A NEW GENERAL LAW OF DEFORMATION.*

BY

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THROUGHOUT the mechanics of elasticity and hydrodynamics frequent use is made of the laws:

I. DEFORMATION PROPORTIONAL TO FORCE APPLIED,

and

II. RATE OF DEFORMATION PROPORTIONAL TO FORCE.

The first of these applies to elastic bodies and is commonly known as Hooke's Law, the second is used in dealing with fluid flow and lacks a specific name though the particular form relating to flow through small tubes is known as Poiseuille's Law.

Hooke's Law is known to hold closely over a considerable range of forces for many solids. Failing to take account of either plastic or viscous yield or any other form of energy dissipation, it applies best to the more nearly perfectly elastic bodies. The viscosity law (II) applies fairly well to many fluids, both highly and but slightly viscous, but not at all well to others. It assumes complete dissipation of the work done in deformation with no elastic storage of energy. Both laws are empirical approximations and must remain so unless they can be deduced from known molecular relations. If one may judge from the literature of the subject, experimentalists have rested in the belief that theoretical work had established the foundation for these laws, while theoretical writers, on the evidence of the experimentalists, have contented themselves with the first term of a Taylor series expansion of an unknown function. Voigt frankly states (Mechanik, 1901, pp. 474 and 495) that the ultimate causes of both elasticity and viscosity are molecular and that the linear relations given rest on the pure assumption that the second order terms are negligible.

In mathematical form, the two laws mentioned, namely:

$$s = kF \tag{1}$$

and

$$s = ktF \tag{2}$$

$$v = ds/dt = kF$$

are at first sight incompatible as applied to the same material. If

* Communicated by the Author.

a steady stress produces a continuous deformation (2) in a homogeneous body, that body cannot possess elastic properties toward the same steady stress and conversely if a body is elastic (1), it is incapable of steady flow. On this basis only heterogeneous bodies are therefore capable of both elastic and viscous deformation under a given stress. The accepted electromagnetic theory of light is based on the assumption of displacements of this nature.

Michelson[1] has carefully investigated the yielding of metals and other solids under a torsional shearing stress. In formulating his results he writes for the displacement the sum of four components: (1) lost motion, (2) elastic, (3) elastico-viscous and (4) viscous displacement. The complete expression is extremely complex and involves nine constants. The term representing the "elastico-viscous" displacement (No. 3) is

$$s_3 = c_3 \left(1 - e^{-a\sqrt{t}} \right) F e^{bF} \tag{3}$$

This does not represent our data as well as the much simpler formula given below.

In order to obtain precise data upon which to formulate a law applicable to slowly yielding media, the writer took up the study of deformation in pitchlike substances subjected to pure coplanar shearing stresses. Many such substances are known to possess both elastic and viscous properties in a high degree and it was hoped that the data obtained might throw light on the fundamental principles upon which (I) and (II) are based. The results indicate that each is a special case of a simple general law, a law quite new in mechanics and of extremely wide application. It is empirical, but so are the laws of elasticity and of viscous flow just mentioned. Carrying the analogy over into electrical theory, another general law of electrical conduction is obtained of which Ohm's Law is a special case.

Considerable data by many observers is available for testing deformation formulas. That obtained by the writer is being presented at the 1921 meeting of the American Society for Testing Materials with complete descriptions of instruments and methods employed which need not be repeated here. Suffice it to state that the shear measured was produced between parallel plates 50 x 100 mm. in area, 2 mm. apart. Displacement readings were taken after ten

[1] A. A. Michelson, *Proc. Nat. Acad.*, May, 1917, and March, 1920.

or more time intervals after each force was applied, and for six to ten different forces and usually several different temperatures. One substance was found (a hard stearine pitch) for which the viscosity as ordinarily computed decreased to one-tenth as the force was increased from 1 to 10 kg. Another, although a hard solid (140° hydrolene), showed true viscosity, *i.e.*, the ratio of force per unit area to rate of shear was practically independent of the force applied.

The displacement-time curves were found to be generalized parabolas, s proportional to the n^{th} power of the time t, hence $\log s$ a linear function of $\log t$. Further, n is the same for all forces, the various s–t curves differing only by a constant factor. A similar relation was found to hold in all cases for the displacement-force curves (time constant), $\log s$ is a linear function $\log F$, s being proportional to the m^{th} power of the force F for any constant time without systematic variation.

