C_l, C_m, ... may be so many of a set of $t+1$ curves which determine q is that C_l, C_m, ... all pass through q, and that no values can be given to the constants λ, μ, ... such that the curve $\lambda C_l + \mu C_m + \ldots$ is not one of a set of $t+1$ curves which determine q.]

A Method for Extending the Accuracy of certain Mathematical Tables. By W. F. Sheppard, M.A., LL.M.

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I. *Introductory.*

1. As an example of the cases to which the following method applies, we may suppose that we have a table of values of

$$u \equiv \tan \tfrac{1}{2}\pi x$$

to seven places of decimals, by intervals of ·01 in x, from $x = {\cdot}00$ to $x = {\cdot}50$, and that we have also a table of values of u to eleven places of decimals, but at larger intervals—say at intervals of ·1. Then our object is to obtain a table which shall have the same intervals as the former, but shall have (approximately) the same accuracy as the latter. For convenience, the table in which the accuracy is less while the intervals are those prescribed may be called the *working table*, while the shorter table, giving the more accurate values of u for a few values of x, may be called the *checking table*.

The method consists in using the working table as the basis for the calculation of the first or second differences in the new table. This latter table is formed by the successive addition of the differences so found; and the values are checked from time to time by means of the more accurate table. The rate at which the accuracy of the table can be extended depends partly on the nature of the function tabulated and partly on the smallness of the successive increments of the argument. Thus, in the case given above, it will be found that the use of first differences, with a certain amount of "smoothing," will extend the table with tolerable accuracy to nine places, while a repetition of the process will extend it to eleven (or certainly to ten) places; or this latter result may be achieved in one operation by the use of second differences.

Before describing the formulæ employed, some preliminary observations are necessary.

2. There are certain well-known cases in which mathematical tables are (or might be) calculated by a formula of derivation, each value in the table being found from the preceding value or from a limited number of preceding values. Thus, for tabulating the function

$$f(x) \equiv e^x$$

by intervals of h in x, we should obviously use the formula

$$f(x+h) = e^h f(x),$$

each value being found from the preceding by multiplying by a constant factor. Similarly, to tabulate

$$f(x) \equiv \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$$

to a larger number of places than would be given by ordinary logarithmic tables, we should first write down the successive values of

$$e^{-h(x+\frac{1}{2}h)},$$

and then use the formula

$$f(x+h) = e^{-h(x+\frac{1}{2}h)} f(x).$$

Or, again, for constructing a table of

$$f(x) \equiv \sin \tfrac{1}{2}\pi x,$$

we might use the formula

$$f(x+h) = 2\cos \tfrac{1}{2}\pi h\, f(x) - f(x-h),$$

each value being thus found from the *two* preceding values. Tables formed in this way must be checked at intervals, by direct calculation of particular values, in order to prevent the accumulation of small errors.

3. In the above cases each value in the table is found from the immediately preceding value, or from a finite number of preceding values. This is not always possible. But a formula of derivation can always be obtained, and, *provided the interval h is small enough*, can always be used, in those cases *in which the differential coefficient of the function tabulated can be expressed in terms of the function itself*,

either alone or in conjunction with the argument. For suppose the function to be

$$u \equiv f(x), \tag{1}$$

and that it satisfies the equation

$$du/dx = \phi(x, u), \tag{2}$$

where $\phi(x, u)$ is a function whose value can be calculated for any given value of x if u is known for that value. Then, writing

$$\left.\begin{aligned} u_0 &= f(x_0) \\ u_{\pm n} &= f(x_0 \pm nh) \end{aligned}\right\},$$

where x_0 is any value of x appearing in the table, we have

$$\begin{aligned} u_1 &= f(x_0+h) \\ &= u_0 + hf'(x_0) + \frac{h^2}{2!}f''(x_0) + \frac{h^3}{3!}f'''(x_0) + \dots . \end{aligned} \tag{3}$$

Also, if we write

$$v \equiv \phi(x, u), \tag{4}$$

we have

$$\left.\begin{aligned} hv_0 &= hf'(x_0) \\ hv_{-1} &= hf'(x_0-h) \\ &= hf'(x_0) - h^2f''(x_0) + \frac{h^3}{2!}f'''(x_0) - \dots \\ hv_{-2} &= hf'(x_0) - 2h^2f''(x_0) + \frac{4h^3}{2!}f'''(x_0) - \dots \\ &\vdots \qquad \vdots \end{aligned}\right\}, \tag{5}$$

and thence

$$\left.\begin{aligned} hv_0 &= hf'(x_0) \\ \Delta\, hv_{-1} &= h^2f''(x_0) - \tfrac{1}{2}h^3f'''(x_0) + \tfrac{1}{6}h^4f^{\text{iv}}(x_0) - \dots \\ \Delta^2\, hv_{-2} &= h^3f'''(x_0) - h^4f^{\text{iv}}(x_0) + \dots \\ \Delta^3\, hv_{-3} &= h^4f^{\text{iv}}(x_0) - \dots \\ &\vdots \qquad \vdots \end{aligned}\right\}. \tag{6}$$

If now we eliminate between (3) and (6), we find

$$u_1 = u_0 + hv_0 + \tfrac{1}{2}\Delta\, hv_{-1} + \tfrac{5}{12}\Delta^2 hv_{-2} + \tfrac{3}{8}\Delta^3 hv_{-3} + \tfrac{251}{720}\Delta^4 hv_{-4} + \tfrac{95}{288}\Delta^5 hv_{-5} + \dots, \tag{7}$$

the coefficients in which may be shown to be the same as those in the expansion—

$$\frac{-\theta}{(1-\theta)\log(1-\theta)} = 1 + \tfrac{1}{2}\theta + \tfrac{5}{12}\theta^2 + \tfrac{3}{8}\theta^3 + \tfrac{251}{720}\theta^4 + \tfrac{95}{288}\theta^5 + \dots .$$

The values of hv_0, Δhv_{-1}, $\Delta^2 hv_{-2}$, ... are known if those of u_0, u_{-1}, u_{-2}, ... are known; and thus each value of u can be calculated from those which precede it.

To illustrate this formula, let us take the example given in § 1. We have then

$$u = \tan \tfrac{1}{2}\pi x,$$

$$v = du/dx = \tfrac{1}{2}\pi \sec^2 \tfrac{1}{2}\pi x$$

$$= \tfrac{1}{2}\pi(1+u^2). \qquad (a)$$

Suppose that the values of u have been found to seven places of decimals by intervals of

$$h = \cdot 01$$

up to $x = \cdot 35$. Then, if we calculated the values of hv, and took their differences, we should have a table of the following form :—

x	u	hv	Δhv	$\Delta^2 hv$	$\Delta^3 hv$	$\Delta^4 hv$
			+	+	+	+
·30	·5095254	·0197860		174		
·31	·5294727	·0201116	3256	182	8	
·32	·5497547	·0204554	3438	193	11	1
·33	·5703899	·0208185	3631	203	10	
·34	·5913984	·0212019	3834	214	11	
·35	·6128008	·0216067	4048			

the fourth difference being found from the average of three or four successive values.

