. 327), we suppose absolute convergence excluded, so that $\int^{\infty} f(x)\,dx$ is divergent. When this integral is convergent, the discussion given here is quite superfluous.

Consequently, by Maclaurin's theorem quoted in the introduction to this paper (p. 327), we see that the series

(5) $$\sum^{\infty} f(n)\,\phi'(n)$$

is also convergent because $f(x)\,\phi'(x)$ tends steadily to zero, in virtue of conditions (α), (γ). Further,

$$\sum_{\mu}^{\nu} \{f(n)-f(n+1)\} = f(\mu)-f(\nu+1),$$

and this tends to the limit $f(\mu)$ as ν tends to infinity; thus the series

(6) $$\sum^{\infty} \{f(n)-f(n+1)\}$$

is also convergent.

It follows from (4), (5), and (6) that the series $\Sigma|X_n|$ converges: and consequently the series ΣX_n is absolutely convergent. But

$$\sum_{\mu}^{\nu} X_n = \int_{\mu}^{\nu} F(x)\,dx - \sum_{\mu}^{\nu} F(n) + \int_{\nu}^{\nu+1} F(x)\,dx$$

and $$\left|\int_{\nu}^{\nu+1} F(x)\,dx\right| < \int_{\nu}^{\nu+1} f(x)\,dx < f(\nu),$$

so that $$\lim_{\nu\to\infty} \int_{\nu}^{\nu+1} F(x)\,dx = 0.$$

Thus

(7) $$\lim_{\nu\to\infty} \left\{\int_{\mu}^{\nu} F(x)\,dx - \sum_{\mu}^{\nu} F(n)\right\} = \sum_{\mu}^{\infty} X_n;$$

and accordingly, since the series on the right has been proved to converge, the limit on the left is also definite.

It follows at once that, *if* $f(x)$ *and* $\phi(x)$ *are subject to the conditions* (α), (β), (γ), *and* (δ), *the series*

$$\sum^{\infty} f(n)\sin\phi(n)$$

and the integral $$\int^{\infty} f(x)\sin\phi(x)\,dx$$

converge, diverge, or oscillate together.

Further, the equation (7) shews that, in case both oscillate, the amplitude of oscillation is the *same* for the series as for the integral; and, in case of divergence, the limit on the left of (7) is still finite.

Under the same conditions (α)–(δ), *the same results apply to the series*

$$\sum^{\infty} f(n)\cos\phi(n)$$

and the integral $$\int^{\infty} f(x) \cos \phi(x)\, dx\,;$$

and, consequently, the same conclusions apply also to the series

$$\sum^{\infty} f(n) \exp\{\pm i\phi(n)\}$$

and the integrals $$\int^{\infty} f(x) \exp\{\pm i\phi(x)\}\, dx.$$

We note as a special case that the conditions (α)–(δ) are certainly satisfied by the function

$$\{M(x)\}^{-\kappa} \quad (\kappa = \beta + i\gamma,\ \beta > 0),$$

provided that $M(x)$ tends steadily to infinity in such a way that $M'(x)/M(x)$ tends steadily to zero. For we have then to take

$$f(x)\, e^{\pm i\phi(x)} = \{M(x)\}^{-x}, \text{ or } f(x) = \{M(x)\}^{-\beta}, \text{ and } \phi(x) = |\gamma| \log\{M(x)\},$$

from which it is evident that the first three conditions are satisfied; as to the fourth condition, we must examine the integral

$$\int^{\infty} f(x)\, \phi'(x)\, dx = \int^{\infty} \frac{|\gamma|\, M'(x)}{\{M(x)\}^{(1+\beta)}}\, dx = |\gamma| \int^{\infty} \frac{dy}{y^{1+\beta}},$$

which is clearly convergent, so that all the four conditions are satisfied.

Again, if the two functions

$$F(x) = f(x)\, e^{\pm i\phi(x)}, \qquad G(x) = g(x)\, e^{\pm i\psi(x)}$$

satisfy the prescribed conditions, their product $F(x)\, G(x)$ will also satisfy the conditions, provided that*

(β') $\phi(x) - \psi(x)$ tends steadily to infinity,

(γ') $\phi'(x) - \psi'(x)$ tends steadily to zero.

