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XXXI. *On the Problem of Random Vibrations, and of Random Flights in one, two, or three Dimensions. By* Lord RAYLEIGH, *O.M., F.R.S.**

WHEN a number (n) of isoperiodic vibrations of unit amplitude are combined, the resultant depends upon the values assigned to the individual phases. When the phases are at random, the resultant amplitude is indeterminate, and all that can be said relates to the *probability* of various amplitudes (r), or more strictly to the probability that the amplitude lies within the limits r and $r+dr$. The important case where n is very great I considered a long time ago† with the conclusion that the probability in question is simply

$$\frac{2}{n}e^{-r^2/n}\,r\,dr. \qquad (1)$$

The phase (θ) of the resultant is of course indeterminate, and all values are equally probable.

The method then followed began with the supposition that the phases of the unit components were limited to 0° and 180°, taken at random, so that the points (r, θ), representative of the vibrations, lie on the axis $\theta=0$, and indifferently on both sides of the origin. The resultant x, being the difference

* Communicated by the Author.

† Phil. Mag. vol. x. p. 73 (1880); Scientific Papers, vol. i. p. 491.

between the number of positive and negative components, is found from Bernoulli's theorem to have the probability

$$\frac{1}{\sqrt{(2\pi n)}} e^{-x^2/2n} dx. \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)^*$$

The next step was to admit also phases of 90° and 270°, the choice between these two being again at random. If we suppose $\frac{1}{2}n$ components at random along $\pm x$, and $\frac{1}{2}n$ also at random along $\pm y$, the chance of the representative point of the resultant lying within the area $dx\,dy$ is evidently

$$\frac{1}{\pi n} e^{-(x^2+y^2)/n} dx\,dy, \quad . \quad . \quad . \quad . \quad (3)$$

or in terms of r, θ,

$$\frac{1}{\pi n} e^{-r^2/n} r\,dr\,d\theta. \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Thus all phases are equally probable, and the chance that the resultant amplitude lies between r and $r+dr$ is

$$\frac{2}{n} e^{-r^2/n} r\,dr. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

This is the same as was before stated, but at present the conditions are limited to a distribution of precisely $\frac{1}{2}n$ components along x and a like number along y. It concerns us to remove this restriction, and to show that the result is the same when the distribution is perfectly arbitrary in respect to all four directions.

For this purpose let us suppose that $\frac{1}{2}n+m$ are distributed along $\pm x$ and $\frac{1}{2}n-m$ along $\pm y$, and inquire how far the result is influenced by the value of m. The chance of the representative point lying in $r\,dr\,d\theta$ is now expressed by

$$\frac{1}{\pi\sqrt{(n^2-4m^2)}} e^{-nr^2/(n^2-4m^2)} e^{-2mr^2\cos 2\theta/(n^2-4m^2)} r\,dr\,d\theta.$$

Since r is of order $\sqrt{n}$, and m/n is small, the exponential containing θ may be expanded. Retaining the first four terms, we have on integration with respect to θ,

$$\frac{2r\,dr}{\sqrt{(n^2-4m^2)}} e^{-nr^2/(n^2-4m^2)} \left\{1+\frac{m^2r^4}{(n^2-4m^2)^2}+\ldots\right\},$$

as the chance of the amplitude lying between r and $r+dr$. Now if the distribution be entirely at random along the four

* See below.

directions, all the values of m of which there is a finite probability are of order not higher than $\sqrt{n}$, n being treated as infinite. But if m is of this order, the above expression becomes the same as if m were zero; and thus it makes no difference whether the number of components along $\pm x$ and along $\pm y$ are limited to be equal, or not. The previous result is accordingly applicable to a thoroughly arbitrary distribution along the four rectangular directions.