To apply (7), we only require the quantities at the end of the table, which may be written thus, Δ' denoting $\Delta/(1+\Delta)$:—

x	u	hv	$\Delta' hv$	$\Delta'^2 hv$	$\Delta'^3 hv$	$\Delta'^4 hv$
·35	·6128008	·0216067	+4048	+214	+11	+1

The formula then gives, for $x = \cdot 36$,

$$u = 10^{-7}(6128008+216067+\tfrac{1}{2} \text{ of } 4048+\tfrac{5}{12} \text{ of } 214+\tfrac{3}{8} \text{ of } 11+\tfrac{251}{720} \text{ of } 1)$$

$$= \cdot 6346193,$$

whence, by (a), $\qquad hv = \cdot 0220342.$

Taking the differences, we get the next line :—

x	u	hv	$\Delta' hv$	$\Delta'^2 hv$	$\Delta'^3 hv$	$\Delta'^4 hv$
·36	·6346193	·0220342	+4275	+227	+13	+1

Continuing the process, I get the following table:—

x	u	hv	$\Delta' hv$	$\Delta'^2 hv$	$\Delta'^3 hv$	$\Delta'^4 hv$
·35	·6128008	·0216067	4048	214	11	1
·36	·6346193	·0220342	4275	227	13	1
·37	·6568772	·0224858	4516	241	14	1
·38	·6795994	·0229628	4770	254	13	1
·39	·7028118	·0234668	5040	270	16	1
·40	·7265425					

The values of u so found are all (practically) correct within 1 in the final figure.

4. The above method might often, I think, be found useful, provided the differences of hv diminish fairly rapidly. But, when this is not the case, there are two objections to be met. In the first place, a great many differences have to be taken into account; and this is troublesome, as the coefficients by which these differences have to be multiplied are not very convenient for calculation. In the second place, the coefficients do not diminish at all rapidly. The effect of this is that the necessary errors in u, due to the results being initially only accurate to seven places, become greatly magnified, and the values have to be checked at very short intervals.

This latter difficulty might be almost entirely removed, in the majority of cases, by taking hv to a larger number of decimal places than u. Thus, in the above example, if the initial values of u are correct to seven places, the initial values of u^2 (up to $x = {\cdot}50$) will be correct within 1×10^{-7}, and the values of hv will therefore be correct within $\frac{1}{2}\pi \times 10^{-9}$. By keeping in the two extra figures, the first differences may be found very accurately to seven places of decimals.

There will, however, still remain the difficulty as to the number of differences to be taken into account. And it may be added that there is a third objection, which will appeal strongly to any one who has had practical experience in constructing tables. The series of calculations by which a value of u is found is always the same, but each of these series of calculations has to be performed independently. A great saving of labour would be effected if the calculations could be taken in sets of similar processes, performed separately on u, hv, $\Delta' hv$, ... This is not possible when the table is being constructed for the first time. But when we possess a "working table" of u, of a less degree of accuracy than that which we are seeking, it becomes possible to simplify the work, by using the method explained in the following sections.

II. *Method as applied to First Differences.*

5. In all cases in which we are concerned with the successive values of a tabulated function, the method of central differences provides us with series which converge very rapidly, and therefore are suitable for numerical calculation. Thus—retaining for the moment the ordinary notation, but using the particular differences which enter into the improved formulæ—our formula of derivation (7) is replaced by

$$\begin{aligned} u_1 &= u_0 + hv_0 + \tfrac{1}{2}\Delta hv_0 - \tfrac{1}{12}\Delta^2 hv_{-1} - \tfrac{1}{24}\Delta^3 hv_{-1} + \tfrac{11}{720}\Delta^4 hv_{-2} \\ &\quad + \tfrac{11}{1440}\Delta^5 hv_{-2} - \dots \\ &= u_0 + \tfrac{1}{2}(hv_0 + hv_1) - \tfrac{1}{24}(\Delta^2 hv_{-1} + \Delta^2 hv_0) \\ &\quad + \tfrac{11}{1440}(\Delta^4 hv_{-2} + \Delta^4 hv_{-1}) - \dots . \end{aligned} \qquad (8)$$

This is obviously more convenient than (7). The apparent difficulty is that we do not know the values of Δhv_0, $\Delta^2 hv_{-1}$, ..., until the values of u_1 and u_2, and perhaps also those of u_3 and u_4, have been calculated. But the point to be noticed is that the unknown quantities all contain h as a factor, and therefore, if h is sufficiently small, they can be obtained with sufficient accuracy from a shorter table of values of u. Suppose, for instance, that the rate of change of u is less, or at any rate not much greater, than that of x, and that $h = \cdot 01$. Then, if we have a working table of u, correct to seven places of decimals, we can deduce a table of hv, practically correct to nine places of decimals; and thus, calculating the successive differences of u from (8), we can build up a new table of u to nine places of decimals. This, again, can be used as a working table for getting a new table to eleven places; and so on, indefinitely.

6. In the absence of any recognized notation for central-difference formulæ, I find it convenient to use* two operators δ and μ, defined by the following relations:—

$$\left.\begin{aligned} \delta f(x) &= f(x+\tfrac{1}{2}h) - f(x-\tfrac{1}{2}h) \\ \mu f(x) &= \tfrac{1}{2}\left\{ f(x+\tfrac{1}{2}h) + f(x-\tfrac{1}{2}h) \right\} \end{aligned}\right\}. \qquad (9)$$

* These operators are more fully discussed in a subsequent paper (*post*, pp. 449–488).

Thus, if $u = f(x)$, and if u_0 and u_1 denote any two successive values of u in a table proceeding by intervals of h in x,

$$\left.\begin{aligned} \delta u_{\frac{1}{2}} &= u_1 - u_0 \\ &= \Delta u_0 \\ \mu u_{\frac{1}{2}} &= \tfrac{1}{2}(u_1 + u_0) \end{aligned}\right\}. \tag{10}$$

Repeating the process denoted by δ, we have

$$\left.\begin{aligned} \delta^2 u_0 &= u_1 - 2u_0 + u_{-1} & \delta^3 u_{\frac{1}{2}} &= u_2 - 3u_1 + 3u_0 - u_{-1} \\ &= \Delta^2 u_{-1}, & &= \Delta^3 u_{-1} \\ \delta^4 u_0 &= \Delta^4 u_{-2}, & \delta^5 u_{\frac{1}{2}} &= \Delta^5 u_{-2} \\ \vdots\ & \quad \vdots & \vdots\ & \quad \vdots \end{aligned}\right\};$$

and, taking μ with powers of δ,

$$\left.\begin{aligned} \mu\delta u_0 &= \tfrac{1}{2}(\delta u_{\frac{1}{2}} + \delta u_{-\frac{1}{2}}) & \mu\delta^2 u_{\frac{1}{2}} &= \tfrac{1}{2}(\delta^2 u_1 + \delta^2 u_0) \\ &= \tfrac{1}{2}(\Delta u_0 + \Delta u_{-1}), & &= \tfrac{1}{2}(\Delta^2 u_0 + \Delta^2 u_{-1}) \\ \mu\delta^3 u_0 &= \tfrac{1}{2}(\delta^3 u_{\frac{1}{2}} + \delta^3 u_{-\frac{1}{2}}) & \mu\delta^4 u_{\frac{1}{2}} &= \tfrac{1}{2}(\delta^4 u_1 + \delta^4 u_0) \\ &= \tfrac{1}{2}(\Delta^3 u_{-1} + \Delta^3 u_{-2}), & &= \tfrac{1}{2}(\Delta^4 u_{-1} + \Delta^4 u_{-2}) \\ \mu\delta^5 u_0 &= \tfrac{1}{2}(\delta^5 u_{\frac{1}{2}} + \delta^5 u_{-\frac{1}{2}}) & \mu\delta^6 u_{\frac{1}{2}} &= \tfrac{1}{2}(\delta^6 u_1 + \delta^6 u_0) \\ &= \tfrac{1}{2}(\Delta^5 u_{-2} + \Delta^5 u_{-3}), & &= \tfrac{1}{2}(\Delta^6 u_{-2} + \Delta^6 u_{-3}) \\ \vdots\ & \quad \vdots & \vdots\ & \quad \vdots \end{aligned}\right\}.$$

In this notation our formula (8) becomes*

$$\begin{aligned} u_1 - u_0 &= \delta u_{\frac{1}{2}} \\ &= \delta \int^{x_{\frac{1}{2}}} v\,dx \\ &= \mu\left(1 - \tfrac{1}{12}\delta^2 + \tfrac{11}{720}\delta^4 - \tfrac{191}{60480}\delta^6 + \tfrac{2497}{3628800}\delta^8 - \dots\right) hv_{\frac{1}{2}}. \end{aligned} \tag{11}$$

The process therefore consists in using the values of u, given by the working table, as the basis for calculating the values of $hv \equiv h\,du/dx$, and then applying the formula (11) to determine the successive first differences of u in the new table. The values so found must be compared from time to time with the values given in the working table, in order to prevent an accumulation of errors.

