ON FUNCTIONS GENERATED BY LINEAR DIFFERENCE EQUATIONS OF THE FIRST ORDER

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1. The most simple solution of the linear difference equation of the first order whose coefficients are meromorphic functions is, in general, a one-valued function with sets of simple sequences of poles tending to infinity. When the coefficients are one-valued functions with essential singularities in the finite part of the plane the solution has, in general, sequences of such singularities.* It is proposed in this paper to show, in connection with the difference equation

$$P(x+1)-P(x) = \chi(x),$$

where $\chi(x)$ is a one-valued analytic function, that, *in general*, its solution cannot be a solution of any differential equation of finite order and dimensions unless either (1) the coefficients of the latter are obtained by differentiation from the solution itself, or (2) from these coefficients and the function $\chi(x)$ and its differentials we can, by the fundamental process of forming finite differences coupled with a finite number of elementary algebraical operations, derive the solution itself.

In these cases we shall say that some of the coefficients of the differential equation belong to a type which embraces the function which is the solution of the difference equation.

The cases of exception to the previous general theorem will be considered. It will also be shown that for the more general equation

$$P(x+1)-\psi(x)P(x) = \chi(x),$$

where $\psi(x)$ and $\chi(x)$ are one-valued analytic functions, a similar result holds good.

The theorem includes as a special case one proved by Hölder† for

* In connection with these statements reference may be made to Guichard, *Ann. de l'Ecole Normale Supérieure*, 5 Sér., T. IV.; Mellin, *Acta Mathematica*, T. XV., pp. 317–384; Hurwitz, *Acta Mathematica*, T. XX., pp. 285–312, and T. XXI., p. 243. I hope to develop the theory in a future paper.

† Hölder, *Mathematische Annalen*, Bd. XXVIII., pp. 1–13; see also Moore, *Mathematische Annalen*, Bd. XLVIII., pp. 49 *et seq.* Hölder's theorem affirms that the gamma function cannot be a solution of a linear differential equation with algebraic coefficients. He states (*loc. cit.*) that the proposition was communicated to him verbally by Weierstrass.

the gamma function and extended by the author* to G and double gamma functions. It is important as showing that the linear difference equation of the first order gives rise to new classes of transcendants which cannot be generated, as are so many functions, by differential equations.

2. If we have a differential equation of finite order and dimensions, we may write it

$$f\{x, y, y^{(1)}, \ldots, y^{(n)}\} = 0, \tag{A}$$

where f is integral in $y, y^{(1)}, \ldots, y^{(n)}$. If the equation be of order n and dimensions m, the terms of *class* s are defined to be terms of the type

$$y^{m_1}(y^{(1)})^{m_2} \ldots (y^{(n)})^{m_{n+1}},$$

where $$m_1+2m_2+\ldots+(n+1)m_{n+1} = s.$$

Terms of zero class will be independent of y and functions solely of the independent variable x. We assume that the differential equation has for its coefficients one-valued analytic functions of x. The variable x is assumed to be real or complex without restriction.

3. THEOREM.—If the solution of the previous differential equation (A) be also a solution of the linear difference equation

$$P(x+1)-P(x) = \chi(x), \tag{B}$$

where $\chi(x)$ is a uniform function of x, the equation (A) can be so reduced that terms of the highest class are of the form

$$f_s(x) \sum_k {}_s\phi_k(x)\, {}_sQ_k,$$

where the ϕ's are simply-periodic functions of x of period unity and ${}_sQ_k$ denotes symbolically some product

$$y^{m_1}(y^{(1)})^{m_2} \ldots (y^{(n)})^{m_{n+1}}$$

of class s.

Let the terms of highest class s in the original equation (A) be (r_s+1) in number. They can be written symbolically

$$R_0(x)\, {}_sQ_0+R_1(x)\, {}_sQ_1+\ldots+R_{r_s}(x)\, {}_sQ_{r_s}$$

where the R's are one-valued functions of x.

