

DISPLACEMENT FROM THE APPARENT VERTICAL IN FREE FALL.

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SYNOPSIS.

A simple method is given of obtaining the displacement by operating in fixed space. It is simple and permits one to visualize the southward component. An appreciable difference in the result is obtained according as the trajectory is treated as a parabola or an ellipse, the expression found in the latter case being only two thirds that found for the former.

IN looking over the various treatments of the equatorward deflection of a freely falling body, the method herein described has not been found. It brings into prominence certain features that are perhaps worthy of record. The case dealt with is that of a body falling towards a homogeneous spherical earth that revolves uniformly about a fixed axis. These assumptions are common to most of the methods found, very few taking account of the earth's equatorial protuberance.

Let the figure represent a sphere of radius R , fixed in space; and let the earth be represented by a second sphere, coincident with the fixed sphere at time $t = 0$, but rotating about the axis OC , with the small uniform angular velocity ω . AF represents a material rod of height h , fixed in the rotating sphere at latitude λ , and normal to the surface at the point A . FD is a plumb-line that takes the position shown due to the fact that it is in uniform circular motion about OC . Its direction gives the *apparent vertical* at the point A , as it is at rest with regard to the rotating sphere. The bob D is supposed to hang just above the spherical surface, and in its rotation to trace out the small circle DBK . Consider now a mass m released from the point F at time $t = 0$, and being at the moment of release at rest with regard to F . *In space* this body has at time $t = 0$ an initial velocity

$$(R + h) \omega \cos \lambda$$

along FH , which is normal to the plane of the circle DAC , and it will experience in space an acceleration \bar{g} directly towards O . Thus its trajectory will lie in the plane $OFHM$ which is fixed in space and which cuts the fixed sphere in the great circle ABL . It will appear later, however, that the body under these conditions always falls on the portion BL .

The diagram consists of two parts. The top part is a vector diagram showing a particle of mass m in a rotating frame. The forces acting on it are T (tension), mg (gravity), and $m\omega^2 r$ (centrifugal force). The angle between T and mg is θ , and the angle between $m\omega^2 r$ and the horizontal is λ . The bottom part shows a particle in a rotating frame with forces T , mg , and $m\omega^2 r$. The angle between T and mg is θ , and the angle between $m\omega^2 r$ and the horizontal is λ . The particle is moving in a circular path with radius r and angular velocity ω . The forces are shown acting on the particle at the end of the string.

(2) its *true weight* $m\bar{g}$ along DO where $m\bar{g}$ measures the force with which the bob is attracted towards the centre O . The relation of these vectors is shown in the upper part of the diagram. From this

or

$$\begin{aligned}\sin \theta &= \frac{r \omega^2 \sin \lambda}{g} \\ &= \frac{R \cos \lambda \omega^2 \sin \lambda}{g}.\end{aligned}$$

And, since θ is small, the $\text{arc } z$ (i.e., AD) = $h \sin \theta = \frac{hR\omega^2 \sin \lambda \cos \lambda}{g}$.

The *angle* z , subtended by the arc z at O , is given by

$$\frac{h\omega^2 \sin \lambda \cos \lambda}{g}$$

and this has a maximum value for $\lambda = 45^\circ$.

In the spherical triangle ABC $\cos a = \cos (b + z) = \cos c \cos b$ since the angle at $A = \pi/2$,

$$\begin{aligned}
 \cos c &= \frac{\cos(b+z)}{\cos b} \\
 &= \cos z - \sin z \cot \lambda, \text{ for } \cot \lambda = \tan b \\
 &= 1 - \frac{z^2}{2!} + \frac{z^4}{4!} \cdots - \left(z + \frac{z^3}{3!} \cdots \right) \cot \lambda.
 \end{aligned}$$

