SOME PROPERTIES OF SYMMETRIC AND ORTHOGONAL SUBSTITUTIONS

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1. Suppose we have a substitution A,

$$x_t = a_{t1}x_1 + a_{t2}x_2 + \ldots + a_{tm}x_m \quad (t = 1, 2, \ldots, m),$$

in the m variables $x_1, x_2, \ldots, x_m$.

It is possible* in an infinity of ways to transform A by a choice of new variables into the canonical substitution

$$x'_t = \lambda_t x_t + \beta_t x_{t+1} \quad (t = 1, 2, \ldots, m), \qquad (a)$$

where $\beta_t = 0$ or 1, and is certainly 0 if $\lambda_t \neq \lambda_{t+1}$.

The expression $$x_1^2 + x_2^2 + \ldots + x_m^2$$

will become a quadratic function $\Sigma c_{ij} x_i x_j$ in the new variables with non-zero determinant. We shall consider in what way the new variables must be chosen so as to transform A into the canonical substitution (a), and to make $\Sigma c_{ij} x_i x_j$ as simple as possible; in the cases in which A is symmetric or orthogonal.

The result is as follows:—

(1) Suppose A is *symmetric* and the canonical substitution N into which A is transformed is the direct product of substitutions of the form

$$x'_1 = ax_1 + x_2, \; \ldots, \; x'_{s-1} = ax_{s-1} + x_s, \; x'_s = ax_s,$$

then the new variables can be chosen so that A becomes N, and $(x_1^2 + x_2^2 + \ldots + x_m^2)$ becomes the sum of functions of the type

$$(x_1 x_s + x_2 x_{s-1} + x_3 x_{s-2} + \ldots + x_s x_1).$$

(2) Suppose A is orthogonal, the canonical substitution N into which A

* *Messenger of Math.*, 1909, p. 24.

is transformed will be the direct product of substitutions of the form

$$\left.\begin{array}{l} x_1' = ax_1 \; + x_2, \; \dots, \; x_{s-1}' = ax_{s-1} \; + x_s, \; x_s' = ax_s \\ y_1' = a^{-1}y_1 + y_2, \; \dots, \; y_{s-1}' = a^{-1}y_{s-1} + y_s, \; y_s' = a^{-1}y_s \end{array}\right\},$$

where $a^2 \neq 1$, or else $a^2 = 1$, and s is even, and of substitutions of the form

$$X_1' = aX_1 + X_2, \; \dots, \; X_{s-1}' = aX_{s-1} + X_s, \; X_s' = aX_s,$$

where $a^2 = 1$, and s is odd.*

Then we can choose the variables x, y, X, so that A becomes N, and $(x_1^2 + x_2^2 + \dots + x_m^2)$ becomes the sum of functions of the type $f_1(x, y)$ and $f_1(X, X)$ respectively, where $f_1(x, y)$ represents the function of x and y denoted by this symbol in *Proc. London Math. Soc.*, Ser. 2, Vol. 10, 1911, p. 426.

2. Suppose that in the substitution

$$\left.\begin{array}{l} x_1' = ax_1 + x_2, \; \dots, \; x_{s-1}' = ax_{s-1} + x_s, \; x_s' = ax_s \\ y_1' = \beta y_1 + y_2, \; \dots, \; y_{s-1}' = \beta y_{s-1} + y_s, \; y_s' = \beta y_s \end{array}\right\}, \qquad (\beta)$$

we replace

$$\left.\begin{array}{lll} x_1 & \text{by} & a_s x_1 + a_{s-1} x_2 + a_{s-2} x_3 + \dots + a_1 x_s \\ x_2 & \text{by} & a_s x_2 + a_{s-1} x_3 + \dots + a_2 x_s \\ x_3 & \text{by} & a_s x_3 + \dots + a_3 x_s \\ \dots & \dots & \dots \\ x_s & \text{by} & a_s x_s \\ y_1 & \text{by} & b_s y_1 + b_{s-1} y_2 + b_{s-2} y_3 + \dots + b_1 y_s \\ y_2 & \text{by} & b_s y_2 + b_{s-1} y_3 + \dots + b_2 y_s \\ y_3 & \text{by} & b_s y_3 + \dots + b_3 y_s \\ \dots & \dots & \dots \\ y_s & \text{by} & b_s y_s \end{array}\right\}. \qquad (\gamma)$$