Divide the equation throughout by $R_0(x)$, and subtract this equation from the one formed by changing x into $(x+1)$. Then, since a solution $f(x)$ satisfies the difference equation (B), the original differential

* *Quarterly Journal of Mathematics*, Vol. XXXI., pp. 310–314; *Phil. Trans. Roy. Soc.*, (A), Vol. CXCVI., pp. 384–387.

equation must be reducible to

$$\sum_{k=1}^{r_s}\left[\frac{R_k(x+1)}{R_0(x+1)}\,{}_sQ_k\{f(x+1)\}-\frac{R_k(x)}{R_0(x)}\,{}_sQ_k\{f(x)\}\right]$$
$$+{}_sQ_0\{f(x+1)\}-{}_sQ_0\{f(x)\}+\text{ terms of lower class }=0,$$

provided this latter equation be not a mere identity.

Now

$${}_sQ_k\{f(x+1)\} = {}_sQ_k\{f(x)+\chi(x)\} = {}_sQ_k\{f(x)\}+\text{ terms of lower class.}$$

Hence $f(x)$ satisfies a differential equation of which terms of highest class are

$$\sum_{k=1}^{r_s}\left[\frac{R_k(x+1)}{R_0(x+1)}-\frac{R_k(x)}{R_0(x)}\right]{}_sQ_k\{f(x)\}.$$

We can now repeat the previous process and reduce the equation to one with fewer terms of class s, unless all the coefficients

$$\frac{R_k(x+1)}{R_0(x+1)}-\frac{R_k(x)}{R_0(x)} \qquad (k=1, 2, \ldots, r_s)$$

vanish identically.

In the latter case the ratios $\dfrac{R_k(x)}{R_0(x)}$ are simply-periodic functions of period unity.

In the former case we either arrive at another alternative of this nature, or obtain a differential equation with a single term of class s.

Finally, therefore, we reduce the differential equation to one in which the terms of the highest class s can be written in the form

$$f_s(x)\sum_k[{}_s\phi_k(x)\,{}_sQ_k(y)],$$

the ϕ's being one-valued simply-periodic functions of x of period unity, and $f_s(x)$ being a one-valued function of x.

4. Suppose now that the reduced differential equation is

$$f(x)\sum_{k=1}^{r_s}\{{}_s\phi_k(x)\,{}_sQ_k(y)\}+\sum_{k=1}^{r_s-1}[{}_{s-1}\psi_k(x)\,{}_{s-1}Q_k(y)]+\text{ terms of lower class }=0,$$

where the ψ's are one-valued functions of x.

Since $y=f(x)$ satisfies the difference equation (B), we have

$$\frac{d^n}{dx^n}[f(x+1)] = y^{(n)}+\chi^{(n)}(x).$$

Hence, when x is changed into $x+1$,

$${}_sQ_k(y) = y^{{}_sm_1}(y')^{{}_sm_2}\ldots(y^{(n)})^{{}_sm_{n+1}}$$

becomes

$$\prod_{r=0}^{n} [\{y^{(r)}+\chi^{(r)}(x)\}^{{}_sm_{r+1}}]$$

$$= {}_sQ_k(y)\left[1+\sum_{r=0}^{n} {}_sm_{r+1}\frac{\chi^{(r)}x}{y^{(r)}}+\sum_{r=0}^{n}\frac{{}_sm_{r+1}({}_sm_{r+1}-1)}{2!}\left\{\frac{\chi^{(r)}(x)}{y^{(r)}}\right\}^2\right.$$

$$\left.+\sum_{\substack{r_1 \\ (r_1 \neq r_2)}}\sum_{r_2} {}_sm_{r_1+1}\,{}_sm_{r_2+1}\frac{\chi^{(r_1)}(x)\,\chi^{(r_2)}(x)}{y^{(r_1)}\,y^{(r_2)}}+\dots\right].$$

Divide the reduced differential equation by $f_s(x)$, change x into $(x+1)$, and subtract the reduced equation divided by $f_s(x)$. We obtain

$$\sum_{k=1}^{r_{s-1}}\left[\frac{{}_{s-1}\psi_k(x+1)}{f_s(x+1)}\,{}_{s-1}Q_k(y)-\frac{{}_{s-1}\psi_k(x)}{f_s(x)}\,{}_{s-1}Q_k(y)\right]$$

$$+\sum_{k=1}^{r_s} {}_sm_1\,{}_sQ_k(y)\frac{\chi(x)}{y}\,{}_s\phi_k(x)+\text{ terms of class }(s-2)$$

$$+\text{terms of lower classes} = 0.$$

5. The terms of class $(s-1)$ in this equation will persist unless we have a series of equations of the type

$$\frac{{}_{s-1}\psi_k(x+1)}{f_s(x+1)}-\frac{{}_{s-1}\psi_k(x)}{f_s(x)} = -{}_sm_1\chi(x)\,{}_s\phi_k(x). \qquad (B_1)$$

Corresponding to some values of k the terms on the right-hand side will vanish. This cannot, however, happen for all the r_s values of k unless there is no factor $y^{{}_sm_1}$ in any of the terms of class s of the original reduced equation.