Our general formula is then

$$s = at^n F^m \tag{4}$$

in which n and m are independent of s, t and F but are functions of the temperature. They are also, of course, independent of the area and thickness of the test piece, independent of the units employed, and also of the method of test so long as it involves only a pure shear. The constant a is independent of s, t and F but depends on the units and method employed.

It is readily seen that the old law of elastic deformation (1) is a special case of (4) for which $n = 0$ and $m = 1$. Also the law of viscous deformation (II) $s = atF$, or $v = aF$ is a special case for which $m = 1$ and $n = 1$. Experimental values of n have been found as low as 0.2 and as high as 0.91 and of m from 0.74 to 3.5. A value of m greater than unity means a plastic yield more than proportional to the force, often of the nature of an internal rupture, while a value less than unity signifies the converse, a consistency resembling quicksand in its nature. All highly fluid liquids of course have n nearly unity, in fact melting might be defined as a temperature at which n has a definite assigned value approaching unity. Similarly a definite low value of n approaching zero might be used to specify the beginning of the solid state.

Alternative forms of the general deformation law (4) may be written

$$\frac{ds}{s} = n\frac{dt}{t} + m\frac{dF}{F} \qquad (5)$$

$$s\left(\frac{l}{s}\frac{ds}{dt}\right)^n = (an)^n F^m \qquad (6)$$

From (5) it is evident that the error in theoretical work on deformation has been the neglect of large second order and time effects not really negligible. Experimental work has been in error through the assumption that deformation is proportional to the force producing it. From (4) and (5) it is evident that a more general form of statement of (I) and (II), including both is

III. PERCENTAGE DEFORMATION IS PROPORTIONAL TO PERCENTAGE CHANGE IN THE FORCE, OR

LOG DEFORMATION IS A LINEAR FUNCTION OF THE LOG OF THE FORCE PRODUCING IT.

However, time must be considered and must be the same for any two or more cases under comparison. Equation (6) is obtained by eliminating t from (4) and its time derivative. It indicates that the quantity $(an)^{-n}$ is a sort of generalized viscosity.

Since the general law of deformation must depend in some manner upon the strain and rupture of molecular bonds, the relaxation from a strained condition is of considerable theoretical interest. Relaxations amounting to half the original shear and continuing for ten minutes were observed in some pitches. The initial portion of the relaxation-time curve is almost identical with the initial part of the displacement-time curve preceding it. The relaxation function is of the same form as the deformation function (4), F being interpreted as the force just previously active. Our data on four widely different pitches indicate that for relaxation the constant m is the same, but a and n different, usually much less than for the preceding deformation. Thus for hard stearine pitch,

$$s = 0.28\, t^{0.56} F^{2.0}$$

$$R = 0.24\, t^{0.33} F^{2.0}$$

In relaxation, as in deformation, the time exponent is independent of the force that has been acting and the force exponent is independent of the time.

Another property of strained materials of great theoretical significance is the automatic release of internal strain with time if simply held in a fixed strained position. This property is associated with and complementary to the relaxation in form mentioned in the last paragraph. Deformed crystals tend to reform in an unstrained condition. Permanently deformed springs weaken gradually with time. We have seen springs of ferrous alloy receive a large permanent set on annealing at only 150° C. when in a strained condition. Amorphous bodies such as pitch, when deformed may be regarded as anisotropic but gradually return to an isotropic condition after a gradual molecular rearrangement requiring some minutes. In solids that rearrangement is much slower, in liquids much more rapid and in gases practically instantaneous. Since rate of deformation $ds/dt = ns/t$ is proportional to deformation s, a rupture theory is indicated in which the (molecular) rupture is self healing at a uniform rate.

The agreement of (4) with data by other observers obtained by other methods is of interest. Data by Green (A. S. T. M., 1919) relate to paints forced through capillary tubes by various static pressures of wide range. Taking his values for rate of discharge as a function of pressure for paints composed of linseed oil mixed with varying proportions of zinc oxide, the constant m in (4) was found to be 3.5 and the same not only for all pressures but for paints ranging from moderately thin mixtures to a thick paste. Data for computing n were not given, although a time effect attributed to inertia was noted. Data taken on an insulating oil with a MacMichael viscosimeter in this laboratory gave a value of $m = 0.77$, again without systematic variation. In this case F is the torque produced by a rotating cup of fluid on a disk suspended in it, and ds/dt is the angular velocity of the cup. Viscosity computed in the ordinary way varied by a factor greater than 2 as the velocity was varied from 1 to 30 R. P. M. This and other forms of absolute viscosimeter have been discredited because they indicated a variation of viscosity with velocity. It appears that the assumption, not the instrument, is in error. Time effects in the strain of elastic solids have been noted by many observers, but no precise data can be found for testing the formula.