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* See p. 480, formula (140).

7. In the preceding section we have supposed that the values of x in the final table are to be the same as the values in the initial or working table. But it will be found simpler, in practice, to take the values in the working table halfway between the values in the final table. Thus, to tabulate u for $x = \cdot 00, \cdot 01, \cdot 02, \ldots$, we should use a working table in which the values of x are $\ldots, \cdot 005, \cdot 015, \cdot 025, \ldots$. Using this table as the basis for calculating the values of $\ldots, hv_{\frac{1}{2}}, hv_{\frac{3}{2}}, \ldots$, we have*

$$\delta u_{\frac{1}{2}} = \left(1 + \tfrac{1}{24}\delta^2 - \tfrac{17}{5760}\delta^4 + \tfrac{367}{967680}\delta^6 - \tfrac{27859}{464486400}\delta^8 + \ldots\right) hv_{\frac{1}{2}}. \quad (12)$$

The coefficients in this formula are a good deal smaller than the coefficients in (11).

8. The principle underlying the method may be made clearer by a geometrical explanation. Let the successive values $\ldots, x_0, x_1, \ldots$ of the argument be represented by abscissæ $\ldots, OM_0, OM_1, \ldots$ measured along a line OX, so that $M_0M_1 = M_1M_2 = \ldots = h$; and at $\ldots, M_0, M_1, \ldots$, let ordinates $\ldots, M_0Q_0, M_1Q_1, \ldots$ be erected, equal to the values of u *given by the working table*. Let the true ordinates of the curve $u = f(x)$ be $\ldots, M_0P_0, M_1P_1, \ldots$; and suppose that each value in the working table is correct within $\pm\frac{1}{2}\rho$. Then, if on M_rQ_r we take

$$q'_rQ_r = Q_rq_r = \tfrac{1}{2}\rho,$$

all that the working table shows us is that P_r lies somewhere between q'_r and q_r; and similarly for P_{r+1}. Now let $\phi_{r+\frac{1}{2}}$ denote the inclination of P_rP_{r+1} to OX, so that

$$M_{r+1}P_{r+1} = M_rP_r + h \tan \phi_{r+\frac{1}{2}}.$$

Then, if we knew the exact position of P_0, and also the exact values of $\phi_{\frac{1}{2}}, \phi_{\frac{3}{2}}, \ldots$, we could (theoretically) determine the exact positions of $P_1, P_2, \ldots$. All that the working table tells us directly about $\tan \phi_{r+\frac{1}{2}}$ is that it lies somewhere between $(M_{r+1}Q_{r+1} - M_rQ_r - \rho)/h$ and $(M_{r+1}Q_{r+1} - M_rQ_r + \rho)/h$; *i.e.*, there is a possible error of $\pm\rho$ in $h \tan \phi_{r+\frac{1}{2}}$. But, if $\tan \phi_{r+\frac{1}{2}}$ can be expressed as a function of M_rP_r and $M_{r+1}P_{r+1}$, and the ordinates immediately preceding and following them, its value can be calculated with a certain degree of accuracy by using $\ldots, M_rQ_r, M_{r+1}Q_{r+1}, \ldots$ in the place of $\ldots, M_rP_r, M_{r+1}P_{r+1}, \ldots$. The limit of the error so introduced will usually be comparable with ρ; and therefore the limit of the error in $h \tan \phi_{r+\frac{1}{2}}$ will be comparable with $h\rho$. If h is so small that this limit is appreciably less than ρ,

* See p. 480, formula (139).

we can substitute the values of $h \tan \phi_{\frac{1}{2}}$, $h \tan \phi_{\frac{3}{2}}$, found in this way, for the values as shown directly by the table; and then, starting from a more accurate position of P_0, we shall arrive at more accurate positions of P_1, P_2, ...

We have supposed, in the above, that the values of x are to be the same in both tables; the formula (11) then gives $h \tan \phi_{r+\frac{1}{2}}$ in terms of ..., $h \tan \psi_r$, $h \tan \psi_{r+1}$, ..., where ψ_r denotes the inclination to OX of the tangent at P_r to the curve $u = f(x)$. But the explanation applies, with the necessary modifications, if the ordinates given in the working table are ..., $M_{-\frac{1}{2}} Q_{-\frac{1}{2}}$, $M_{\frac{1}{2}} Q_{\frac{1}{2}}$, $M_{\frac{3}{2}} Q_{\frac{3}{2}}$, ... ; the value of $h \tan \phi_{r+\frac{1}{2}}$, in terms of ..., $h \tan \psi_{r-\frac{1}{2}}$, $h \tan \psi_{r+\frac{1}{2}}$, $h \tan \psi_{r+\frac{3}{2}}$, ..., is then given by (12).

Let the ordinates whose more accurate values are given by the checking table be $M_0 P_0$, $M_n P_n$, $M_{2n} P_{2n}$, ... Then, starting from the given position of P_0, and proceeding by the successive steps, we may or may not hit the given position of P_n. If we do not, one or more of the values of $h \tan \phi$ must be altered. But, even then, there is always the possibility that our path may in the interval have steadily diverged, and then steadily come back again. It is therefore necessary that the limit of the error in $h \tan \phi$, as deduced from the ordinates given by the working table, should be *appreciably* less than ρ. Suppose, for instance, that this limit is $\frac{1}{10}\rho$, and that $n = 10$. Then, starting with the accurate value of $M_0 P_0$, the deduced value of $M_1 P_1$ is correct within $\frac{1}{10}\rho$. But the errors in $M_0 Q_0$, $M_1 Q_1$, ..., which give rise to the errors in $\tan \phi$, are independent, and therefore the errors in $\tan \phi_{\frac{1}{2}}$, $\tan \phi_{\frac{3}{2}}$, ... are practically independent. We can therefore only be sure that $M_2 P_2$ is correct within $\frac{1}{5}\rho$; and, similarly, we can only be sure that $M_5 P_5$ is correct within $\frac{1}{2}\rho$, *i.e.*, we cannot be sure that it is more correct than the original value in the working table.

In practice, however, these difficulties do not arise, on account of the tendency of independent errors to balance one another. Suppose, for instance, that by taking $h = \cdot 01$, and checking at intervals of $10h \equiv \cdot 1$, we can extend a seven-place table to nine places. Then, if we took $h = \cdot 001$ (a seven-place table at these intervals being supposed to exist), and only checked at intervals of $100h \equiv \cdot 1$, the possible error at each step would be divided by 10, but the number of steps would be multiplied by 10. The *possible* limit of error at the middle of the checking interval would therefore be unaltered. But the *probable* error would be about $1/\sqrt{10}$ of what it was before, so that, with some care in smoothing, the table would be tolerably correct to ten places. It might, at any rate, be used to ten places for the purpose of getting a new table to twelve or thirteen places.