Hence, *either* some of the coefficients ${}_{s-1}\psi_k(x)$ are such that $\frac{{}_{s-1}\psi_k(x)}{f_s(x)}$ is a function which is a solution of a difference equation of the type

$$P(x+1)-P(x) = -\,{}_sm_1\chi(x)\,{}_s\phi_k(x);$$

or ${}_sm_1 = 0$ for each of the terms ${}_sQ_k(y)$, and the differential equation may be written

$$f_s(x)\left[\sum_{k=1}^{r_s} {}_s\phi_k(x)\,{}_sQ_k(y)+\sum_{k=1}^{r_{s-1}} {}_{s-1}\phi_k(x)\,{}_{s-1}Q_k(y)\right]+\text{ terms of lower class} = 0,$$

where the ϕ's are simply-periodic functions of x of period unity. In this differential equation the terms ${}_sQ_k(y)$ do not involve y apart from its differentials with regard to x.

Take the latter of the two alternatives thus presented, and let terms of class $(s-2)$ in the differential equation just written be

$$\sum_{k=1}^{r_{s-2}} \left[{}_{s-2}\psi_k(x)\, {}_{s-2}Q_k(y)\right].$$

If from this differential equation we now form the reduced equation, it will be of lower class than $(s-1)$ and the terms of class $(s-2)$ will be

$$\sum_{k=1}^{r_{s-2}} \left\{\frac{{}_{s-2}\psi_k(x+1)}{f_s(x+1)} - \frac{{}_{s-2}\psi_k(x)}{f_s(x)}\right\} {}_{s-2}Q_k(y) + \sum_{k=1}^{r_s} {}_s\phi_k(x)\, {}_sQ_k(y)\, {}_sm_2 \frac{\chi^{(1)}(x)}{y^{(1)}}$$

$$+ \sum_{k=1}^{r_{s-1}} {}_{s-1}\phi_k(x)\, {}_{s-1}Q_k(y)\, {}_{s-1}m_1 \frac{\chi(y)}{y}.$$

These terms will persist unless the functions $\frac{{}_{s-2}\psi_k(x)}{f_s(x)}$ (or some of them) satisfy a difference equation of the type

$$P(x+1) - P(x) = -{}_sm_2\, {}_s\phi_k(x)\, \chi^{(1)}(x) - {}_{s-1}m_1\, {}_{s-1}\phi_k(x)\, \chi(x).$$

Both the constants ${}_sm_2$ and ${}_{s-1}m_1$ may vanish when, and only when, none of the terms of class s in the original reduced differential equation involve y or y' apart from their differentials with regard to x, and none of the terms of class $(s-1)$ involve y.

6. Repeating the process, we see that ultimately :—(1) Either some of the coefficients of the differential equation, when written in the form

$$\sum_{k=1}^{r_s} {}_s\phi_k(x)\, {}_sQ_k(y) + \text{ terms of lower class } = 0,$$

must satisfy a difference equation of the type

$$P(x+1) - P(x) = \sum_{r=0} a_r \chi^{(r)}(x)\, \phi_r(x) \tag{C}$$

where the constants a do not all vanish; or (2) the differential equation must be of the form

$${}_s\phi(x)\,(y^{(n)})^{{}^sm_{n+1}} + \sum_k {}_{s-1}\phi_k(x)\, {}_{s-1}Q_k\left\{y^{(n-1)},\ y^{(n)}\right\}$$

$$+ \sum_k {}_{s-2}\phi_k(x)\, {}_{s-2}Q_k\left\{y^{(n-2)},\ y^{(n-1)},\ y^{(n)}\right\} + \dots$$

$$+ \sum_k \frac{{}_{s-n}\psi_k(x)}{f_s(x)}\, {}_{s-n}Q_k\left\{y,\ y^{(1)},\ \dots,\ y^{(n)}\right\}$$

$$+ \text{ terms of lower class than } (s-n) = 0,$$

where $s = (n+1)\,{}_sm_{n+1}$ and the ϕ's are one-valued simply-periodic functions of period unity; or (3) the differential equation can be reduced to one of lower class, in which terms of highest class persist.

