ON A MODIFIED FORM OF PURE RECIPROCANTS POSSESSING THE PROPERTY THAT THE ALGEBRAICAL SUM OF THE COEFFICIENTS IS ZERO

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Prof. Sylvester at the close of his Sixteenth Lecture on the "Theory of Reciprocants," remarks:*—"It will be already seen from an inspection of the fundamental forms that there is no law for the coefficients of reciprocants akin to that of their algebraical sum being zero in invariants."

I wish to shew in this short communication that the forms may be so modified that the vanishing of the algebraical sum of the coefficients is in evidence.

Denoting with Sylvester dy/dx by t and dx/dy by τ, also

$$\frac{1}{(s+2)!}\frac{d^{s+2}y}{dx^{s+2}} \text{ by } a_s, \qquad \frac{1}{(s+2)!}\frac{d^{s+2}x}{dy^{s+2}} \text{ by } \alpha_s,$$

we have the well known formulæ

$$\alpha_0 = -a_0 \div t^3,$$

$$\alpha_1 = (-a_1 t + 2a_0^2) \div t^5,$$

$$\alpha_2 = (-a_2 t^2 + 5a_0 a_1 t - 5a_0^3) \div t^7,$$

$$\alpha_3 = \{-a_3 t^3 + (6a_0 a_2 + 3a_1^2)\, t^2 - 28a_0^2 a_1 t + 14a_0^4\} \div t^9,$$

$$\alpha_4 = \{-a_4 t^4 + (7a_0 a_3 + 7a_1 a_2)\, t^3 - (28a_0^2 a_2 + 28a_0 a_1^2)\, t^2 + 84a_0^3 a_1 t - 42a_0^5\} \div t^{11},\dagger \quad \dots .$$

In these formulæ we may interchange the sets of symbols

$$t, a_0, a_1, \dots, \qquad \tau, \alpha_0, \alpha_1, \dots .$$

* *Mathematical Papers*, Vol. IV, p. 398.

† Observe that in the *Mathematical Papers*, Vol. IV, p. 311, the term $28a_0 a_1^2$ is erroneously printed $28a_1^2$.

A pure reciprocant is a homogeneous and isobaric function of

$$a_0,\ a_1,\ a_2,\ \ldots,$$

which is equal, the sign being disregarded, to the same function of

$$\alpha_0,\ \alpha_1,\ \alpha_2,\ \ldots,$$

multiplied by some power of t.

If we form from the above relations any such function of $\alpha_0, \alpha_1, \alpha_2, \ldots$, each term of the function, when expressed in terms of $t, a_0, a_1, a_2, \ldots$, gives rise to a term which is simply a power of a_0 multiplied by a power of t, and in the combination of terms which stands for a pure reciprocant these powers of a_0 must in combination vanish.

Ex. gr.,

$$\alpha_0^2\alpha_3-3\alpha_0\alpha_1\alpha_2+2\alpha_1^3=\ldots+\left(-\frac{a_0}{t^3}\right)^2\left(14\,\frac{a_0^4}{t^9}\right)$$
$$-3\left(-\frac{a_0}{t^3}\right)\left(2\,\frac{a_0^2}{t^5}\right)\left(-5\,\frac{a_0^3}{t^7}\right)+2\left(2\,\frac{a_0^2}{t^5}\right)^3,$$

and since the right-hand side is equal to $a_0^2a_3-3a_0a_1a_2+2a_1^3$ with a negative sign and a power of t, it is clear that the terms written on the right-hand side must vanish identically.

Hence, if $f(a_0, a_1, a_2, a_3, \ldots)$ be a pure reciprocant,

$$f(-a_0,\ 2a_0^2,\ -5a_0^3,\ 14a_0^4,\ 42a_0^5,\ \ldots)$$

must vanish identically.

The general a_0 term here is

$$(-)^s\,\frac{1}{s}\binom{2s}{s-1}\,a_0^s.$$

Hence, if we write $$a_s=(-)^{s+1}\,\frac{1}{s+1}\binom{2s+2}{s}\,b_s,$$

that is to say $$b_s=(-)^{s+1}\,\frac{(s+1)!}{(2s+2)!}\,\frac{d^{s+2}y}{dx^{s+2}},$$

and denote the pure reciprocant by

$$\phi(b_0,\ b_1,\ b_2,\ \ldots),$$

the form is such that the algebraical sum of the coefficients vanishes.

