

## ON FOURIER'S COEFFICIENTS OF BOUNDED FUNCTIONS\*

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It is well known that,  $f(t)$  and its square  $f^2(t)$  being integrable functions, the series

$$(1) \quad 2\pi a_0^2 + \pi \sum_{n=1}^{\infty} (a_n^2 + b_n^2) = \int_0^{2\pi} f^2(t) dt$$

is convergent if  $a_n, b_n$  are the Fourier's coefficients of  $f(t)$ , i.e. if

$$a_0 = \frac{1}{2\pi} \int_0^{2\pi} f(t) dt, \quad a_n = \frac{1}{\pi} \int_0^{2\pi} f(t) \cos nt dt, \quad b_n = \frac{1}{\pi} \int_0^{2\pi} f(t) \sin nt dt$$

( $n = 1, 2, \dots$ ).

For bounded functions, that is to say for such functions  $f(t)$  that

$$|f(t)| \leq M \quad (0 \leq t \leq 2\pi),$$

$M$  being independent of  $t$ , the following question presents itself naturally :

Is it possible to diminish the exponent 2 without depriving the series (1) of its convergence, by putting  $2-\epsilon$  ( $\epsilon > 0$ ) instead of 2? If no positive  $\epsilon$  exists, which suffices to render

$$(2) \quad \sum_{n=1}^{\infty} (|a_n|^{2-\epsilon} + |b_n|^{2-\epsilon})$$

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\* [The manuscript of this paper was received by me, after considerable delay in the post, on December 8th, 1919. The author was unacquainted with two recent memoirs bearing on the same point, viz. :—

T. Carleman, "Über die Fourierkoeffizienten einer stetigen Funktion," *Acta Mathematica*, Vol. 41 (1918), pp. 377-384 ;

E. Landau, "Bemerkungen zu einer Arbeit von Herrn Carleman," *Math. Zeitschrift*, Vol. 5 (1919), pp. 147-153.

In these memoirs the problem put by Mr. Steinhaus is solved in a number of different manners, and it is shown that the same conclusion holds even for *continuous* functions. But the solution given by Mr. Steinhaus is different, and interesting in itself.—G. H. H.]

convergent for all bounded functions, is it at least always possible to find, for every *given* bounded function, an appropriate positive  $\epsilon$  rendering (2) convergent?

The answer is *negative*, even if we choose the second formulation of our problem. In fact, the following statement is true:

*There exists a bounded integrable function  $f(t)$ , independent of  $\epsilon$ , which renders the series (2) divergent for every positive  $\epsilon$ , however small.*

The object of this paper is to establish this proposition. For the sake of brevity, we write, for any integrable function  $f(t)$ ,

$$\frac{1}{\pi} \int_0^{2\pi} f(t) \cos nt dt = f_{2n-1}, \quad \frac{1}{\pi} \int_0^{2\pi} f(t) \sin nt dt = f_{2n} \quad (n = 1, 2, \dots),$$

and we call  $\{f_n\}$  shortly the "Fourier sequence of  $f$ ". In a former publication\* the following theorem was proved:

*The summability (C1) of*

$$(3) \quad \sum_{k=1}^{\infty} h_k x_k,$$

*for all Fourier's sequences  $\{x_k\}$  of bounded functions  $x(t)$ , is a necessary and sufficient condition that  $\{h_k\}$  should be the Fourier's sequence of an integrable function  $h(t)$ .*

We draw the immediate conclusion that, if a given sequence  $\{h'_k\}$  is *not* a Fourier's sequence, there exists another sequence  $\{x'_k\}$  which is the Fourier's sequence of a bounded function and which renders

$$(3') \quad \sum_{k=1}^{\infty} h'_k x'_k$$

not summable (C1), and, *a fortiori*, divergent.

Messrs. Hardy and Littlewood have shown in their important and beautiful papers on Diophantine Approximation,<sup>†</sup> that

$$(4) \quad \sum_{\nu=1}^{\infty} \frac{\cos \nu^2 t}{\nu^{\frac{1}{2}}}$$

is not a Fourier's series.

\* H. Steinhaus, "Additive und stetige Funktionaloperationen," *Mathematische Zeitschrift*, Vol. 5 (1919), pp. 186–221, Parallelsatz 3, Umkehrung des Parallelsatzes 3.

† G. H. Hardy and J. E. Littlewood, "Some Problems of Diophantine Approximation," *Acta Mathematica*, Vol. 38 (1914), Part 2, p. 237.

The sequence  $\{\bar{h}_k\}$  defined by

$$(5) \quad \begin{aligned} \bar{h}_{2n} &= 0 & (n = 1, 2, \dots), \\ \bar{h}_{2n-1} &= \frac{1}{\nu^n} & (n = 1, 2, \dots, \nu^2, \dots), \\ \bar{h}_{2n-1} &= 0 & (n \neq \nu^2), \end{aligned}$$

is therefore not a Fourier's sequence; nevertheless

$$(6) \quad \sum_{k=1}^{\infty} |\bar{h}_k|^{2+\delta}$$

is convergent for every positive  $\delta$ , as results from (5). The conclusion we had drawn from our theorem quoted above permits us to affirm the existence of a bounded integrable function  $\bar{x}(t)$ , whose Fourier's sequence  $\{\bar{x}_k\}$  renders

$$(7) \quad \sum_{k=1}^{\infty} \bar{h}_k \bar{x}_k$$

divergent. We can now assert the divergence of

$$(8) \quad \sum_{k=1}^{\infty} |\bar{x}_k|^{2-\epsilon}$$

for  $1 > \epsilon > 0$  (which implies the divergence of the same series for  $\epsilon \geq 1$ , and consequently for all  $\epsilon > 0$ ). In fact, the convergence of (8), *i.e.* of  $\sum_{k=1}^{\infty} |\bar{x}_k|^{1+(1-\epsilon)}$ , and the convergence of (6), which holds for every positive  $\delta$ , and therefore for

$$\delta = \frac{\epsilon}{1-\epsilon},$$

implies the absolute convergence of (7),\* contrary to what has been shown; and  $\bar{x}(t)$  is therefore the bounded integrable function, the existence of which we proposed to demonstrate.

\* If  $\sum_{k=1}^{\infty} |x_k|^{1+j}$  and  $\sum_{k=1}^{\infty} |h_k|^{1+1/j}$  are convergent, then  $\sum_{k=1}^{\infty} |h_k x_k|$  is convergent, for  $j > 0$ . Cf. F. Riesz, "Les systèmes d'équations linéaires à une infinité d'inconnues" (Paris, Gauthier-Villars, 1913), Chap. 3, § 33, pp. 43-45. In our case  $j = 1-\epsilon$ . Riesz puts  $1+j = \rho > 1$ .