On Plane Cubics. *By* A. C. Dixon.

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This note contains some further developments of the theory of corresponding points on a cubic, as given in Salmon's *Higher Plane Curves*, and of the closely connected theory of three conics.

1. The theory of corresponding points on a plane cubic may very well be introduced by a discussion of the problem to inscribe a complete quadrilateral in the curve, that is to find four straight lines whose six points of intersection lie on the curve.

Let us distinguish the points on the curve by their elliptic arguments, taking a point of inflexion as zero, so that three points u, v, w are collinear if $u+v+w\equiv 0$. The change from this notation to the language of the theory of residuation is quite easy.

Let u, v, w, u', v', w' be the six vertices of a complete quadrilateral, so that $u'vw$, $uv'w$, uvw', $u'v'w'$ are collinear triads of points. Then

$$u'+v+w\equiv 0,$$
$$u+v'+w\equiv 0,$$
$$u+v+w'\equiv 0,$$
$$u'+v'+w'\equiv 0.$$

Hence $$2u\equiv 2u',$$

and $$u'-u\equiv v'-v\equiv w'-w\equiv \text{a half-period, say } \omega.$$

Thus when one vertex is given the opposite vertex has three possible positions, there being three half-periods, and there are three series of inscribed complete quadrilaterals. Each series is doubly infinite, since two vertices, not opposite, may be arbitrarily chosen. In each series the choice of one vertex, u, determines the opposite vertex, u', uniquely; the relation between these points is reciprocal, and they may be called "corresponding points." Since $2u\equiv 2u'$, the tangents at corresponding points meet on the curve.

By supposing $v=u$, or otherwise, we find that $-u-u'$, the residual point of u, u', corresponds to $-2u$, the point of intersection of the tangents at u, u'.

Now let u_1, v_1, w_1, u_1', v_1', w_1' be the vertices of another quadrilateral of the same system. Then

$$u+v+w \equiv \omega, \quad u_1+v_1+w_1 \equiv \omega,$$

$$u+v+w+u_1+v_1+w_1 \equiv 0,$$

and the six points u, v, w, u_1, v_1, w_1 lie on a conic.

Hence the six lines vw, wu, uv, v_1w_1, w_1u_1, u_1v_1 touch a conic. So do the six lines vw, wu, uv, $v_1'w_1'$, $w_1'u_1$, u_1v_1' in the same way. These conics have five tangents in common, and are therefore the same; similarly $u'v'w'$ touches the same conic. Hence the sides of any two quadrilaterals of the same system touch one conic S.

Again, let any tangent to S cut the cubic in u_2', v_2', w_2', and let u_2, v_2, w_2 be the corresponding points. Then u_2, v_2, w_2, u_2', v_2', w_2' are the vertices of a complete quadrilateral of the system, and the eight lines $u'vw$, $uv'w$, uvw', $u'v'w'$, $u_2'v_2w_2$, $u_2v_2'w_2$, $u_2v_2w_2'$, $u_2'v_2'w_2'$ touch one conic, which must again be S. Thus we have a singly infinite series of quadrilaterals of the system, all circumscribed to S. The same argument would prove the existence of such a series circumscribed to any conic touching the four lines $u'vw$, $uv'w$, uvw', $u'v'w'$ or touching the four sides of any other quadrilateral of the system. Hence the conic S is one of an infinite system, and it is uniquely defined when it is made to touch two arbitrary straight lines, since each of these may be taken as the side of a quadrilateral of the system, so that thus eight tangents are given. The tangential equation to S is therefore of the form $\lambda U+\mu V+\nu W = 0$, U, V, W being definite expressions and λ, μ, ν arbitrary coefficients.

If two of the quadrilaterals circumscribed to S have a pair of opposite vertices in common, the conic S must reduce to this pair of points. The cubic is therefore the locus of the pairs of points included in the system of conics $\lambda U+\mu V+\nu W = 0$, and the two points of any pair are corresponding points on the cubic.

Conversely, if any three conics are given, the tangential equations being $U = 0$, $V = 0$, $W = 0$, the locus of the pairs of points included in the system $\lambda U+\mu V+\nu W = 0$ is a cubic. For, if any straight line α be taken, the conics of the system which touch it touch also three other lines β, γ, δ, and the only points in which α can meet the locus are its intersections with β, γ, δ. The four lines α, β, γ, δ form an inscribed complete quadrilateral, as in the above theory.

Reciprocally, if $U = 0$, $V = 0$, $W = 0$ are ordinary equations, the envelope of the pairs of straight lines included in the system $\lambda U+\mu V+\nu W = 0$ is a curve of the third class.