If now we consider case (2) and reduce the equation last written, we get a differential equation of class $(s-n)$, of which the terms of highest class are

$$\sum_k \left\{ \frac{{}_{s-n}\psi_k(x+1)}{f_s(x+1)} - \frac{{}_{s-n}\psi_k(x)}{f_s(x)} \right\} {}_{s-n}Q_k(y) + {}_s\phi(x)\,{}_sm_{n+1}\,\chi^{(n)}(x)\,[y^{(n)}]^{m_{n+1}-1}$$

$$+\Sigma\,{}_{s-1}\phi(x)\,{}_{s-1}m_n\,\chi^{(n-1)}(x)\,[y^{(n)}]^{{}_{s-1}m_{n+1}}\,[y^{(n-1)}]^{{}_{s-1}m_n-1}+\ldots.$$

These terms of highest class exist unless some of the functions $\dfrac{{}_{s-n}\psi_k(x)}{f_s(x)}$ satisfy difference equations of the form

$$P(x+1)-P(x) = -{}_sm_{n+1}\,{}_s\phi(x)\,\chi^{(n)}(x)+\sum_{r=0}^{n-1} a_r\chi^{(r)}(x)\,\phi_r(x),$$

which is of the same type as equation (C).

In this last difference equation the terms on the right-hand side only vanish when $\chi(x)$ satisfies a differential equation of the form

$${}_sm_{n+1}\,\frac{d^ny}{dx^n}\,{}_s\phi(x) = \sum_{r=0}^{n-1} a_r\,\frac{d^ry}{dx^r}\,\phi_r(x).$$

Thus either (1) the original differential equation can be reduced to one of lower class in which terms of the highest class persist, or (2) it has coefficients which, when the terms of highest class are written in the form

$$\sum_{k=1}^{r_s} {}_s\phi_k(x)\,{}_sQ_k(y),$$

are solutions of a difference equation of the type

$$P(x+1)-P(x) = \sum_{r=0}^{n} a_r\phi_r(x)\,\chi^{(r)}(x), \tag{C}$$

in which all the coefficients a_r on the right-hand side are certainly not zero.

7. In the first case we can again reduce the equation to one of lower class, and so on indefinitely, unless the second alternative occurs again. Ultimately, we either are forced to the second alternative, or we get an equation whose class is unity, that is, an equation

$$h(x)\,y+k(x) = 0,$$

which is not a differential equation, and whose coefficients must be such that $-\dfrac{k(x)}{h(x)}$ is a solution of the difference equation (B).

Suppose now that the antepenultimate reduced equation to the one just written is

$$y'+p_1(x)y+p_2(x)y^2+p_3(x)=0, \tag{1}$$

which is of class 2, and the most general form of equation of this class. We reduce it to

$$y^2+\frac{p_1(x+1)-p_1(x)-2p_2(x+1)\chi}{p_2(x+1)-p_2(x)}y$$
$$+\frac{p_2(x+1)\chi^2+p_3(x+1)-p_3(x)+\chi'}{p_2(x+1)-p_2(x)}=0,$$

or (say)