9. Suppose that we are using (12), and that u_0 and u_n are two consecutive values in the checking table. Then, taking u_0 and u_n to the same number of places as hv, and calculating $\delta u_{\frac{1}{2}}$, $\delta u_{\frac{3}{2}}$, ... from (12), we obtain successively

$$\left.\begin{aligned} u_1 &= u_0 \quad + \delta u_{\frac{1}{2}} \\ u_2 &= u_1 \quad + \delta u_{\frac{3}{2}} \\ &\vdots \qquad\quad \vdots \\ u_n &= u_{n-1} + \delta u_{n-\frac{1}{2}} \end{aligned}\right\}.$$

If the sum of the calculated values of $\delta u_{\frac{1}{2}}$, $\delta u_{\frac{3}{2}}$, ..., $\delta u_{n-\frac{1}{2}}$ is not equal to $u_n - u_0$, one or more of them will require correction. But it is not always easy to decide whether the correction should be in one of the tabulated values of

$$hv,$$

or in one of the values of

$$\left(\tfrac{1}{24}\delta^2 - \tfrac{17}{5760}\delta^4 + \ldots\right) hv.$$

To avoid this difficulty, and the similar difficulty which arises in using (11), it is better to convert each formula into the corresponding formula for central summation.

In the notation which we shall adopt, the successive values of any function $f(x)$, for values of x proceeding by a constant difference h, are regarded as the first differences (δ) of another function, denoted by

$$\sigma f(x).$$

This function therefore satisfies the relation

$$\delta\, \sigma f(x) = f(x), \tag{13}$$

so that

$$\left.\begin{aligned} &\vdots \qquad\qquad\qquad\qquad \vdots \\ \sigma f(x - \tfrac{1}{2}h) - \sigma f(x - \tfrac{3}{2}h) &= f(x-h) \\ \sigma f(x + \tfrac{1}{2}h) - \sigma f(x - \tfrac{1}{2}h) &= f(x) \\ \sigma f(x + \tfrac{3}{2}h) - \sigma f(x + \tfrac{1}{2}h) &= f(x+h) \\ &\vdots \qquad\qquad\qquad\qquad \vdots \end{aligned}\right\}. \tag{14}$$

Replacing $f(x)$ by u, this gives

$$\left.\begin{aligned} &\vdots \qquad\qquad \vdots \\ \sigma u_{-\frac{1}{2}} &= \ldots + u_{-2} + u_{-1} \\ \sigma u_{\frac{1}{2}} &= \ldots + u_{-2} + u_{-1} + u_0 \\ \sigma u_{\frac{3}{2}} &= \ldots + u_{-2} + u_{-1} + u_0 + u_1 \\ &\vdots \qquad\qquad \vdots \end{aligned}\right\}. \tag{15}$$

We may adopt (15) as the definition of the operator σ, and we then have also

$$\left.\begin{aligned} &\vdots \qquad\qquad \vdots \\ \mu\sigma u_{-1} &= \ldots + u_{-2} + \tfrac{1}{2}u_{-1} \\ \mu\sigma u_0 &= \ldots + u_{-2} + u_{-1} + \tfrac{1}{2}u_0 \\ \mu\sigma u_1 &= \ldots + u_{-2} + u_{-1} + u_0 + \tfrac{1}{2}u_1 \\ &\vdots \qquad\qquad \vdots \end{aligned}\right\}. \tag{16}$$

The operation represented by σ involves the introduction of an arbitrary constant. If this constant is properly chosen, we have

$$\begin{aligned} \sigma\delta u_0 &= \ldots + \delta u_{-\frac{3}{2}} + \delta u_{-\frac{1}{2}} \\ &= u_0, \end{aligned}$$

and, similarly,

$$\begin{aligned} \sigma\delta^2 u_{\frac{1}{2}} &= \delta u_{\frac{1}{2}}, \\ \sigma\delta^3 u_0 &= \delta^2 u_0, \\ &\text{\&c.} \end{aligned}$$

Comparing these with (13), we see that σ combines with powers of δ according to the laws of algebra in the same way as if

$$\sigma = \delta^{-1}; \tag{17}$$

and it also combines according to these laws with powers of μ.

With this notation, the formulæ (11) and (12) become respectively, by successive additions,

$$u = \mu\left(\sigma - \tfrac{1}{12}\delta + \tfrac{11}{720}\delta^3 - \tfrac{191}{60480}\delta^5 + \tfrac{2497}{3628800}\delta^7 - \ldots\right) hv, \tag{18}$$

$$u = \left(\sigma + \tfrac{1}{24}\delta - \tfrac{17}{5760}\delta^3 + \tfrac{367}{967680}\delta^5 - \tfrac{27859}{464486400}\delta^7 + \ldots\right) hv. \tag{19}$$

To apply these latter formulæ, we write down the values of hv_0, hv_1, ..., or of $hv_{\frac{1}{2}}$, $hv_{\frac{3}{2}}$, ..., as the case may be, and take their differences. We then calculate the value of $\sigma hv_{\frac{1}{2}}$ or of σhv_0 from the value of u_0 given in the checking table, by writing (18) or (19) in the form

$$\sigma hv_{\frac{1}{2}} = u_0 + \tfrac{1}{2}hv_0 + \tfrac{1}{12}\mu\delta\, hv_0 - \tfrac{11}{720}\mu\delta^3\, hv_0 + \dots \tag{20}$$

or

$$\sigma hv_0 = u_0 - \tfrac{1}{24}\delta\, hv_0 + \tfrac{17}{5760}\delta^3\, hv_0 - \dots, \tag{21}$$

and perform a similar process for $\sigma hv_{n+\frac{1}{2}}$, $\sigma hv_{2n+\frac{1}{2}}$, ..., or for σhv_n, σhv_{2n}, ... If we use (18), we ought then to have

$$\left.\begin{aligned} \sigma hv_{n+\frac{1}{2}} - \sigma hv_{\frac{1}{2}} &= hv_1 + hv_2 + \dots + hv_n \\ \sigma hv_{2n+\frac{1}{2}} - \sigma hv_{n+\frac{1}{2}} &= hv_{n+1} + hv_{n+2} + \dots + hv_{2n} \\ \vdots \qquad &\qquad \vdots \end{aligned}\right\} \tag{22}$$

or

$$\left.\begin{aligned} \sigma hv_n - \sigma hv_0 &= hv_{\frac{1}{2}} + hv_{\frac{3}{2}} + \dots + hv_{n-\frac{1}{2}} \\ \sigma hv_{2n} - \sigma hv_n &= hv_{n+\frac{1}{2}} + hv_{n+\frac{3}{2}} + \dots + hv_{2n-\frac{1}{2}} \\ \vdots \qquad &\qquad \vdots \end{aligned}\right\}. \tag{23}$$

If these conditions are not satisfied, one or more of the values of hv must be altered, by inspection of the differences. The necessary alterations having been made, the intermediate values of σhv are found by successive addition of the values of hv, and (18) or (19) is then applied for calculating the values of u.

If any values of hv are altered, it will not usually be necessary to make the corresponding alterations in the first or higher differences, since the coefficients of δhv, $\delta^3 hv$, ... in the formula used are so small that the resulting terms are hardly affected by the alterations.

It should be noticed that the arbitrary constant in σhv has the same value in (18) as in (19). As soon, therefore, as the series of values of σhv has been obtained, we can apply either formula indifferently. Or we may, if we like, apply both formulæ, and thereby obtain a table in which the values of u proceed by intervals of $\frac{1}{2}h$ in x.

Some numerical examples, illustrating the method, will be found in the following sections.