2. It is known also that the intersections of these pairs of lines lie on a cubic, the Jacobian of the three conics, and the points of this cubic may be taken as "conjugate" in pairs in such a way that the polars with respect to $U = 0$, $V = 0$, $W = 0$ of either of the conjugates meet in the other.

Let A', B', C' be any three collinear points of the Jacobian; A, B, C their respective conjugates; and take $U = 0$, $V = 0$, $W = 0$ to be the three pairs of lines which cut in A', B', C' respectively. Then A lies on the polar of A' with respect to V, that is on the line through B' harmonically conjugate to $B'A'$ with respect to the lines V. C lies on the polar of C' with respect to V. Hence A, C, B' are collinear; so in like manner are B, C, A' and A, B, C', and the six points thus form a complete quadrilateral inscribed in the Jacobian, so that A, A'; B, B'; C, C' are pairs of corresponding points belonging to one system.

The four lines $V = 0$, $W = 0$ form a quadrilateral whose third diagonal is $B'C'$. From the above harmonic relations it follows that the other two diagonals meet in A. These form a conic of the system which we may without loss of generality take to be $V - W = 0$, since it passes through the intersections of $V = 0$, $W = 0$. In like manner we may take $W - U = 0$ to be a pair of straight lines; these will meet in B, and, if k, k' are so chosen that

$$U + kV = 0, \quad V - W + k'(W - U) = 0$$

break up into pairs of lines, the lines of each pair will meet in C. Hence

$$k'(U + kV) + \{V - W + k'(W - U)\} = 0$$

or

$$V(1 + kk') - W(1 - k') = 0$$

will also break up into a pair of lines meeting in C. But this is impossible in general, unless C coincides with B', C' or A, which is not true, and therefore the last equation must be an identity. Thus $k' = 1$, $k = -1$ and $U - V = 0$ represents a pair of lines meeting in C.

The whole figure may be somewhat simplified by projecting $A'B'C'$ to infinity. Then $U = 0$, $V = 0$, $W = 0$ are pairs of parallel lines, say

$$Q_4 Q_1 R_1 R_4 \quad \text{and} \quad Q_3 Q_2 R_3 R_2,$$

$$R_4 R_2 P_2 P_4 \quad \text{and} \quad R_1 R_3 P_1 P_3,$$

$$P_4 P_3 Q_3 Q_4 \quad \text{and} \quad P_2 P_1 Q_2 Q_1.$$

The lines $V-W=0$, $W-U=0$, $U-V=0$ are the pairs of diagonals of the parallelograms $P_1P_2P_4P_3$, $Q_1Q_2Q_3Q_4$, $R_1R_3R_2R_4$, and the following sets of lines are concurrent:—

$$P_1P_4,\quad Q_2Q_4,\quad R_3R_4;$$
$$P_1P_4,\quad Q_1Q_3,\quad R_1R_2;$$
$$P_2P_3,\quad Q_2Q_4,\quad R_1R_2;$$
$$P_2P_3,\quad Q_1Q_3,\quad R_3R_4.$$

Now let the conic $\lambda U+\mu V+\nu W=0$ be "associated" with the point whose coordinates are (λ, μ, ν). Take ABC as triangle of reference, and let the system of coordinates be so chosen that the equation $x+y+z=0$ represents $A'B'C'$. Then the conics associated with A, B, C, A', B', C' are pairs of lines meeting in A', B', C', A, B, C respectively.

Since the lines U form a harmonic pencil with $x+y+z=0$ and $x=0$, we may put

$$U\equiv p\,(x+y+z)^2-qx^2;$$

similarly, we have

$$V\equiv p\,(x+y+z)^2-ry^2;$$
$$W\equiv p\,(x+y+z)^2-sz^2;$$

the coefficients p are all the same since $V-W$, $W-U$, $U-V$ break up into lines.

Then the conic associated with any point in the plane is the polar conic of that point with respect to the cubic

$$p\,(x+y+z)^3-qx^3-ry^3-sz^3=0,$$

which is the locus of a point that lies on its associated conic.

This cubic might have been taken as the foundation of the whole construction, and, if A, B' are made to approach and ultimately coincide, we have the figure discussed in Salmon's *Higher Plane Curves*, p. 153.

It may be noticed that the polar conic of $A'B'C'$ with respect to this last cubic must pass through A, B, C, since it is the locus of points whose polar conics touch $A'B'C'$. The polar lines of A', B', C' touch the Hessian at A, B, C, and, since they are by definition tangents to the polar conic of $A'B'C'$, it follows that this conic touches the Hessian at A, B, C.