For then the product $f(x)\, g(x)$ will obviously tend steadily to zero, and each of the integrals

$$\int^{\infty} f(x)\, g(x)\, \phi'(x)\, dx, \qquad \int^{\infty} f(x)\, g(x)\, \psi'(x)\, dx$$

is convergent, because $\lim f(x) = 0$, $\lim g(x) = 0$.

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* These additional conditions are superfluous when the signs in the two exponential functions are the same.

2. *Examination of certain Special Series.*

1. Consider first the simple case*

$$\Sigma \frac{1}{n^{1+ai}} \quad (a \text{ real}).$$

Here we can take $M(x) = x$, $\beta = 1$, in the result at the end of § 1; then $M(x)$ tends steadily to infinity, while $M'(x)/M(x)$ tends steadily to zero.

Consequently, the behaviour of the series is determined by that of the integral

$$\int_1^\nu \frac{dx}{x^{1+ai}} = \frac{1}{ai}\left\{1 - \frac{1}{\nu^{ai}}\right\}.$$

Now, as $\nu \to \infty$, this integral oscillates, the amplitude (both for real and for imaginary parts) being $2/a$; and so the same is true of the series. The theorem of § 1 can also be applied to cases such as

$$M(x) = \log x, \quad \log(\log x), \quad \ldots,$$

but the range of oscillation for the corresponding integrals and series is infinite.

2. Secondly, let us consider the type of series which is obtained by introducing a complex index in the general logarithmic series: that is, we consider the series

$$\Sigma \frac{1}{n \,.\, l_1 n \,.\, l_2 n \,\ldots\, l_{k-1} n \,(l_k n)^{1+ai}},$$

where † $\qquad l_k x = |\log(l_{k-1} x)|, \qquad l_1 x = \log x.$

Here we can take $\qquad \phi(x) = a\, l_{k+1} x, \quad f(x) = \phi'(x)/a$

or ‡ $\qquad f(x) = 1/\{x \,.\, l_1 x \,.\, l_2 x \ldots l_k x\}.$

Then we can find a constant K, so that

$$f(x)\,\phi'(x) < K/x^2,$$

and so condition (δ) of § 1 is satisfied and the other conditions are

* This series can be discussed by Weierstrass's rule depending on the quotient of two consecutive terms in the series (see my *Infinite Series*, p. 204). The particular case of the rule which is needed here is, however, rather troublesome to establish; and it would be almost impossible to use a similar method in the other cases given below.

† Of course, after a certain stage, the logarithms are all positive, and the sign of the modulus may then be omitted; it is often simpler to suppose that the earlier terms are left out from the series so as to avoid this complication.

‡ Because, using accents for differential coefficients, we have

$$l'_{k+1} = \frac{l'_k}{l_k}, \quad l'_k = \frac{l'_{k-1}}{l_{k-1}}, \quad \ldots, \quad l'_1 = \frac{1}{x}.$$

evidently satisfied. Thus we have to consider the integral

$$\int_\mu^\nu f(x)\, e^{-i\phi(x)}\, dx = \frac{1}{ia}\,(e^{-i\phi(\mu)} - e^{-i\phi(\nu)}),$$

which again oscillates (as $\nu \to \infty$) with an amplitude $2/a$; and so the series has the same range of oscillation.

3. It follows without further proof that, *if $\psi(x)$ is any function tending steadily to zero (as $x \to \infty$) the series*

$$\Sigma\,\frac{\psi(n)}{n^{1+ai}}, \qquad \Sigma\,\frac{\psi(n)}{n\,.\,l_1 n\,.\,l_2 n\,\ldots\,l_{k-1}n.(l_k n)^{1+ai}}$$

are both convergent, in virtue of Dirichlet's test of convergence* and the results obtained in (1), (2) above.

Thus, as a simple example, we may note that the series

$$\Sigma\,\frac{1}{n^{1+ai}(\log n)^\beta}$$

is convergent if β is positive.

We can generalise these results still further by supposing that $\psi(x)$ *is complex, but tends to zero as x tends to infinity in such a way that*

$$\int^\infty |\,\psi'(x)\,|\, dx$$

is convergent. For then we have the inequality †

$$\left|\int_{X_1}^{X_2} F(x)\,\psi(x)\,dx\right| < HV \quad (X_2 > X_1),$$

where

$$V = \int_{X_1}^\infty |\,\psi'(x)\,|\,dx,$$

and H is the upper limit to the integral

$$\left|\int_{X_1}^{X} F(x)\,dx\right| \quad (X_1 \leqslant X \leqslant X_2).$$

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* See my *Infinite Series*, Art. 20.