If a term of the reciprocant be as regards the literal portion

$$a_0^{\lambda_0}a_1^{\lambda_1}a_2^{\lambda_2}\ldots,$$

this, by the substitution, acquires the sign

$$(-)^{\lambda_0+\lambda_2+\lambda_4+\dots},$$

which is the same for *every* term because

$$(-)^{\lambda_0+\lambda_2+\lambda_4+\dots} = (-)^{(\lambda_0+\lambda_1+\lambda_2+\dots)-(\lambda_1+2\lambda_2+3\lambda_3+4\lambda_4+\dots)},$$

and the right-hand side is constant for every term of the reciprocant. Hence we may take the substitution to be

$$a_s = \frac{1}{s+1}\binom{2s+2}{s} b_s,$$

and then the algebraical sum of the coefficients vanishes. Otherwise we may assert that a pure reciprocant vanishes on putting

$$a_s = \frac{1}{s+1}\binom{2s+2}{s}.$$

The same result is deducible from the writer's theorem given in the *Transactions of the Cambridge Philosophical Society*, Vol. XXI, No. 6, pp. 143–170. The paper is entitled "The Operator Reciprocants of Sylvester's Theory of Reciprocants," and the theorem will be found on p. 166.

It states that the transformation

$$a_0 = \frac{1}{c_0},$$

$$a_1 = -\frac{2c_1}{c_0^3},$$

$$a_2 = -\frac{3c_2}{c_0^4} + \frac{8c_1^2}{c_0^5},$$

$$a_3 = -\frac{4c_3}{c_0^5} + \frac{30c_1c_2}{c_0^6} - \frac{40c_1^3}{c_0^7},$$

&c.,

converts all pure reciprocants in the elements

$$a_0,\ a_1,\ a_2,\ a_3,\ \dots,$$

into seminvariants in the elements

$$c_0,\ c_1,\ c_2,\ c_3,\ \dots .$$

The reader will observe that on the dexter of the above relations the sums of the coefficients are respectively

$$1,\ -2,\ +5,\ -14,\ +42,\ \ldots,\ (-)^s \frac{1}{s+1}\binom{2s+2}{s},$$

and that knowing that in seminvariants the sum of the coefficients vanishes, we at once proceed to the theorem of this paper.

The substitution of

$$\frac{1}{s+1}\binom{2s+2}{s} b_s \text{ for } a_s$$

in pure reciprocants leads to a remarkable simplification in the numerical values of the coefficients which must be of importance to any investigator who desires to study the mutual relations of the forms.

Below are given (1) Sylvester's forms, (2) the forms which arise by the modification which has been explained.

$$\begin{cases} a_0 \\ b_0 \end{cases}$$

$$\begin{cases} 4a_0a_2-5a_1^2 \\ b_0b_2-b_1^2 \end{cases}$$

$$\begin{cases} a_0^2a_3-3a_0a_1a_2+2a_1^3 \\ 7b_0^2b_3-15b_0b_1b_2+8b_1^3 \end{cases}$$

$$\begin{cases} 50a_0^2a_4-175a_0a_1a_3+28a_0a_2^2+105a_1^2a_2 \\ 3b_0^2b_4-7b_0b_1b_2+b_0b_2^2+3b_1^2b_2 \end{cases}$$

$$\begin{cases} 800a_0^2a_2a_4-875a_0^2a_3^2-1000a_0a_1^2a_4+2450a_0a_1a_2a_3-1344a_0a_2^3-35a_1^2a_2^2 \\ 48b_0^2b_2b_4-49b_0^2b_3^2-48b_0b_1^2b_4+98b_0b_1b_2b_3-48b_0b_2^3-b_1^2b_2^2 \end{cases}$$

$$\begin{cases} 125a_0^3a_3^2-750a_0^2a_1a_2a_3+256a_0^2a_2^3+165a_0a_1^2a_2^2+500a_1^3a_3-300a_1^4a_2 \\ 49b_0^3b_0^2-210b_0^2b_1b_2b_3+64b_0^2b_2^3+33b_0b_1^2b_2^2+112b_1^3b_3-48b_1^4b_2 \end{cases}$$

$$\begin{cases} \quad 625a_0^3a_4^2-4375a_0^2a_1a_3a_4-49700a_0^2a_2^2a_4+55125a_0^2a_2a_3^2+128625a_0a_1^2a_2a_4 \\ -61250a_0a_1^2a_3^2-156800a_0a_1a_2^2a_3+84868a_0a_2^4 \\ -78750a_1^4a_4 \quad +183750a_1^3a_2a_3 \quad -102165a_1^2a_2^3 \\ \quad 9b_0^3b_4^2-42b_0^2b_1b_3b_4-426b_0^2b_2^2b_4+441b_0^2b_2b_3^2+882b_0b_1^2b_2b_4 \\ -392b_0b_1^2b_3^2-896b_0b_1b_2^2b_3+433b_0b_2^4 \\ -432b_1^4b_4 \quad +840b_1^3b_2b_3 \quad -417b_1^2b_2^3 \end{cases}$$