Now even if we take h as 40,000 cm., which is the greatest value for which I can find¹ any experiments, and if we consider these to have been made in the latitude of greatest southerly deviation (45°), z is found to be 1.07×10^{-7} radians, and the terms in z^2 would appear only in the fourteenth place of decimals. Thus to this degree of approximation

$$\cos c = 1 - z \cos \lambda$$

which on substituting the value of z gives

$$\cos c = \frac{1 - h\omega^2 \cos^2 \lambda}{g}$$

but from the cosine series

$$\cos c = 1 + \frac{c^2}{2!}$$

to terms in c^2 , therefore on comparison

$$c = \omega \sqrt{\frac{2h}{g}} \cos \lambda$$

and the arc c (*i.e.*, AB) = $R\omega \sqrt{\frac{2h}{g}} \cos \lambda$, showing that B is the point

occupied by the plumb-bob at the instant that the falling body strikes the spherical surface (the angle ABD being small, as it always is) for this is the angle that D moves through in the time of fall ($\sqrt{2h/g}$ when we neglect terms in ω^2).

Consider next the trajectory in the fixed plane $OFHM$. It would appear at first sight that on account of the very small rotation of the earth during the fall no appreciable difference would result from treating this as a parabola or as an ellipse. If we denote the point of impact on the line BL from a parabolic path as N , and that from an elliptical path as N' the two distances AN and AN' will differ only by a very small part of either line, but when we realise that it is the difference BN or BN' that we seek, and that AB is nearly as long as AN or as AN' it becomes evident that our result depends on the *differences* between quantities of the same order of magnitude, and that quantities otherwise insignificant become of importance.²

¹ Rundell, quoted by E. H. Hall, *PHYS. REV.*, Vol. XVII., p. 186, 1903.

² Even in the case of the extreme height quoted the distances are as follows: $AB = 295955$ cm., $AN = 295973$ cm., $AN' = 295967$ cm., differences of the order of 5/1000 of a per cent. but $AN - AB = 18$ cm., $AN' - AB = 12$ cm., and these differ by 66 per cent.

Taking the line of action of the acceleration as constant, we have a constant horizontal velocity of $(R + h)\omega \cos \lambda$ during the fall (\bar{t}) and the resulting value of AB will be $(R + h)\omega \bar{t} \cos \lambda$ and subtracting the value of $AB = \omega \bar{t} \cos \lambda$ we have

$$\begin{aligned} BN &= (R + h)\omega \bar{t} \cos \lambda - R\omega \bar{t} \cos \lambda \\ &= h\omega \bar{t} \cos \lambda \\ &= \frac{1}{2}\omega g \bar{t}^3 \cos \lambda \text{ since } h = \frac{1}{2}g\bar{t}^2. \end{aligned}$$

This is of course along the great circle, and we get the easterly and the southerly displacements by multiplying this by the cosine and the sine of the angle ABD .

$$\begin{aligned} ABD &= \sin^{-1} \frac{z}{AB} \\ &= \sin^{-1} \frac{hR\omega^2 \sin \lambda}{g(R + h)\omega \bar{t}} \text{ or very nearly,} \\ &= \sin^{-1} \frac{h\omega \sin \lambda}{g\bar{t}}. \end{aligned}$$

Turning now to the elliptical path, take axes of X and Y respectively as OF and OM . We have for the component acceleration parallel to X

$$\frac{d^2x}{dt^2} = -\frac{gR^2x}{(x^2 + y^2)^{3/2}},$$

also the initial velocity along FH

$$\left(\frac{dx}{dt}\right)_0 = (R + h)\omega \cos \lambda$$

Writing

$$x = x_0 + \left(\frac{dx}{dt}\right)_0 t + \left(\frac{d^2x}{dt^2}\right)_0 t^2 + \left(\frac{d^3x}{dt^3}\right)_0 t^3 \dots$$

we get

$$\bar{x} = (R + h)\omega \bar{t} \cos \lambda - \frac{g\omega \bar{t}^3 \cos \lambda}{6}$$

where \bar{x} and \bar{t} indicate the values of these two quantities at the instant of impact. Now treating this value ($BN' = \bar{x}$) as in the case of the parabolic path we find for BN'

$$\frac{1}{3}\omega g \bar{t}^3 \cos \lambda$$

which is the Gaussian expression usually obtained by a transformation to rotating axes. It differs from the value obtained on the parabolic path simply by the factor two thirds as explained above.