† See *Proc. London Math. Soc.*, Vol. 6, 1907, p. 65; it is perhaps worth while to remark that there the integral is proved to be less than HV', where

$$V' = \int_{X_1}^{X_2} |\,\psi'(x)\,|\,dx + |\,\psi(X_2)\,|.$$

But

$$\psi(X_2) = -\int_{X_2}^\infty \psi'(x)\,dx\,;$$

and so

$$|\,\psi(X_2)\,| \leqslant \int_{X_2}^\infty |\,\psi'(x)\,|\,dx.$$

Thus

$$V' \leqslant \int_{X_1}^\infty |\,\psi'(x)\,|\,dx,$$

and so the value of V given in the text is larger than V'.

Thus, since $H \leqslant 2/a$ [see (1), (2) above], we can choose X_1 so that

$$\left|\int_{X_1}^{X_2} F(x)\,\psi(x)\,dx\right| < \epsilon \quad (\text{if } X_2 > X_1),$$

because, by proper choice of X_1, the integral V can be made as small as we please. Consequently the integral

$$\int^{\infty} F(x)\,\psi(x)\,dx$$

is convergent: and so the same is true of the series

$$\sum^{\infty} F(n)\,\psi(n),$$

provided that the conditions of § 1 are satisfied by the function $F(x)\,\psi(x)$.

As an illustration of the last result, we may take

$$F(x) = x^{-(1+ai)}, \quad \psi(x) = (\log x)^{-\kappa} \quad (\kappa = \beta + i\gamma,\ \beta > 0).$$

For then
$$|\psi'(x)| = \frac{|\kappa|}{x(\log x)^{1+\beta}},$$

and so
$$\int^{\infty} |\psi'(x)|\,dx = \int^{\infty} \frac{|\kappa|\,dx}{x(\log x)^{1+\beta}},$$

which converges when β is positive, because the indefinite integral is

$$\frac{|\kappa|}{\beta}(\log x)^{-\beta}.$$

Further, as was pointed out in (1) above, the conditions of § 1 are satisfied by the functions $F(x)$, $\psi(x)$; and so the conditions are satisfied also by their product, since the function in the exponential is here

$$a \log x + \gamma \log(\log x).$$

Thus we see that the series

$$\sum \frac{1}{n^{1+ai}(\log n)^{\beta+\gamma i}} \quad (\beta > 0)$$

is convergent; and the same method can be easily extended to more complicated cases. In this way it can be proved that the series*

$$\sum \frac{(l_2 n)^{\kappa_2}(l_3 n)^{\kappa_3}\ldots(l_k n)^{\kappa_k}}{n^{1+ai}(l_1 n)^{\beta+\gamma i}} \quad (\beta > 0)$$

is convergent, whatever the indices $\kappa_2, \kappa_3, \ldots, \kappa_k$ may be (real or complex).

* These examples were suggested to me by Mr. Hardy.

4. As a last example, we shall find an asymptotic formula for the series

$$\sum_1^\nu \frac{(\log n)^\kappa}{n^{1+ai}} \quad (\kappa = \beta + i\gamma,\ \beta > 0),$$

which has been discussed in the special case $\kappa = 1$, by Mertens.*

Here we have to put

$$f(x) = \frac{(\log x)^\beta}{x}, \qquad \phi(x) = a \log x - \gamma \log(\log x),$$

and so we find
$$f'(x) = -\frac{(\log x)^\beta}{x^2}\left(1 - \frac{\beta}{\log x}\right),$$

$$\phi'(x) = \frac{1}{x}\left(a - \frac{\gamma}{\log x}\right).$$

Thus the first three conditions of § 1 will be satisfied, as soon as $\log x$ is greater than both β and γ/a; further, the integral (δ) of § 1 will converge, provided that

$$\int^\infty \frac{(\log x)^\beta}{x^2}\, dx$$

is convergent; but this reduces to the known integral

$$\int^\infty \xi^\beta e^{-\xi}\, d\xi \quad (\text{if } \xi = \log x),$$

and so all the conditions of § 1 are satisfied here.