The next point to notice is that the result is symmetrical and independent of the directions of the rectangular axes, from which we may conclude that it has a still higher generality. If a total of n components, to be distributed along one set of rectangular axes, be divided into any number of large groups, it makes no difference whether we first obtain the probabilities of various resultants of the groups separately and afterwards of the final resultants, or whether we regard the whole n as one group. But the probability in each group is the same, notwithstanding a change in the system of rectangular axes; so that the probabilities of various resultants are unaltered, whether we suppose the whole number of components restricted to one set of rectangular axes or divided in any manner between any number of sets of axes. This last state of things is equivalent to no restriction at all; and we conclude that if n unit vibrations of equal pitch and of thoroughly arbitrary phases be compounded, then when n is very great the probability of various resultant amplitudes is given by (1).

If the amplitude of each component be l, instead of unity, as we have hitherto supposed for brevity, the probability of a resultant amplitude between r and $r+dr$ is

$$\frac{2}{\pi l^2} e^{-r^2/nl^2} r\, dr. \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

In 'Theory of Sound,' 2nd edition, § 42a (1894), I indicated another method depending upon a transition from an equation in finite differences to a partial differential equation and the use of a Fourier solution. This method has the advantage of bringing out an important analogy between the present problems and those of gaseous diffusion, but the demonstration, though somewhat improved later*, was incomplete, especially in respect to the determination of a constant multiplier. At the present time it is hardly worth while to pursue it further, in view of the important improvements effected by Kluyver and Pearson. The latter

* Phil. Mag. vol. xlvii. p. 246 (1899); Scientific Papers, vol. v. p. 370.

was interested in the "Problem of the Random Walk," which he thus formulated:—"A man starts from a point O and walks l yards in a straight line; he then turns through any angle whatever and walks another l yards in a second straight line. He repeats this process n times. I require the probability that after these n stretches he is at a distance between r and $r+dr$ from his starting point O."

"The problem is one of considerable interest, but I have only succeeded in obtaining an integrated solution for *two* stretches. I think, however, that a solution ought to be found, if only in the form of a series in powers of $1/n$, when n is large"*. In response, I pointed out that this question is mathematically identical with that of the unit vibrations with phases at random, of which I had already given the solution for the case of n infinite†, the identity depending of course upon the vector character of the components.

In the present paper I propose to consider the question further with extension to *three* dimensions, and with a comparison of results for one, two, and three dimensions‡. The last case has no application to random vibrations but only to random *flights*.

One Dimension.

In this case the required information for any finite n *is* afforded by Bernoulli's theorem. There are $n+1$ possible resultants, and if we suppose the component amplitudes, or stretches, to be unity, they proceed by intervals of *two* from $+n$ to $-n$, values which are the largest possible. The probabilities of the various resultants are expressed by the corresponding terms in the expansion of $(\frac{1}{2}+\frac{1}{2})^n$. For instance the probabilities of the extreme values $\pm n$ are $(1/2)^n$. And the probability of a combination of a positive and b negative components is

$$(\tfrac{1}{2})^n \frac{n!}{a!\,b!}, \quad \ldots\ldots \quad (6)$$

in which $a+b=n$, making the resultant $a-b$. The largest values of (6) occur in the middle of the series, and here a distinction arises according as n is even or odd. In the

* 'Nature,' vol. lxxii. p. 294 (1905).

† 'Nature,' vol. lxxii. p. 318 (1905); Scientific Papers, vol. v. p. 256.

‡ It will be understood that we have nothing here to do with the direction in which the vibrations take place, or are supposed to take place. If that is variable, there must first be a resolution in fixed directions, and it is only after this operation that our present problems arise.

former alternative there is a unique middle term when $a=b=\frac{1}{2}n$; but in the latter a and b cannot be equated, and there are two equal middle terms corresponding to $a=\frac{1}{2}n+\frac{1}{2}$, $b=\frac{1}{2}n-\frac{1}{2}$, and to $a=\frac{1}{2}n-\frac{1}{2}$, $b=\frac{1}{2}n+\frac{1}{2}$. The values of the second fraction in (6) are the series of integers in what is known as the "arithmetical triangle."