$$y^2+q_1(x)y+q_2(x)=0.$$

And, reducing this, we get

$$y\{q_1(x+1)-q_1(x)-2\chi\}+q_1(x+1)\chi+\chi^2+q_2(x+1)-q_2(x)=0.$$

We see therefore that by taking the coefficients of the equation (1), forming similar functions when $(x+1)$ is substituted for x, and taking rational combinations of these quantities and χ and χ', we can form a solution of the difference equation (B). Hence the coefficients of (1) must, all or some of them, be functions from which, by the fundamental process of forming finite differences coupled with a finite number of elementary algebraical operations (addition, subtraction, multiplication, or division), solutions of the difference equation (B) can, with the aid of $\chi(x)$ and its derivates, be built up. Some of them must therefore be one-valued functions with infinite sequences of zeros or poles arranged in a manner which is in strict correlation to the distribution of zeros and poles which characterises the nature of the solution of the difference equation. The coefficients *may*, of course, be more complex functions than the functions which are solutions of the difference equation; for the process of forming finite differences may materially simplify the distribution of zeros and poles. This would be the case when the coefficients are derived from linear difference equations. And so, for instance, the gamma function might be a solution of a differential equation whose coefficients were built up with the aid of double gamma functions. But the gamma function can be derived, without the intervention of differential equations, by the ordinary process of forming finite differences from the double gamma function. Thus this possibility does not affect the main object of this paper, which is to show *not* that by means of differential equations we can go from functions which are more to functions which are less complex, but that we cannot proceed *vice versa*, the more complex functions being obtained from the less complex by the process of difference integration. If we proceed successively backwards, we see that all the previous remarks about the nature of the coefficients of the differential equation (1) must be true of the coefficients of the original differential equation.

8. Consider now the second alternative of § 6.

We have the difference equation

$$P(x+1)-P(x) = \sum_{r=0} a_r \phi_r(x)\, \chi^{(r)}(x), \tag{C}$$

in which any but not all of the a's may vanish.

[When the upper limit for r is n, a_n is not zero.]

Let the solution of $\quad P(x+1)-P(x) = \chi(x)$

be

$$G(x)+\phi_0(x),$$

$\phi_0(x)$ being a simply-periodic function of x of period unity. The solution of

$$P(x+1)-P(x) = \chi^{(r)}(x)$$

will be

$$G^{(r)}(x)+\phi_1(x).$$

Hence the solution of the difference equation (C) will be

$$\sum_{r=0} a_n \phi_r(x)\, G^{(r)}(x)+\phi(x).$$

Thus in this case the coefficients of the original differential equation must be functions which may be obtained from the solution of the original difference equation (B) by the process of differentiation.

9. We have, however, still to consider the particular case when the difference equation (C) reduces to

$$P(x+1)-P(x) = 0.$$

This will happen if $\chi(x)$ satisfies the linear differential equation

$$\sum_{r=0}^{n} a_r \phi_r(x) \frac{d^r y}{dx^r} = 0, \tag{C_1}$$

in which the functions ϕ are uniform functions simply-periodic of period unity.

Consider the case in which the ϕ's are meromorphic functions with a single essential singularity at infinity. This is equivalent to assuming that the coefficients of the original differential equation (A) are also meromorphic. In this case one of the fundamental solutions of (C_1) is* given by an aggregate of the type $\sum_{k=0}^{m-1} x^k \Phi_k(x)$, where $\Phi_k(x)$ denotes a simply-periodic function of the second kind which satisfies the difference equation

$$\Phi_k(x+1) = e^\theta \Phi_k(x),$$

where e^θ is the l-th root r_l-ply repeated of the fundamental equation of (C_1) and $(1 \leqslant m \leqslant r_l)$.

The number r_l which intervenes must of course be $\leqslant n$. When the roots of the fundamental equation of (C_1) are all different the numbers r_l are all unity.

We see then that it is possible that the solution of the difference equation

$$P(x+1) - P(x) = \sum_{k=0}^{m-1} x^k \Phi_k(x), \tag{D}$$

or a sum of solutions of such equations, may satisfy a differential equation of finite order and dimensions whose coefficients, supposed meromorphic, are not of a type which, in the language of § 1, embraces the solution of the difference equation.

10. Let us consider the solution of this difference equation.

Since $\Phi_k(x)$ satisfies $\quad \Phi_k(x+1) = e^\theta \Phi_k(x)$,

we have

$$\Phi_k(x) = e^{\theta x} \wp_k(x),$$

where $\wp_k(x)$ is a simply-periodic function of period unity.

Hence a solution of $\quad P(x+1) - P(x) = \Phi_k(x)$

is

$$\frac{\Phi_k(x)}{e^\theta - 1} + \wp(x).$$

The solution of (D) is composed of the sum of solutions of equations of the type

$$P(x+1) - P(x) = e^{\theta x} x^k \wp_k(x). \tag{E}$$

Put

$$P(x) = e^{\theta x} \wp_k(x)\, Q(x),$$

and this equation becomes

$$e^\theta Q(x+1) - Q(x) = x^k,$$

of which a solution is

$$Q(x) = -\frac{\iota \Gamma(1+k)}{2\pi} \int \frac{e^{-xz}}{1 - e^{\theta - z}} (-z)^{-k-1} dz,$$

* Forsyth, *Differential Equations*, Part III., Vol. IV. (1902), p. 415.

the integral being taken round the usual contour for the Riemann ζ function.* This solution may be readily verified by substitution.