10. For a simple example, let us take, as in § 3,

$$u = \tan \tfrac{1}{2}\pi x,$$

so that

$$v = du/dx = \tfrac{1}{2}\pi\,(1+u^2). \qquad (\alpha)$$

We may suppose that we have a seven-place table of u by intervals of ·01 in x, and a nine-place table by intervals of ·1, and that we require a nine-place table by intervals of ·01. For x = ·30 and x = ·40 our checking (nine-place) table gives

x	u
·30	·50952 5449
·40	·72654 2528

(β)

From the seven-place table, with a little smoothing of differences, I get a table of hv, of which the following is a portion:—

x	u	hv	δhv	$\delta^2 hv$	$\delta^3 hv$	$\delta^4 hv$
		+	+	+	+	+
			30 8176		838	
·30	·5095254	·01978 6005		1 7375		64
			32 5551		902	
·31	·5294727	·02011 1556		1 8277		74
			34 3828		976	
·32	·5497547	·02045 5384		1 9253		80
			36 3081		1056	
·33	·5703899	·02081 8465		2 0309		83
			38 3390		1139	
·34	·5913984	·02120 1855		2 1448		96
			40 4838		1235	
·35	·6128008	·02160 6693		2 2683		103
			42 7521		1338	
·36	·6346193	·02203 4214		2 4021		111
			45 1542		1449	
·37	·6568779	·02248 5756		2 5470		125
			47 7012		1574	
·38	·6795993	·02296 2768		2 7044		134
			50 4056		1708	
·39	·7028118	·02346 6824		2 8752		152
			53 2808		1860	
·40	·7265425	·02399 9632		3 0612		166
			56 3420		2026	

(γ)

From (β) and (γ), by means of (20), we get, for x = ·305,

$$\sigma hv = 10^{-9}\,(50952\ 5449 + \tfrac{1}{2} \text{ of } 1978\ 6005 + \tfrac{1}{24} \text{ of } 63\ 3727 - \tfrac{11}{1440} \text{ of } 1740)$$

$$= {\cdot}51944\ 484\dot{3};$$

and, for $x = \cdot 405$,

$$\sigma hv = \cdot 73858\ 7990.$$

The difference of these is ·21914 3147, which is equal to the sum of the calculated values of hv from $x = \cdot 31$ to $x = \cdot 40$, so that no correction is needed in these values. Finding the successive values of σhv by successive additions of hv, and applying (19), we get the result shown below. The second and fourth differences of hv are omitted, for convenience of printing. All the values of u, as shown in the last column, are correct within 1 in the final figure.

x	σhv	hv	δhv	$\delta^3 hv$	u
·295	·49965 8838		30 8176	838	·49967 1676
		1978 6005			
·305	·51944 4843		32 5551	902	·51945 8405
		2011 1556			
·315	·53955 6399		34 3828	976	·53957 0722
		2045 5384			
·325	·56001 1783		36 3081	1056	·56002 6908
		2081 8465			
·335	·58083 0248		38 3390	1139	·58084 6219
		2120 1855			
·345	·60203 2103		40 4838	1235	·60204 6968
		2160 6693			
·355	·62363 8796		42 7521	1338	·62365 6606
		2203 4214			
·365	·64567 3010		45 1542	1449	·64569 1820
		2248 5756			
·375	·66815 8766		47 7012	1574	·66817 7637
		2296 2768			
·385	·69112 1534		50 4056	1708	·69114 2531
		2346 6824			
·395	·71458 8358		53 2808	1860	·71461 0553
		2399 9632			
·405	·73858 7990		56 3420	2026	·73861 1460

If we wished the values of x in our final table to be ..., ·30, ·31, ·32, ..., we should have to use (18). To do this, we must calculate

$$\left(\sigma - \tfrac{1}{12}\delta + \tfrac{11}{720}\delta^3 - \ldots\right) hv \qquad (\epsilon)$$

for the intermediate values ..., ·295, ·305, ·315, ..., and then take the arithmetic mean of each pair of consecutive values. When, of two consecutive values given by (ϵ), one ends with an odd integer, and the other with an even integer, their arithmetic mean is doubtful. The doubt might be avoided by originally calculating $\frac{1}{2}hv$ instead of hv, and adding consecutive values of

$$\left(\sigma - \tfrac{1}{12}\delta + \tfrac{11}{720}\delta^3 - \ldots\right) \tfrac{1}{2}hv\,;$$

or, having calculated hv, we may keep in one or two extra figures in (ϵ), these figures being dropped after the arithmetic means have been found. The following shows the process, from $x = \cdot 30$ to $x = \cdot 35$:—

x	$(\sigma - \frac{1}{12}\delta + \frac{11}{720}\delta^3 - \ldots)\, hv$	u
·295	·49963 3169 47	....
·300	....	·50952 5449
·305	·51941 7727 61	....
·310	....	·52947 2745
·315	·53952 7761 49	....
·320	....	·54975 4652
·325	·55998 1542 38	....
·330	....	·57038 9929
·335	·58079 8316 15	....
·340	....	·59139 8351
·345	·60199 8385 45	....
·350	....	·61280 0788
·355	·62360 3189 69	....

These values, like the former, are correct within 1 in the final figure.

11. One class of functions to which the method is readily applicable comprises functions of the form

$$u = e^{\int P\,dx},$$

where P is some simple function of x. For we have, then,

$$du/dx = Pe^{\int P\,dx} = Pu,$$

so that hv is very easily calculated. If, for instance, we had constructed by means of logarithmic tables a table of values of

$$u = \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}x^2},$$

approximately correct to seven places of decimals, we could easily extend it, the formula being

$$hv = -\,hxu.$$

12. The method is also specially useful in dealing with the *inversion of integrals*. If x is given in terms of u by

$$x = \int^{u} \phi(u)\, du, \qquad (24)$$

and if the values of x in terms of u are tabulated, we can by ordinary methods of approximation construct a table of u in terms of x. The accuracy of this latter table will be limited by the accuracy of the former. But we have, from (24),

$$dx/du = \phi(u);$$

and therefore $$hv = hdu/dx = h/\phi(u). \tag{25}$$

Hence, if h is sufficiently small, we can substitute in (25) the values already found for u, and then apply (18) or (19) to get a more accurate table.

The example in § 10 may be regarded as coming under this head. For we have

$$\tan^{-1} u = \int^u du/(1+u^2),$$

and our original seven-figure table of $u \equiv \tan \frac{1}{2}\pi x$ may be supposed to have been obtained by inversion from a table of $x \equiv 2/\pi \tan^{-1} u$.

A case of special interest is that in which $\phi(u)$ is an exponential function of u. Suppose that

$$\phi(u) = e^{\int Q du}, \tag{26}$$

where Q is some simple function of u. Then we have, as in § 11,

$$\phi'(u) = Q\phi(u). \tag{27}$$

But, by (24), $$dx/du = \phi(u),$$

whence $$du/dx = 1/\phi(u). \tag{28}$$

Combining (27) and (28), we find that

$$d\{\phi(u)\}/dx = Q. \tag{29}$$

Now the working table gives u in terms of x, at intervals of h. Calculating the values of

$$hQ,$$

and applying (18) or (19) to (29), we get the values of $\phi(u)$. Then, calculating the values of $hdu/dx = h/\phi(u)$, and applying (18) or (19) again, we get back to a table of x, but with more accurate values.

Consider, for instance, the integral

$$\alpha \equiv 2\int_0^x z\,dx,$$

where $$z = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}.$$

Here we have
$$da/dx = 2z,$$
$$dz/dx = -xz,$$
so that
$$d(2z)/da = -x,$$
$$dx/da = 1/(2z).$$