We have now to consider the values of

$$\frac{n!}{a!\,b!} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

to be found in the neighbourhood of the middle of the series. If n be even, the value of the term counted s onwards from the unique maximum is

$$\frac{n!}{(\frac{1}{2}n-s)!\,(\frac{1}{2}n+s)!}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

If n be odd, we have to choose between the two middle terms. Taking for instance, $a=\frac{1}{2}n+\frac{1}{2}$, $b=\frac{1}{2}n-\frac{1}{2}$, the sth term onwards is

$$\frac{n!}{\{\frac{1}{2}n-(s-\frac{1}{2})\}!\,\{\frac{1}{2}n+(s-\frac{1}{2})\}!}. \quad . \quad . \quad . \quad (9)$$

The expressions (8) and (9) are brought into the same form when we replace s by the resultant amplitude x. When n is even, $x=-2s$; when s is odd, x is $-2(s-\frac{1}{2})$, so that in both cases we have on restoration of the factor $(\frac{1}{2})^n$

$$\frac{n!}{2^n.(\frac{1}{2}n-\frac{1}{2}x)!\,(\frac{1}{2}n+\frac{1}{2}x)!}. \quad . \quad . \quad . \quad (10)$$

The difference is that when n is even, x has the $(n+1)$ values

$$0, \qquad \pm 2, \qquad \pm 4, \qquad \pm 6, \ldots . \pm n;$$

and when n is odd, the $(n+1)$ values

$$\pm 1, \qquad \pm 3, \qquad \pm 5, \ldots . \pm n.$$

The expression (10) may be regarded as affording the complete solution of the problem proposed; it expresses the probability of any one of the possible resultants, but for practical purposes it requires transformation when we contemplate a very great n.

The necessary transformation can be obtained after Laplace with the aid of Stirling's theorem. The process is detailed in Todhunter's 'History of the Theory of Probability,' p. 548, but the corrections to the principal term there exhibited (of the first order in x) do not appear here where the probabilities of the *plus* and *minus* alternatives are equal. On account of the symmetry, no odd powers of x can occur. I have calculated the resulting expression with retention of the terms which are of the order $1/n^2$ in comparison with the principal term. The resultant x itself may be considered to be of order not higher than $\sqrt{n}$.

By Stirling's theorem

$$n\,! = \sqrt{(2\pi)}\,n^{n+\frac{1}{2}}e^{-n}C_n, \quad . \quad . \quad . \quad . \quad (11)$$

where

$$C_n = 1 + \frac{1}{12n} + \frac{1}{288n^2} + \ldots, \quad . \quad . \quad . \quad (12)$$

with similar expressions for $(\frac{1}{2}n - \frac{1}{2}x)\,!$ and $(\frac{1}{2}n + \frac{1}{2}x)\,!$ For the moment we omit the correcting factors C. Thus

$$\frac{1}{(\frac{1}{2}n-\frac{1}{2}x)\,!\;(\frac{1}{2}n+\frac{1}{2}x)\,!} = \frac{e^n}{2\pi}\left(\frac{n}{2}\right)^{-n-1}\left(1-\frac{x^2}{n^2}\right)^{-\frac{1}{2}n-\frac{1}{2}}\left(\frac{1-x/n}{1+x/n}\right)^{\frac{1}{2}x}.$$

For the logarithm of the product of the two last factors, we have

$$\frac{n+1}{2}\left\{\frac{x^2}{n^2} + \frac{x^4}{2n^4} + \frac{x^6}{3n^6} + \ldots\right\} - \frac{x^2}{n} - \frac{x^4}{3n^3} - \frac{x^6}{5n^5} - \ldots$$

$$= -\frac{x^2}{2n} + \frac{x^2}{2n^2} - \frac{x^4}{4n^3}\left(\frac{1}{3} - \frac{1}{n}\right) - \frac{x^6}{6n^5}\left(\frac{1}{5} - \frac{1}{n}\right) - \ldots,$$

and for the product itself

$$e^{-x^2/2n}\left\{1 + \frac{1}{2n}\left(\frac{x^2}{n} - \frac{x^4}{6n^2}\right) + \frac{1}{8n^2}\left(\frac{3x^4}{n^2} - \frac{3x^6}{5n^3} + \frac{x^8}{36n^4}\right)\right\}. \quad (13)$$