This will not alter (β), for (γ) is a substitution permutable with (β).†

Suppose

$$\left.\begin{array}{l} \phi_1(x, y) \equiv x_1 y_s + x_2 y_{s-1} + x_3 y_{s-2} + \dots + x_{s-2} y_3 + x_{s-1} y_2 + x_s y_1 \\ \phi_2(x, y) \equiv x_2 y_s + x_3 y_{s-1} + \dots + x_{s-1} y_3 + x_s y_2 \\ \phi_3(x, y) \equiv x_3 y_s + \dots + x_s y_3 \\ \dots \quad \dots \quad \dots \\ \phi_s(x, y) \equiv x_s y_s \end{array}\right\}.$$

* *Proc. London Math. Soc.*, Ser. 2, Vol. 10, 1911, pp. 429, 432.

† *Messenger of Math.*, 1911, p. 111.

The substitution (γ) replaces

$$k_1\phi_1(x, y)+k_2\phi_2(x, y)+k_3\phi_3(x, y)+\ldots+k_s\phi_s(x, y)$$

by $$K_1\phi_1(x, y)+K_2\phi_2(x, y)+K_3\phi_3(x, y)+\ldots+K_s\phi_s(x, y),$$

where
$$\left.\begin{aligned} K_1 &= k_1\phi_s(a, b) \\ K_2 &= k_1\phi_{s-1}(a, b)+k_2\phi_s(a, b) \\ K_3 &= k_1\phi_{s-2}(a, b)+k_2\phi_{s-1}(a, b)+k_3\phi_s(a, b) \\ \ldots & \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\ K_s &= k_1\phi_1(a, b)+k_2\phi_2(a, b)+\ldots+k_s\phi_s(a, b) \end{aligned}\right\}.$$

The proof, which is easy, is left to the reader.

3. Again, suppose in (β) we take $a\beta = 1$; and let $f_1(x, y), f_2(x, y), \ldots, f_s(x, y)$ be the invariants of (β) given in *Proc. London Math. Soc.*, Ser. 2, Vol. 10, 1911, pp. 426, 427.

Then the substitution (γ) replaces $f_1(x, y)$ by

$$\{f_s(a, b)f_1(x, y)+f_{s-1}(a, b)f_2(x, y)+f_{s-2}(a, b)f_3(x, y)+\ldots+f_1(a, b)f_s(x, y)\},$$

when s is even, and by

$$\{f_s(a, b)f_1(x, y)+f_{s-1}(a, b)f_2(x, y)+f_{s-2}(a, b)f_3(x, y)+\ldots+f_1(a, b)f_s(x, y)\}$$
$$+\tfrac{1}{4}\{f_{s-1}(a, b)f_4(x, y)+f_{s-3}(a, b)f_6(x, y)+\ldots+f_4(a, b)f_{s-1}(x, y)\},$$

when s is odd.