Thus the asymptotic formulæ for the series

$$\sum_1^\nu \frac{(\log n)^\kappa}{n^{1+ai}}$$

is given by the asymptotic formula for the integral

$$\int_1^\nu \frac{(\log x)^\kappa}{x^{1+ai}}\, dx = \int_0^{\log \nu} \xi^\kappa e^{-ai\xi}\, d\xi.$$

Thus, on integrating by parts, we get the formula

$$\int_1^\nu \frac{(\log x)^\kappa}{x^{1+ai}}\, dx = -\frac{1}{\nu^{ai}}\left[\frac{(\log \nu)^\kappa}{ai} + \kappa \frac{(\log \nu)^{\kappa-1}}{(ai)^2} + \kappa(\kappa-1)\frac{(\log \nu)^{\kappa-2}}{(ai)^3} + \dots \right.$$
$$\left. + \kappa(\kappa-1)\dots(\kappa-m+1)\frac{(\log \nu)^{\kappa-m}}{(ai)^{m+1}}\right]$$
$$+ \int_0^{\log \nu} \frac{\kappa(\kappa-1)\dots(\kappa-m)}{(ai)^{m+1}} \xi^{\kappa-m-1} e^{-ai\xi}\, d\xi,$$

provided that $\beta - m$ is positive.

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* *Göttingen Nachrichten*, 1887, p. 266; the method adopted by Mertens is to differentiate the first series of p. 332 with respect to a.

When β is an integer, the formula just obtained is only true if $m = \beta - 1$; to extend it to $m = \beta$, we divide the remainder integral into two, from 0 to 1 and from 1 to $\log \nu$. The second of these integrals may be again integrated by parts, and so we obtain the term given by $m = \beta$ together with a constant; the new remainder integral is then proportional to

$$\int_1^{\log \nu} \xi^{i\gamma - 1} e^{-ai\xi} \, d\xi,$$

which is easily seen to converge to a definite value as ν tends to infinity.* Consequently, if we write $m = \beta$ in the expression in square brackets at the foot of p. 335, the difference between this formula and the sum of the series will tend to a definite limit as ν tends to infinity.

When β is not an integer, we take m as the integer next less than β, and the remainder integral can then be proved to converge (as $\nu \to \infty$) by a method similar to that used in the last case.

Summing up, we have now the result that

$$\sum_1^{\nu} \frac{(\log n)^\kappa}{n^{1+ai}} \sim -\frac{1}{\nu^{ai}} \left[\frac{(\log \nu)^\kappa}{ai} + \kappa \frac{(\log \nu)^{\kappa-1}}{(ai)^2} + \kappa(\kappa-1) \frac{(\log \nu)^{\kappa-2}}{(ai)^3} + \dots \right.$$
$$\left. + \kappa(\kappa-1) \dots (\kappa - m + 1) \frac{(\log \nu)^{\kappa - m}}{(ai)^{m+1}} \right],$$

where m is the integral part of β. Thus in particular we have Mertens's result

$$\sum_1^{\nu} \frac{\log n}{n^{1+ai}} \sim -\frac{1}{\nu^{ai}} \left\{ \frac{\log \nu}{ai} + \frac{1}{(ai)^2} \right\}.$$

These asymptotic equations imply that the difference between the expressions on the two sides of the symbol $\sim$ has a finite limit as ν tends to infinity.

It would be easy to multiply examples of this type by introducing more logarithms, but enough has been said to indicate the scope of the method.

3. *A Different Test for Series which contain Periodic Factors.*

Suppose that we wish to discuss the series

$$\Sigma f(n) \, v(n),$$

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* In fact, the real part of the integral can be written in the form

$$\text{const.} + \int_c^{\log \nu} \left\{ \left(a - \frac{\gamma}{\xi} \right) \cos(a\xi - \gamma \log \xi) \right\} \frac{d\xi}{a\xi - \gamma} \quad (ac > \gamma),$$

to which we can apply Dirichlet's test for convergence (*Infinite Series*, Art. 169). Similarly the imaginary part of the integral can be proved to converge.

where $f(x)$ has the same properties as in § 1, but $v(n)$ has the period ω, so that

$$v(n+\omega) = v(n).$$

Then the necessary and sufficient condition for convergence is that

$$\sum_1^{\omega} v(n) = 0.$$

For, suppose that
$$\Omega = \sum_1^{\omega} v(n),$$

then
$$v(\omega) = \Omega - v(1) - v(2) - \ldots - v(\omega-1).$$

Then, since $v(r\omega+s) = v(s)$, we find that

$$\sum_1^{\lambda\omega} f(n)\, v(n) = \Omega \sum_{r=1}^{\lambda} f(r\omega) + \sum_{s=1}^{\omega-1} v(s)\, S,$$

where $S = f(s) - f(\omega) + f(s+\omega) - f(2\omega) + \ldots + f[s+(\lambda-1)\,\omega] - f(\lambda\omega)$.