The principal term in (10) is

$$\frac{\sqrt{(2\pi)}\,.\,n^{n+\frac{1}{2}}e^{-n}}{2^n}\cdot\frac{e^n}{2\pi}\left(\frac{n}{2}\right)^{-n-1}e^{-x^2/2n} = \sqrt{\left(\frac{2}{n\pi}\right)}e^{-x^2/2n}.$$

There are still the factors C to be considered. We have

$$\frac{C_n}{C_{\frac{1}{2}(n-x)}C_{\frac{1}{2}(n+x)}} = \left\{1 + \frac{1}{12n} + \frac{1}{288n^2}\right\}$$

$$\left\{1 + \frac{1}{6(n-x)} + \frac{1}{72(n-x)^2}\right\}^{-1}\left\{1 + \frac{1}{6(n+x)} + \frac{1}{72(n+x)^2}\right\}^{-1}$$

$$= \left\{1 + \frac{1}{12n} + \frac{1}{288n^2}\right\}\left\{1 - \frac{1}{3n} + \frac{1}{3n^2}\left(\frac{1}{6} - \frac{x^2}{n}\right)\right\}$$

$$= 1 - \frac{1}{4n} + \frac{1}{32n^2}\left(1 - \frac{32x^2}{3n}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

Finally we obtain

$$\sqrt{\left(\frac{2}{n\pi}\right)}e^{-x^2/2n}\left\{1-\frac{1}{4n}\left(1-\frac{2x^2}{n}+\frac{x^4}{3n^2}\right)\right.$$
$$\left.+\frac{1}{32n^2}\left(1-\frac{44x^2}{3n}+\frac{38x^4}{3n^2}-\frac{12x^6}{5n^3}+\frac{x^8}{9n^4}\right)\right\}, \quad . \quad (15)$$

as the probability when n is large of the resultant amplitude x. It is to be remembered that x is limited to a series of discrete values with a common difference equal to 2, and that our approximation has proceeded upon the supposition that x is not of higher order than $\sqrt{n}$.

If the component amplitudes or stretches be l, in place of unity, we have merely to write x/l in place of x.

The special value of the series (15) is realized only when n is very great. But it affords a closer approximation to the true value than might be expected when n is only moderate. I have calculated the case of $n=10$, both directly from the exact expression (10) and from the series (15) for all the admissible values of x.

TABLE I.

$n=10$.

x.	From (10).	From (15).
0	·24609	·24608
2	·20508	·20509
4	·11719	·11722
6	·04394	·04392
8	·00977	·00975
10	·00098	·00102

The values for $x=0$ and twice those belonging to higher values of x should total unity. Those above from (10) give 1·00001 and those from (15) give 1·00008. It will be seen that except in the extreme case of $x=10$, the agreement between the two formulæ is very close. But, even for much higher values of n, the actual calculation is simpler from the exact formula (10).

When l is very small, while n is very great, we may be able for some purposes to disregard the discontinuous character of the probability as a function of x, replacing the isolated points by a continuous representative curve. The difference between the abscissæ of consecutive isolated points is $2l$; so that if dx be a large multiple of l, we may take

$$\sqrt{\left(\frac{1}{2n\pi}\right)}e^{-x^2/2nl^2}dx/l, \quad . \quad . \quad . \quad . \quad (16)$$

as the approximate expression of the probability that the resultant amplitude lies between x and $x+dx$.

Two Dimensions.

If there is but one stretch of length l, the only possible value of r is of course l.