In fact, since $f_1(x, y)$ is an invariant of (β), and (γ) is permutable with (β), therefore (γ) replaces $f_1(x, y)$ by an invariant of (β), *e.g.*, by

$$k_1f_1(x, y)+k_2f_2(x, y)+k_3f_3(x, y)+\ldots+k_{s-1}f_{s-1}(x, y)+k_sf_s(x, y).$$

The above theorem may be verified by direct substitution when $s = 1, 2, 3, 4, 5$. Assume it true for all values of s up to the one considered. If we put x_s and y_s zero in $f_t(x, y)$, we get the result of changing $x_1, x_2, x_3, \ldots, x_s$ into $0, x_1, x_2, \ldots, x_{s-1}$ and $y_1, y_2, y_3, \ldots, y_s$ into $0, y_1, y_2, \ldots, y_{s-1}$ in $-f_{t+2}(x, y)$. Thence by our assumption $k_1, k_2, \ldots, k_{s-2}$ have the values required by the given theorem. It is at once seen by direct substitution that $k_s = f_1(a, b)$. We leave to the reader the proof that $k_{s-1} = f_2(a, b)$, when s is even, and

$$k_{s-1} = f_2(a, b)+\tfrac{1}{4}f_4(a, b),$$

when s is odd.

4. We now prove the theorem (1) of § 1 when A is *symmetric*.

Suppose N is the canonical substitution into which A can be transformed [cf. (a) of § 1], so that $P^{-1}NP = A$. Let $PP' = C$, where P' is the substitution transposed to P and C is

$$x' = c_{t1}x_1+c_{t2}x_2+\ldots+c_{tm}x_m \quad (t = 1, 2, \ldots, m).$$

When A is transformed into N, $(x_1^2+x_2^2+\ldots+x_m^2)$ is transformed into $\Sigma c_{ij} x_i x_j$.

Now, since A is symmetric, we prove, as in *Messenger of Math.*, 1912, p. 152, that new variables can be chosen so that N is unaltered, and $\Sigma c_{ij} x_i x_j$ is the sum of quadratic forms with matrices of the type

$$\begin{vmatrix} 0 & 0 & \ldots & 0 & 1 \\ 0 & 0 & \ldots & 1 & k_2 \\ 0 & 0 & \ldots & k_2 & k_3 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 1 & \ldots & k_{s-2} & k_{s-1} \\ 1 & k_2 & \ldots & k_{s-1} & k_s \end{vmatrix},$$

each corresponding to one of the substitutions of the type

$$x_1' = ax_1+x_2, \ \ldots, \ x_{s-1}' = ax_{s-1}+x_s, \ x_s' = ax_s,$$

of which N is the direct product.

Apply (γ) to this, putting in (γ) $x_t = y_t$ and $a_t = b_t$; this does not alter N. Then, by § 2, we get the sum of quadratic forms with matrices of the type

$$\begin{vmatrix} 0 & 0 & \ldots & 0 & K_1 \\ 0 & 0 & \ldots & K_1 & K_2 \\ 0 & 0 & \ldots & K_2 & K_3 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ K_1 & K_2 & \ldots & K_{s-1} & K_s \end{vmatrix},$$

where
$$K_1 = a_s^2, \quad K_2 = 2a_s a_{s-1}+a_s^2 k_2,$$
$$K_3 = (2a_{s-2}a_s+a_{s-1}^2)+2a_s a_{s-1} k_2+a_s^2 k_3, \ \ldots .$$

Now choose in turn $a_s, a_{s-1}, a_{s-2}, \ldots,$ so that

$$K_1 = 1, \quad K_2 = K_3 = \ldots = 0;$$

and the required transformation of A into N and $x_1^2+x_2^2+\ldots+x_m^2$ into the standard form of § 1 (1) is completed.

It follows from this theorem (1) of § 1 that any two symmetric substitutions with the same invariant-factors (and therefore with the same canonical substitution) can be transformed into one another by an orthogonal substitution.

For suppose $PAP^{-1} = N$, where $PP' = C$; and suppose that when we put

$$b_{t1}x_1 + b_{t2}x_2 + \ldots + b_{tm}x_m \quad \text{for} \quad x_t,$$

N remains unaltered while $\Sigma c_{ij}x_ix_j$ takes the standard form of § 1 (1), which we call $\Sigma l_{ij}x_ix_j$. Then, if L, B are

$$x'_t = l_{t1}x_1 + l_{t2}x_2 + \ldots + l_{tm}x_m,$$

and

$$x'_t = b_{t1}x_1 + b_{t2}x_2 + \ldots + b_{tm}x_m,$$

respectively, we have $BCB' = L$, and $BN = NB$.