The current through some leaky dielectrics attains a steady value only after several seconds or even many minutes from the time a steady E. M. F. is applied. This retardation is difficult to account for on the basis of any accepted theory of conduction; induction effects are not involved, and both displacement and Ohm's Law currents reach a steady value in a minute fraction of a second. Polarization, caused by an accumulation of ions, can be effective only near the electrodes while the effect of secondary ionization by collision is to increase rather than to decrease the current with time.

The mathematical theory of such conduction is similar in form to that applicable to deformable bodies, but the relation of this mechanical law to the electrical case under consideration is not merely a formal one. The electrical displacement consists of motions of electric charges impelled by potential gradients and if these charges are carried by material particles, the motions of the charges will be governed by the motions of the particles with which they are associated, the impelling force being electrical instead of mechanical. Corresponding with mechanical strain or displacement s in the above formulas, is q the electrical displacement; current density $i = dq/dt$, corresponds to ds/dt while potential gradient X corresponds to stress F in the mechanical equations. The tentative forms for electrical displacements are therefore

$$q = at^n X^m$$
$$i = ant^{n-1} X^m$$
$$di/dX = anmt^{n-1} X^{m-1}$$

Ohm's Law is a special form for which both m and $n=1$ corresponding to pure viscosity in fluid motion. The special case of $m=1$ and $n=0$ is that of purely elastic displacements in ideal dielectrics.

The value of the new form lies in its flexibility and generality. Note that both q and i are *additive* ($q = q_1 + q_2 +$), being in general sums of terms of the same form but with different constants. Consider the case of leakage through mica. The current even after 30 minutes (7.10^{-13}) was still twice as great as after an hour and $^1/_3$ the value after 5 minutes. An elastic displacement current ($i = dq/dt$) would have come to zero in a fraction of a second. Adding a conduction current $(E-P)/R$ such that

$$i = \frac{dq}{dt} + \frac{E-P}{R}$$

does not represent the facts any better since the current does not drop suddenly to a steady value but decreases very gradually for a long time. The new formula represents this decay curve for mica very well if n is small and negative but not zero ($n = -0.2$).

Mr. J. E. Shrader has studied the variation of current with voltage and with time. The only material for which both are reported is xylene. For this liquid the equations

$$q = 3.7 \times 10^{-12} t^{0.33} X^{1.10}$$
$$i = 1.23 \times 10^{-12} t^{-0.67} X^{1.10}$$

have been deduced. For illustration, other rough values may be mentioned: varnished cambric $m = 0.70$, fuller board $m = 0.73$, cement paper $m = 0.54$. For castor oil $n = 1.0$, high flash oil $n = 0.23$, bakelite test cup $n = 0.74$, white India mica $n = -0.2$.

Both theory and experimental data indicate that the best technical dielectrics are those for which $n = 0$ or the smallest possible with m nearly unity and a very small over the range of operating temperatures. The watt loss per cycle of course depends directly upon n since it is a time integral over a cycle and approaches zero as n approaches zero. The possibility of establishing technical criteria of quality appears worth following up.

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The Common Occurrence of Aurora in the South of England. LORD RAYLEIGH. (*Nature*, March 31, 1921.)—There is a certain green line of wave-length 5578 Angstrom units, which is found in the light of the aurora. Several observers have noted that this line is of common occurrence in the light of the sky at night. Two years ago Slipher identified it at the Lowell Observatory, Cal., on every clear night on which he sought for it. Lord Rayleigh has renewed his quest for this line in England and has found it on many nights. It is, however, not always obtainable. One "fairly clear" night it was absent, while it appeared on many cloudy nights. He attributes his success to the use of Marion's "iso-record" plates especially sensitive in the green region of the spectrum. The experimenter proposes a program of "systematic comparison of the auroral intensity with sun-spots and magnetic disturbances, and also a comparison of its different intensities in Great Britain and elsewhere."

G. F. S.

The output of solid mineral fuel in France for November, 1920, was 2374 thousands of tons as against a monthly average in the year before the war of 3404. From January to December of last year the number of mine workers rose from 188,000 to 224,000. The deficiency of coal was supplied in order of decreasing importance by Great Britain, Germany, the Sarre Basin, the United States, and Belgium. (*La Nature*, April 2, 1921.)

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