I have used the above method for constructing tables of values of $2z$ and x in terms of a, by intervals of $\theta \equiv \cdot 01$ in a, from $a = \cdot 00$ to $a = \cdot 80$. The two tables given below show a portion of the work. The working table was obtained, to seven places of decimals, from Kramp's tables of $\log_{10} \int_t^\infty e^{-t^2} dt$; and, for the checking table,

a	x	$\sigma(-\theta x)$	$-\theta x$	δ	δ^2	δ^3	$2z$
			−	−	−	−	
		·6979952 47		1432 70		47	·6979892 78
·40	·5244005		52440 05		10 85		
		·6927512 42		1443 55		47	·6927452 28
·41	·5388360		53883 60		11 32		
		·6873628 82		1454 87		49	·6873568 20
·42	·5533847		55338 47		11 81		
		·6818290 35		1466 68		51	·6818229 24
·43	·5680515		56805 15		12 32		
		·6761485 20		1479 00		54	·6761423 57
·44	·5828415		58284 15		12 86		
		·6703201 05		1491 86		57	·6703138 89
·45	·5977601		59776 01		13 43		
		·6643425 04		1505 29		58	·6643362 31
·46	·6128130		61281 30		14 01		
		·6582143 74		1519 30		63	·6582080 43
·47	·6280060		62800 60		14 64		
		·6519343 14		1533 94		65	·6519279 22
·48	·6433454		64334 54		15 29		
		·6455008 60		1549 23		69	·6454944 04
·49	·6588377		65883 77		15 98		
		·6389124 83		1565 21		71	·6389059 61
·50	·6744898		67448 98		16 69		
		·6321675 85		1581 90		79	·6321609 94

a	$\sigma(\theta/2z)$	$\theta/2z$	δ	δ^3	δ^5	x
			+	+	+	
·40	·5243959 9438		1084 5374	1 9416	87	·5244005 1271
		144353 2138				
·41	·5388313 1576		1131 6306	2 0784	127	·5388360 3028
		145484 8444				
·42	·5533798 0020		1180 8022	2 2279	127	·5533847 1955
		146665 6466				
·43	·5680463 6486		1232 2019	2 3901	134	·5680514 9833
		147897 8485				
·44	·5828361 4971		1285 9917	2 5657	168	·5828415 0725
		149183 8402				
·45	·5977545 3373		1342 3472	2 7581	173	·5977601 2603
		150526 1874				
·46	·6128071 5247		1401 4608	2 9678	192	·6128129 9102
		151927 6482				
·47	·6279999 1729		1463 5422	3 1967	220	·6280060 1444
		153391 1904				
·48	·6433390 3633		1528 8203	3 4476	249	·6433454 0540
		154920 0107				
·49	·6588310 3740		1597 5460	3 7234	269	·6588376 9274
		156517 5567				
·50	·6744827 9307		1669 9951	4 0261	306	·6744897 5020

the values of x and z for $a = \cdot 1, \cdot 2, \ldots$ were determined very accurately. Calculating the initial and checking values of $\sigma(-\theta x)$ by means of (20), the application of (19) gives a table of $2z$ to nine places of decimals for $a = \cdot 005, \cdot 015, \ldots$. Thence the values of $\theta/(2z)$ are found (with a little smoothing) correct to eleven or ten places of decimals: and our final table is formed by a second application of (19). The first table on the preceding page shows the calculation of $2z$ for $a = \cdot 395, \cdot 405, \ldots, \cdot 505$, x being taken initially correct to seven places, and the first two and last two values of $\sigma(-\theta x)$ being adjusted so as to give $2z$ correct to nine places for $a = \cdot 40$ and $a = \cdot 50$. In the second table the values of $\theta/(2z)$ are inserted as found from the first table, and $\sigma\theta/(2z)$ is adjusted for $a = \cdot 40$ and $a = \cdot 50$. For convenience of printing, the even differences of $\theta/(2z)$ have been omitted.

13. The last class of cases which we shall consider under this head comprises those in which we are dealing with a definite integral, but, for convenience of interpolation, the quantity tabulated is not the integral itself, but is some function of the integral, or of the integral and the argument. If, for instance, we have

$$a = \sqrt{\frac{2}{\pi}} \int_0^x e^{-\frac{1}{2}x^2}\, dx,$$

it will be found that when x becomes tolerably great the higher differences of a become (relatively) very great. It is therefore more convenient to tabulate either

$$u_1 = e^{\frac{1}{2}x^2} \int_x^\infty e^{-\frac{1}{2}x^2}\, dx$$

or

$$u_2 = \log_{10} \sqrt{\frac{2}{\pi}} \int_x^\infty e^{-\frac{1}{2}x^2}\, dx.$$

The method can then be applied for extending the table either of u_1 or of u_2.

Generally, let
$$u = \int_x^\infty z\, dx,$$

where
$$z = e^{-\int P\, dx},$$

and let us write
$$u_1 = z^{-1} u,$$
$$u_2 = \log_{10} u.$$

Then it is easily shown that for extending a table of u_1 we have

$$h\,du_1/dx = -(h - hPu_1)\,;$$

while for extending a table of u_2 we have

$$h\,du_2/dx = -\log_{10} e \,.\, hz/u,$$

or

$$\log_{10}(-h\,du_2/dx) = \log_{10}(\log_{10} e \,.\, hz) - u_2.$$

The values of hP in the first case, and of $\log_{10} e \,.\, hz$ or $\log_{10}(\log_{10} e \,.\, hz)$ in the second, are calculated directly from those of x, and may be taken, at once, to any degree of accuracy we require. When they have been found, the process of extension may be repeated indefinitely, with very little trouble.

14. The method fails whenever u is changing so rapidly that, even though h is small, the working table does not give $h\,du/dx$ to a much greater degree of accuracy than that of the first differences of u as actually shown. As a general rule, we should not apply the method to cases in which the first difference of u contains about the same number of significant figures as u itself; but this rule is open to a good many exceptions. In particular, we should notice whether the greater part of the first difference depends on a function of x not involving u; if so, it may still be possible to extend the table.

The failure of the method, however, is only a failure in its practical utility; theoretically we could go on applying it by taking smaller and smaller values of h.

III. *Extension to use of Second Differences.*

15. The formula (19) may be written

$$u = (\sigma + \tfrac{1}{24}\delta - \tfrac{17}{5760}\delta^3 + \ldots)\,hD\,u,$$

where, as usual, D denotes differentiation. If we apply this to a table of u for $x = \ldots x_0 - h,\ x_0,\ x_0 + h,\ \ldots$, so as to get a table for $x = \ldots x_0 - \tfrac{1}{2}h,\ x_0 + \tfrac{1}{2}h,\ \ldots$, and then repeat the process, the result of the double operation may be denoted by

$$u = (\sigma + \tfrac{1}{24}\delta - \tfrac{17}{5760}\delta^3 + \ldots)\,hD\,(\sigma + \tfrac{1}{24}\delta - \tfrac{17}{5760}\delta^3 + \ldots)\,hD\,u.$$

The operators σ, δ, and hD combine according to the ordinary laws of algebra, so that the result is equivalent to

$$u = (\sigma + \tfrac{1}{24}\delta - \tfrac{17}{5760}\delta^3 + \ldots)^2 h^2 D^2 u,$$

or*
$$u = (\sigma^2 + \tfrac{1}{12} - \tfrac{1}{240}\delta^2 + \tfrac{31}{60480}\delta^4 - \tfrac{289}{3628800}\delta^6 + \ldots)\, h^2 D^2 u. \qquad (30)$$

This formula is quite general, and it enables us to extend the table of u, when u satisfies an equation of the form

$$d^2u/dx^2 = \psi(x, u), \qquad (31)$$

even if du/dx cannot be expressed as a simple function of u and x. The symbol σ^2 represents the operation of a double summation, introducing two arbitrary constants. Its relation to the operations represented by σ and by powers of δ is shown by the following table :—

x	$\sigma^2 f(x)$	1st diff.	2nd diff.	3rd diff.	4th diff.	...
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
$x_0 - h$	$\sigma^2 f(x_0 - h)$		$f(x_0 - h)$		$\delta^2 f(x_0 - h)$	
		$\sigma f(x_0 - \tfrac{1}{2}h)$		$\delta f(x_0 - \tfrac{1}{2}h)$		...
x_0	$\sigma^2 f(x_0)$		$f(x_0)$		$\delta^2 f(x_0)$	
		$\sigma f(x_0 + \tfrac{1}{2}h)$		$\delta f(x_0 + \tfrac{1}{2}h)$		...
$x_0 + h$	$\sigma^2 f(x_0 + h)$		$f(x_0 + h)$		$\delta^2 f(x_0 + h)$	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	

To apply the formula, we must first tabulate the values of

$$h^2 w \equiv h^2 d^2u/dx^2,$$

by means of (31), using the values of u given in the working table. We then, by successive additions, construct a table of values of

$$\sigma\, h^2 w,$$

and, repeating the process, we get the values of

$$\sigma^2\, h^2 w,$$

from which u is given by

$$u = (\sigma^2 + \tfrac{1}{12} - \tfrac{1}{240}\delta^2 + \tfrac{31}{60480}\delta^4 - \tfrac{289}{3628800}\delta^6 + \ldots)\, h^2 w. \qquad (32)$$

* See p. 483, formula (145).