Then, if $Q = BP$, $QAQ^{-1} = N$, and $QQ' = L$, *i.e.*, we can transform A into N, and $x_1^2 + x_2^2 + \ldots + x_m^2$ into $\Sigma l_{ij}x_ix_j$ simultaneously.

Take D so that $DD' = L$ (*Messenger of Math.*, 1912, p. 147).

Then we have $$(D^{-1}Q)\,A\,(D^{-1}Q)^{-1} = D^{-1}ND,$$

and

$$(D^{-1}Q)(D^{-1}Q)' = E,$$

i.e., $D^{-1}Q$ is orthogonal.

Similarly any symmetric substitution with the same invariant-factors as A is transformable into $D^{-1}ND$ by an orthogonal substitution, and is therefore transformable into A by an orthogonal substitution.

For another proof, see Bôcher's *Introduction to Higher Algebra*, p. 302.

5. The theorem (2) of § 1 when A is *orthogonal* is established in a somewhat similar manner by making use of § 3 instead of § 2, though the details of the proof are more troublesome.

In the *Messenger of Math.*, 1912, pp. 153, 154, it is shown that, without altering N, $(x_1^2 + x_2^2 + \ldots + x_m^2)$ can be expressed as the sum of invariants of the type

$$k_1f_1(x, y) + k_2f_2(x, y) + k_3f_3(x, y) + \ldots, \quad \text{when} \quad a^2 \neq 1,$$

or $$k_1f_1(x, y) + k_2f_2(x, y) + k_3f_3(x, y) + \ldots + m_2f_2(x, x) + m_4f_4(x, x)$$
$$+ m_6f_6(x, x) + \ldots + n_2f_2(y, y) + n_4f_4(y, y) + n_6f_6(y, y) + \ldots,$$

when $a^2 = 1$, and s is even, or

$$k_1f_1(X, X) + k_3f_3(X, X) + k_5f_5(X, X) + \ldots,$$

when $a^2 = 1$, and s is odd.*

Apply the substitution equivalent to (γ), and we get another invariant

* Remembering that $f_t(x, x)$, $f_t(y, y)$, and $f_t(X, X)$ are zero if $s - t$ is odd.

of N. Then we can choose $a_s, a_{s-1}, a_{s-2}, \ldots$, and $b_s, b_{s-1}, b_{s-2}, \ldots$, in turn to reduce this new invariant to $f_1(x, y)$, $f_1(x, y)$, or $f_1(X, X)$ respectively as in § 4.

It follows from this theorem (2) of § 1, just as in § 4, that any two orthogonal substitutions with the same invariant-factors can be transformed into one another by an orthogonal substitution.

More generally, whenever we have two substitutions which are such that either can be transformed into a given substitution at the same time that $x_1^2+x_2^2+\ldots+x_m^2$ is transformed into a given quadratic form, the two substitutions are transformable into one another by an orthogonal substitution.

Again :—

Any quadratic invariant with non-zero determinant of any substitution can be changed into any other quadratic invariant with non-zero determinant by a change of variables which does not alter the substitution.

If the substitution is the canonical substitution N, it has just been established that each of the two. quadratic invariants can be transformed without altering N into a standard form [which is the sum of invariants such as $f_1(x, y)$ and $f_1(X, X)$]. Hence one quadratic invariant can be transformed into the other without altering N.

In the general case suppose A is any substitution with two quadratic invariants of non-zero determinant. Let $A = P^{-1}NP$, where N is the canonical form of A ; and let Q be a substitution permutable with N transforming one of the invariants of N corresponding to one of the invariants of A into the other.

Then $P^{-1}QP$ transforms one invariant of A into the other.