Hence the solution of the equation (E) is

$$\wp_k(x)\, e^{\theta x} \left\{ \frac{-\iota}{2\pi} \int \frac{e^{-xz}\,(k)!}{1-e^{\theta-z}}\,(-z)^{-k-1}\,dz \right\} + \wp(x).$$

The integral is equal to the residue at the origin of $\dfrac{k!\, e^{-xz}}{\{1-e^{\theta-z}\}\{-z\}^{k+1}}$. It is thus an extended Bernoullian number, which we may denote by $S_k(x, \theta)$. It is evidently, in general, a polynomial in x of degree k. [It is of degree $k+1$ when $e^\theta = 1$ and we have real Bernoullian numbers.]

The solution of (E) is therefore

$$\wp_k(x)\, e^{\theta x} S_k(x, \theta) + \wp(x).$$

Hence the solution of (D) is

$$\sum_{k=0}^{m-1} [e^{\theta x} \wp_k(x)\, S_k(x, \theta)] + \wp(x),$$

which is the type of function generated by linear differential equations with simply-periodic coefficients.

11. Thus it is possible that a differential equation may admit as a solution a sum of terms of the type

$$e^{\theta x} \wp_k(x)\, S_k(x, \theta),$$

which sum is a solution of a linear difference equation of the type (D), when its coefficients are not of a type which embraces the solution.

The linear differential equation with constant coefficients is an example of this peculiarity. This equation admits terms of the type $e^{\theta x} x^k$ as solutions. Such terms satisfy a linear difference equation of the type (B), but it would be absurd to say that some of the constant coefficients of the differential equation belong, in the language of § 1, to a type which embraces the solution $\Sigma e^{\theta x} x^k$.

As another example, we may take the linear differential equation with meromorphic simply-periodic coefficients of period unity. This equation, when the roots of its fundamental equation are all different has simply-periodic functions of the second kind as its independent solutions. These functions are each a solution of a difference equation

$$P(x+1) - P(x) = \chi(x),$$

* See, for instance, a paper by the author, *Messenger of Mathematics*, Vol. XXIX., pp. 64–128.

where $\chi(x)$ is a simply-periodic function of the second kind. And the simply-periodic coefficients of the linear differential equation obviously cannot be said to belong to a type which embraces, in the sense we have defined in § 1, the simply-periodic functions of the second kind which are solutions.

12. We have, finally, the theorem :—

When the solution of the difference equation

$$P(x+1)-P(x)=\chi(x), \tag{B}$$

where $\chi(x)$ is a meromorphic function, is not the type of function that can be obtained as a solution of a linear differential equation with uniform simply-periodic coefficients of period unity, it cannot be obtained as the solution of any differential equation of finite order and dimensions with meromorphic coefficients, unless either (1) these coefficients are obtained by differentiation from the function itself, or (2) from these coefficients and $\chi(x)$ and its differentials we can by the fundamental process of forming finite differences coupled with a finite number of elementary algebraical operations, derive the solution itself.

In the more general case when $\chi(x)$ is not meromorphic, but has essential singularities in the finite part of the plane, it must satisfy a linear differential equation with simply-periodic coefficients of period unity if the solution of the difference equation is also to be a solution of a differential equation of finite order and dimensions with uniform coefficients whose coefficients do not some of them belong to a type which embraces the solution itself.

13. We may now extend the previous theorem to the case when the difference equation (B) is of the more general type

$$P(x+1)-\psi(x)\,P(x)=\chi(x). \tag{F}$$

Let $G(x)$ be a particular solution of the equation

$$P(x+1)-\psi(x)\,P(x)=0,$$

and let $H(x)+\phi(x)$, where $\phi(x)$ is simply periodic of period unity, be the general solution of

$$P(x+1)-P(x)=\frac{\chi(x)}{G(x+1)}.$$

Then the complete solution of the equation (F) is obviously

$$P(x) = G(x)\,\{H(x)+\phi(x)\}.$$

By the theorem just proved $\dfrac{P(x)}{G(x)}$ can, in general, only be a solution of a differential equation some of whose coefficients embrace $H(x)$. That is to say, either the coefficients of the differential equation for $\dfrac{P(x)}{G(x)}$ are obtained by differentiation from $\dfrac{P(x)}{G(x)}$, or from them and $\dfrac{\chi(x)}{G(x+1)}$ and its derivates, by the fundamental process of forming finite differences coupled with a finite number of elementary algebraical operations, we can deduce $\dfrac{P(x)}{G(x)}$.