The determination of the values $\sigma h^2 w_{\frac{1}{2}}$, $\sigma h^2 w_{n+\frac{1}{2}}$, $\sigma h^2 w_{2n+\frac{1}{2}}$, ..., by which the table of $\sigma h^2 w$ has to be checked, involves the knowledge of the accurate values of u_0, u_n, ..., and also of either u_1, u_{n+1}, ... or $(du/dx)_0$, $(du/dx)_n$, ... If the values known are u_1, u_{n+1}, ..., we must calculate $\sigma^2 h^2 w_0$, $\sigma^2 h^2 w_n$, ..., and also $\sigma^2 h^2 w_1$, $\sigma^2 h^2 w_{n+1}$, ..., by means of (32), written in the form

$$\sigma^2 h^2 w_0 = u_0 - \tfrac{1}{12} h^2 w_0 + \tfrac{1}{240} \delta^2 h^2 w_0 - \dots, \tag{33}$$

and then take

$$\left.\begin{aligned} \sigma h^2 w_{\frac{1}{2}} &= \sigma^2 h^2 w_1 - \sigma^2 h^2 w_0 \\ \sigma h^2 w_{n+\frac{1}{2}} &= \sigma^2 h^2 w_{n+1} - \sigma^2 h^2 w_n \\ \vdots \quad & \quad \vdots \end{aligned}\right\}. \tag{34}$$

But in most cases du/dx can be expressed in terms of u and x, so that $(du/dx)_0$, $(du/dx)_n$, ... can be calculated from the checking values of u_0, u_n, ... We have, for these cases,

$$\sigma h^2 w_{\frac{1}{2}} = h\,(du/dx)_0 + \tfrac{1}{2} h^2 w_0 + (\tfrac{1}{12}\mu\delta - \tfrac{11}{720}\mu\delta^3 + \dots)\, h^2 w_0, \tag{35}$$

which is obtained from (20) by writing $h\,du/dx$ for u.

16. The process of checking may be performed by first tabulating $\sigma h^2 w$, with any necessary alterations of $h^2 w$, and then making any further alterations which are necessary to make the values of $\sigma^2 h^2 w_n$, ... agree with those found from (33). But it is better to consider the checking values of $\sigma h^2 w$ and of $\sigma^2 h^2 w$ simultaneously. We should have, if the values of $h^2 w$ were exact,

$$\sigma h^2 w_{n+\frac{1}{2}} = \sigma h^2 w_{\frac{1}{2}} + h^2 w_1 + h^2 w_2 + h^2 w_3 + \dots + h^2 w_n, \tag{36}$$

$$\sigma^2 h^2 w_n = \sigma^2 h^2 w_0 + n\sigma h^2 w_{\frac{1}{2}} + (n-1)\, h^2 w_1 + (n-2)\, h^2 w_2 + \dots + h^2 w_{n-1}. \tag{37}$$

If these equations are satisfied by the tabulated values of $h^2 w$, no correction is necessary; but, if they are not satisfied, one or more values of $h^2 w$ must be altered accordingly, it being noted that an alteration of ± 1 in $h^2 w_r$ makes a difference of ± 1 in $\sigma h^2 w_{n+\frac{1}{2}}$, and of $\pm (n-r)$ in $\sigma^2 h^2 w_n$.

To illustrate this by a simple example, let us take

$$u = e^x,$$

which gives

$$w = d^2u/dx^2 = u.$$

Taking $h = \cdot 01$, and starting with $x = 1\cdot 00$, we have the following table to seven places of decimals:—

x	u	δ	δ^2	δ^3
		+	+	+
		270473		29
1·00	2·7182818		2719	
		273192		27
1·01	2·7456010		2746	
		275938		26
1·02	2·7731948		2772	
		278710		30
1·03	2·8010658		2802	
		281512		27
1·04	2·8292170		2829	
		284341		29
1·05	2·8576511		2858	
		287199		28
1·06	2·8863710		2886	
		290085		30
1·07	2·9153795		2916	
		293001		28
1·08	2·9446796		2944	
		295945		30
1·09	2·9742741		2974	
		298919		31
1·10	3·0041660		3005	
		301924		29

The values to eleven places for $x = 1\cdot 00$ and $x = 1\cdot 10$ are

x	u
1·00	2·7182818 2846
1·10	3·0041660 2395

Hence, by (35), since $h^2 w = h^2 u$,

$$\sigma h^2 w_{\frac{1}{2}} = 10^{-11}\left\{271828\ 1828 + 1359\ 1409 + \tfrac{1}{24} \text{ of } 54\ 3665 - \tfrac{11}{1440} \text{ of } 56\right\}$$

$$= \cdot 0273189\ \ 5889,$$

$$\sigma h^2 w_{n+\frac{1}{2}} = \cdot 0301921\ \ 1889.$$

The difference of these (omitting decimal point) is 28731 6000, and the sum of the corresponding values of $h^2 w$ comes to 28731 5999 ; one value must therefore be increased by 1. Again, by (33), we find

$$\sigma^2 h^2 w_0 = 2\cdot 7182591\ \ 7622,$$

$$\sigma^2 h^2 w_n = 3\cdot 0041409\ \ 8936,$$

the difference of the two being

$$2858818\ \ 1314.$$

If with the above value of $\sigma h^2 w_{\frac{1}{2}}$ we calculate

$$10\,\sigma h^2 w_{\frac{1}{2}} + 9 h^2 w_1 + 8 h^2 w_2 + \ldots + h^2 w_9,$$

we get 2858818 1303 ;

so that the sum must be increased by 11. If we take this $= 7+6-2$, so that two values of h^2w are increased by 1, and one is diminished by 1, we get the result shown below. The odd differences of h^2w are omitted, and the last column starts with the figure in the sixth decimal place of u. The altered values of h^2w are indicated by asterisks.

x	$\sigma^2 h^2w$	σh^2w	h^2w	$\delta^2 h^2w$	u
1·00	2·7182591 7622		2718 2818	2719	... 18 2846
		273189 5889			
1·01	2·7455781 3511		2745 6010	2746	... 10 1501
		275935 1899			
1·02	2·7731716 5410		2773 1948	2773	... 47 6394
		278708 3847			
1·03	2·8010424 9257		2801 0659*	2801	... 58 3467
		281509 4506			
1·04	2·8291934 3763		2829 2171*	2828	... 70 1432
		284338 6677			
1·05	2·8576273 0440		2857 6511	2859	... 11 1804
		287196 3188			
1·06	2·8863469 3628		2886 3710	2886	... 09 8925
		290082 6898			
1·07	2·9153552 0526		2915 3795	2915	... 94 9997
		292998 0693			
1·08	2·9446550 1219		2944 6795*	2946	... 95 5106
		295942 7488			
1·09	2·9742492 8707		2974 2741	2973	... 40 7256
		298917 0229			
1·10	3·0041409 8936		3004 1660	3005	... 60 2395
		301921 1889			

These values are all correct within 3 (or $3\frac{1}{2}$) in the last figure.