Now, in virtue of the decreasing character of $f(x)$, the sum S has a definite limit $\phi(s)$ less than $f(s)$, as λ tends to infinity, and so

$$\lim_{\lambda\to\infty} \sum_{s=1}^{\omega-1} v(s)\, S = \sum_{s=1}^{\omega-1} v(s)\, \phi(s).$$

But
$$\sum_{r=1}^{\lambda} f(r\omega) > \int_1^{\lambda} f(\omega\xi)\, d\xi = \int_1^{\omega\lambda} f(x)\, dx,$$

and so this sum tends to infinity with λ, since $\int^{\infty} f(x)\, dx$ is divergent.

It follows that the sum
$$\sum_1^{\lambda\omega} f(n)\, v(n)$$

also tends to infinity with λ unless Ω is zero; in the latter case, the sum has the finite limit

$$\sum_{s=1}^{\omega-1} v(s)\, \phi(s).$$

Now, if $0 < \mu < \omega$, we have

$$\left| \sum_{\lambda\omega+1}^{\lambda\omega+\mu} f(n)\, v(n) \right| < V\mu f(\lambda\omega),$$

where V denotes the largest of the values $|v(1)|$, $|v(2)|$, ..., $|v(\omega)|$. Consequently

$$\lim_{\lambda\to\infty} \sum_{\lambda\omega+1}^{\lambda\omega+\mu} f(n)\, v(n) = 0,$$

and so the behaviour of the general series

$$\sum^{\infty} f(n)\, v(n)$$

is the same as that of $$\lim_{\lambda\to\infty} \sum_{1}^{\lambda\omega} f(n)\, v(n).$$

Thus the series diverges unless Ω is zero, and converges if Ω is zero.

As a simple example, we can infer the convergence of the series

$$\Sigma f(n) \sin nh\, \Theta(\sin^2 nh),$$

where $h = p\pi/q$ (p, q being positive integers), and Θ is any continuous function; because here $\omega = 2q$ if p is odd, or $\omega = q$ if p is even, and in either case $\Omega = 0$.

But the series $\Sigma f(n) \cos nh\, \Theta(\sin^2 nh)$

can be made to diverge by adjustment of Θ if p is even and q is odd.*

Similarly the series $\Sigma(-1)^{[nh]} f(n)$,

where $[\xi]$ denotes the integral part of ξ, will diverge if $h = p/q$, where p is even and q is odd,† because again $\omega = q$ and so $\Omega = 1$.

The applications to series of the type

$$\Sigma f(n) \sin(n^s h), \qquad \Sigma f(n) \cos(n^s h),$$

where s is a positive integer, are equally obvious. Thus, for example, the series

$$\Sigma f(n) \sin\left(n^2 \frac{2\pi}{q}\right), \quad \Sigma f(n) \cos\left(n^2 \frac{2\pi}{q}\right)$$

can converge only if

$$\sum_{0}^{q-1} \sin\left(n^2 \frac{2\pi}{q}\right) = 0, \qquad \sum_{0}^{q-1} \cos\left(n^2 \frac{2\pi}{q}\right) = 0,$$

respectively.

Thus, when q is of the form $4k+1$ the sine-series converges, but the cosine-series diverges; but if q is of the form $4k+3$, the cosine-series converges, while the sine-series is divergent; if q is of the form $4k$, both series diverge, but if q is of the form $4k+2$, both converge. These results follow from the values found by Gauss for $\Sigma \exp(2\pi i n^2/q)$ in his investigations on quadratic residues.‡

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* For instance, if we take $p = 2$, $q = 3$, the value of Ω is easily seen to be

$$\Theta(0) - \Theta(\tfrac{3}{4}),$$

which, of course, may have any value.

† Bromwich and Hardy, *l.c.*, p. 240.

‡ *Werke*, Bd. II., p. 9 (§ 19).