The twelve lines $U=0$, $V=0$, $W=0$, $V-W=0$, $W-U=0$, $U-V=0$ all touch the Cayleyan. A rule is given (Salmon, Art. 181) for finding the points of contact; they may be found from our figure directly as follows. With the notation that was used above, the points Q_1, Q_2, Q_3, Q_4 are common to the polar conics of all points on AC, and R_1, R_2, R_3, R_4 to those of all points on AB. Let any line through any point X meet AC in Y and AB in Z. Then the polar conics of X, Y, Z have four common points, and therefore meet any line, say $Q_3Q_2R_3R_2$, in pairs of points in involution; that is, the polar conic of X meets $Q_3Q_2R_3R_2$ in points in involution with the pairs Q_2, Q_3 and R_2, R_3. Here X may be any point; let it be taken on the Hessian near to A. Then its polar conic consists of two lines, one consecutive to Q_2R_2, and the other to Q_1R_1. In the limit these meet Q_2R_2 in its point of contact with the Cayleyan and in A'. Hence the point of contact is conjugate to A' in the involution to which the pairs Q_2, Q_3 and R_2, R_3 belong. The foci of this involution are points on the Hessian, and so this agrees with the construction given in Salmon.

3. In the above work we have met with three doubly infinite series of conics related to the cubic, namely, the polar conics of lines, all of which have triple contact with the Hessian, the polar conics of points, and the conics which are inscribed in complete quadrilaterals inscribed in the curve. The general equation of a conic of the last series may be found as follows. Let the given cubic be the Hessian of

$$x^3+y^3+z^3+6mxyz = 0;$$

this is to be taken so that the relations between two corresponding points (x_1, y_1, z_1), (x_2, y_2, z_2) of our system are

$$x_1x_2+m(y_1z_2+y_2z_1) = 0,$$

$$y_1y_2+m(z_1x_2+z_2x_1) = 0,$$

$$z_1z_2+m(x_1y_2+x_2y_1) = 0.$$

The tangential equation to these two points is

$$(\xi x_1+\eta y_1+\zeta z_1)(\xi x_2+\eta y_2+\zeta z_2) = 0,$$

(ξ, η, ζ) being written for current tangential coordinates. This equation may be written

$$x_1x_2(m\xi^2-\eta\zeta)+y_1y_2(m\eta^2-\zeta\xi)+z_1z_2(m\zeta^2-\xi\eta) = 0$$

in virtue of the above conditions, and thus for any conic of the system the equation is

$$\lambda(m\xi^2-\eta\zeta)+\mu(m\eta^2-\zeta\xi)+\nu(m\zeta^2-\xi\eta) = 0,$$

since the system includes an infinity of such pairs of corresponding points. The left-hand side of this equation is the first emanant of the contravariant $m(\xi^3+\eta^3+\zeta^3)-3\xi\eta\zeta$, which may be expressed in Salmon's notation as $4SQ-3TP$, after multiplication by $(1+8m^3)^2$.

Elementary Proof of a Theorem for Functions of several Variables.

By H. F. Baker. Received and read February 13th, 1902.

We are in the habit of assuming that, if an ordinary power series in any number of variables does not vanish for zero values of the variables, the inverse of the series can be expanded in a converging series.

In the following note it is proved that the new series has at least the same range of convergence as the original, provided no zero of the original is contained in this range. A trivial particular case is the convergence of the expansion for $(1+x)^{-n}$ when $|x| < 1$, n being a positive integer.

Incidentally a *volume*, of ellipsoidal shape, is found about a point, at which a function of several variables does not vanish, within which no zero of the function exists.

It is pointed out in conclusion that the theorem that every equation has a root is a corollary from the general results.

1. Suppose that the ordinary power series

$$f(x, y) = \sum_{h=0,\, k=0} a_{h,k}\, x^h y^k$$

converges for $|x| < R$, $|y| < S$; let $r < R$, $s < S$, and suppose that for $|x| = r$, $|y| = s$ the derived series

$$f_1(X, Y) = \sum_{m=0} \sum_{n=0} \frac{f^{(m,n)}(x, y)}{m!\, n!}(X-x)^m (Y-y)^n$$

all converge uniformly for $|X-x| = D$, $|Y-y| = E$; let the values of all $|f_1(X, Y)|$ for ev[e]ry set of values

$$|x| = r, \quad |y| = s, \quad X = x+De^{i\theta}, \quad Y = y+Ee^{i\phi}$$