17. This method will be found useful for constructing a table of values of

$$u = \int^x v\, dx,$$

when we already possess a table of values of v at the required intervals, but not of sufficient accuracy to give us u to the number of places we desire. If we have

$$dv/dx = \phi(x, v),$$

we can apply (19) to determine a more accurate table of v; and a second application of (19) will give u. But, if we have no particular use for the more accurate table of v, we can omit the calculations represented by the formula

$$v = (\sigma + \tfrac{1}{24}\delta - \tfrac{17}{5760}\delta^3 + \dots)\, h\phi(x, v),$$

and proceed at once to the calculation of u by the formula

$$u = (\sigma^2 + \tfrac{1}{12} - \tfrac{1}{240}\delta^2 + \tfrac{31}{60480}\delta^4 - \dots)\, h^2\phi(x, v).$$

Suppose, for instance, that

$$v = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2},$$

and that we have tabulated v to seven places by intervals of ·01 in x. Then we have

$$dv/dx = -xv,$$

and, by using the formula

$$u = (\sigma^2 + \tfrac{1}{12} - \tfrac{1}{240}\delta^2 + \dots)(-h^2 xv),$$

we shall get u practically correct to eleven places, instead of merely to nine places.

18. If the values of x in the original table of u (or, in the case considered in the last section, in the original table of v) are halfway between the values to be adopted in the final table, we must use the formula*

$$u_{r+\frac{1}{2}} = \mu\,(\sigma^2 - \tfrac{1}{24} + \tfrac{17}{1920}\delta^2 - \tfrac{367}{193536}\delta^4 + \tfrac{27859}{66355200}\delta^6 - \dots)\,h^2 w_{r+\frac{1}{2}}. \quad (38)$$

This formula, written in the form

$$\sigma^2 h^2 w_0 = u_{\frac{1}{2}} - \tfrac{1}{2}\sigma\, h^2 w_{\frac{1}{2}} + (\tfrac{1}{24}\mu - \tfrac{17}{1920}\mu\delta^2 + \dots)\, h^2 w_{\frac{1}{2}}, \quad (39)$$

may also be used for calculating the checking values $\sigma^2 h^2 w_0$, $\sigma^2 h^2 w_n$, ..., when the values of u in the checking table are $u_{\frac{1}{2}}$, $u_{n+\frac{1}{2}}$, ..., instead of u_0, u_n, ...

IV. *Generalizations.*

19. Suppose that u satisfies a differential equation

$$d^2u/dx^2 - \phi\,(x, u)\, du/dx - \psi\,(x, u) = 0, \quad (40)$$

or, more generally,

$$d^2u/dx^2 = F\,(x, u, du/dx), \quad (41)$$

and that we have a table of u by intervals h in x. For any tabulated value, as x_0, we have†

$$h\,(du/dx)_0 = (\mu\delta - \tfrac{1}{6}\mu\delta^3 + \tfrac{1}{30}\mu\delta^5 - \tfrac{1}{140}\mu\delta^7 + \dots)\, u_0, \quad (42)$$

and similarly for $x = x_1, x_2, \dots$. If by substituting from (42) in (41) we get $h^2 d^2u/dx^2$ to a greater degree of accuracy than is given by the original table, we can apply (32) to obtain a more accurate table of u.

In a great many cases, however, the coefficient of du/dx in the

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* See p. 484, formula (152). † See p. 465, formulæ (74).

differential equation does not involve u. If, then, we have

$$d^2u/dx^2 - f(x)\, du/dx - \psi(u) = 0, \tag{43}$$

it is simpler to write

$$U = e^{-\frac{1}{2}\int^x f(x)\,dx}\, u, \tag{44}$$

and we have, for the differential equation of U,

$$d^2U/dx^2 = e^{-\frac{1}{2}\int^x f(x)\,dx}\, \psi\left(x,\ e^{\frac{1}{2}\int^x f(x)\,dx}\, U\right) - \left[\tfrac{1}{2}f'(x) - \tfrac{1}{4}\{f(x)\}^2\right] U. \tag{45}$$

By tabulating U instead of u, we are able to proceed at once to the application of the method of §§ 15 and 16.

20. For example, consider Bessel's function of order 1,

$$u = J_1(x),$$

which satisfies the equation

$$x^2\, d^2u/dx^2 + x\, du/dx + (x^2 - 1)\, u = 0.$$

Writing $$U = \sqrt{x}\,.\,u,$$

we have $$d^2U/dx^2 + (1 - 3/4x^2)\, U = 0,$$

or $$h^2 d^2U/dx^2 = -h^2 (1 - 3/4x^2)\, U,$$

so that we can apply (32) for values of x exceeding $\sqrt{3/8} = \cdot 612\ldots$.

Thus, taking $h = \cdot 1$, Lommel's table* of $J_1(x)$ gives the following values of U:—

x	u	U	x	u	U
·8	·368842	·329902	1·6	·569896	·720868
·9	·405950	·385118	1·7	·577765	·753314
1·0	·440051	·440051	1·8	·581517	·780187
1·1	·470902	·493886	1·9	·581157	·801070
1·2	·498289	·545848	2·0	·576725	·815612
1·3	·522023	·595198	2·1	·568292	·823534
1·4	·541948	·641243	2·2	·555963	·824627
1·5	·557937	·683331			

By direct calculation, I find

x	U	dU/dx
1·0	·44005 05857	·54517 23937
2·0	·81561 20449	·11272 63651

* E. Lommel, *Studien über die Bessel'sche Functionen* (Leipzig, 1868), p. 127.

Hence, using (35), we get

x	$\sigma^2 h^2 d^2 U/dx^2$	$\sigma h^2 d^2 U/dx^2$	$h^2 d^2 U/dx^2$	δ	δ^2	δ^3	δ^4	U
		+	−	−	+	+	±	
1·0	·44014242		110013		3741		+ 212	·44005059
		5390081		77745		210		
1·1	·49404323		187758		3951		128	·49388660
		5202323		73794		338		
1·2	·54606646		261552		4289		66	·54584832
		4940771		69505		404		
1·3	·59547417		331057		4693		30	·59519809
		4609714		64812		434		
1·4	·64157131		395869		5127		+ 2	·64124121
		4213845		59685		436		
1·5	·68370976		455554		5563		− 18	·68332990
		3758291		54122		418		
1·6	·72129267		509676		5981		29	·72086769
		3248615		48141		389		
1·7	·75377882		557817		6370		42	·75331371
		2690798		41771		347		
1·8	·78068680		599588		6717		53	·78018686
		2091210		35054		294		
1·9	·80159890		634642		7011		56	·80106974
		1456568		28043		238		
2·0	·81616458		662685		7249		− 58	·81561204

The approximation would of course be more rapid if we took x by intervals of ·01, and calculated some of the intermediate values. The process can be repeated, the multipliers of U only requiring to be calculated once.

21. More generally, suppose that u satisfies a differential equation

$$d^n u/dx^n = F(x, u, du/dx, d^2u/dx^2, \ldots, d^{n-1}u/dx^{n-1}). \tag{46}$$

Then, if u is tabulated by intervals of h in x, the values of $h\,du/dx$, $h^2 d^2u/dx^2$, ... are given by (42) and similar formulæ. Substituting these values, as found from the table of u, in the expression given by (46), we get a series of values of $h^n d^n u/dx^n$; and we have then*

$$u = (\sigma + \tfrac{1}{24}\delta - \tfrac{17}{5760}\delta^3 + \ldots)^n\, h^n\, d^n u/dx^n \tag{47}$$

or

$$u = \mu\,(1 + \tfrac{1}{4}\delta^2)^{-\frac{1}{2}}\,(\sigma + \tfrac{1}{24}\delta - \tfrac{17}{5760}\delta^3 + \ldots)^n\, h^n\, d^n u/dx^n, \tag{48}$$

the powers of σ and of δ in the expanded series being combined in accordance with the relation

$$\sigma = \delta^{-1}.$$

* For the coefficients in these expansions, up to $n = 8$, see pp. 484-5, formulæ (153)-(156).