Therefore the coefficients of the differential equation for $P(x)$ are either obtained by the finite combination of a finite number of differentials of $P(x)$ and $G(x)$, or from them and a finite number of successive differentials of $\chi(x)$, $G(x+1)$, and $G(x)$ we can, by the fundamental process of forming finite differences coupled with a finite number of elementary algebraical operations, derive $P(x)$.

Now, in general, $G(x)$, which is the solution of

$$f(x+1)-\psi(x)f(x) = 0,$$

is a much more simple type* of function than $P(x)$, which is the solution of

$$P(x+1)-\psi(x)P(x) = \chi(x). \tag{F}$$

Therefore we may say that, in this case, as in the previous one, the solution of the difference equation cannot, in general, be the solution of a differential equation of finite order and dimensions, unless some of the coefficients of this differential equation are of a type which embraces $P(x)$.

14. The fundamental case of exception occurs when $\dfrac{\chi(x)}{G(x+1)}$ can be expressed as the sum of one or more aggregates of the type $\sum\limits_{k=0}^{n-1} x^k e^{\theta x}\wp_k(x)$.

In this case $$\chi(x) = G(x+1)\,\Sigma \sum_{k=0}^{n-1} e^{\theta x}x^k\wp_k(x),$$

and the difference equation (F) is resoluble into a sum of others of the type

$$P(x+1)-\psi(x)P(x) = G(x+1)\sum_{k=0}^{m-1} e^{\theta x}x^k\wp_k(x),$$

* The further discussion of this point must be reserved for the investigation referred to in § 1.

whose solution is $G(x)\left\{\sum_{k=0}^{m-1}\left[e^{\theta x}S_k(x,\theta)\wp_k(x)\right]+\wp(x)\right\}$.

Now $\dfrac{G'(x)}{G(x)}$ is a solution of the difference equation

$$P(x+1)-P(x)=\frac{\psi'(x)}{\psi(x)}.$$

Hence the differential equation whose solution is $\dfrac{G'(x)}{G(x)}$ must have as coefficients functions belonging to a type which embraces $\dfrac{G'(x)}{G(x)}$, *i.e.*, which embraces $G(x)$, unless $\dfrac{\psi'(x)}{\psi(x)}$ is typified by $\sum\limits_{l=0} e^{\theta' x}x^l\wp_l(x)$. Therefore, unless $\dfrac{\psi'(x)}{\psi(x)}$ is of this character, the differential equation whose solution is of the form

$$\Sigma\, G(x)\left\{\sum_{k=0}^{m-1} e^{\theta x}S_k(x,\theta)\,\wp_k(x)+\wp(x)\right\}$$

must have as coefficients functions belonging to a type which embraces solutions of the difference equation (F) when $\chi(x)$ is zero.

When, however,
$$\frac{\psi'(x)}{\psi(x)}=\sum_{l=0} e^{\theta' x}x^l\wp_l(x),$$

we have
$$\frac{G'(x)}{G(x)}=\sum_{l=0} e^{\theta' x}S_l(x,\theta')\,\wp_l(x)\,;$$

and therefore
$$G(x)=\exp\left[\int^x \Sigma\, e^{\theta' x}\,S_l(x,\theta')\,\wp_l(x)\right].$$

Thus, for the complete case of exception to arise, the difference equation (F) must be resoluble into a sum of others of the type

$$P(x+1)-\exp\left[\int^x \sum_{l=0} e^{\theta' x}x^l\wp_l(x)\right]P(x)$$
$$=\left[\sum_{k=0}^{m-1} e^{\theta x}x^k\wp_k(x)+\wp(x)\right]\exp\left[\int^{x+1}\sum_{l=0} e^{\theta' x}S_l(x,\theta')\,\wp_l(x)\right].$$