When there are two stretches of lengths l_1 and l_2, r may vary from l_2-l_1 to l_2+l_1, and then if θ be the angle between them

$$r^2=l_1{}^2+l_2{}^2-2l_1l_2\cos\theta, \quad . \quad . \quad . \quad . \quad (17)$$

and

$$\sin\theta\, d\theta=r\,dr/l_1l_2. \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

Since all angles θ between 0 and π are deemed equally probable, the chance of an angle between θ and $\theta+d\theta$ is $d\theta/\pi$. Accordingly the chance that the resultant r lies between r and $r+dr$ is

$$\frac{r\,dr}{\pi l_1 l_2 \sin\theta}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

or if with Prof. Pearson * we refer the probability to unit of area in the plane of representation,

$$\phi_2(r^2)=\frac{1}{2\pi^2 l_1 l_2 \sin\theta}$$
$$=\frac{1}{\pi^2\sqrt{\{2r^2(l_1{}^2+l_2{}^2)-r^4-(l_1{}^2-l_2{}^2)^2\}}}, \quad . \quad . \quad (20)$$

$\phi_2(r^2)dA$ denoting the chance of the representative point lying in a small area dA at distance r from the origin.

If the stretches l_1 and l_2 are equal, (20) reduces to

$$\phi_2(r^2)=\frac{1}{\pi^2 r\sqrt{\{4l^2-r^2\}}}, \quad . \quad . \quad . \quad . \quad (21)$$

Prof. Pearson's expression, applicable when $r<2l$. When $r>2l$, $\phi_2(r^2)=0$.

When there are three equal stretches ($n=3$), $\phi_3(r^2)$ is

* Drapers' Company Research Memoirs, Biometric Series III., London, 1906.

expressible by elliptic functions * with a discontinuity in form as r passes through l.

For values of n from 4 to 7 inclusive, Pearson's work is founded upon the general functional relation †

$$\phi_{n+1}(r^2) = \frac{1}{\pi}\int_0^{\pi} \phi_n(r^2 + l^2 - 2rl\cos\theta)d\theta. \quad . \quad . \quad (22)$$

Putting $r=0$, he deduces the special conclusion that

$$\phi_{n+1}(0) = \phi_n(l^2), \quad . \quad . \quad . \quad . \quad . \quad (23)$$

as is indeed evident *a priori*.

From (22) the successive forms are determined graphically. For values of n higher than 7 an analytical expression proceeding by powers of $1/n$ is available, and will be further referred to later.

A remarkable advance in the theory of random vibrations and of flights in two dimensions, when the number (n) is finite, is due to J. C. Kluyver ‡, who has discovered an expression for the probability of various resultants in the form of a definite integral involving Bessel's functions. His exposition is rather concise, and I think I shall be doing a service in reproducing it with some developments and slight changes of notation. It depends upon the use of a discontinuous integral evaluated by Weber, viz.

$$\int_0^{\infty} J_1(bx)J_0(ax)dx = u \text{ (say)}.$$

To examine this we substitute from

$$\pi \, . \, J_1(bx) = 2\int_0^{\frac{1}{2}\pi} \cos\theta \sin(bx\cos\theta)d\theta \text{ §},$$

and take first the integration with respect to x. We have ||

$$\int_0^{\infty} dx \sin(bx\cos\theta)J_0(ax) = 0, \qquad \text{if } a^2 > b^2\cos^2\theta,$$

or

$$= (b^2\cos^2\theta - a^2)^{-\frac{1}{2}}, \qquad \text{if } b^2\cos^2\theta > a^2,$$

Thus, if $a^2 > b^2$, $u=0$. If $b^2 > a^2$,

$$u = \frac{2}{\pi}\int \frac{d\theta\cos\theta}{\sqrt{(b^2\cos^2\theta - a^2)}} = \frac{2}{\pi b}\sin^{-1}\frac{b\sin\theta}{\sqrt{(b^2-a^2)}}.$$

* Pearson (*l. c.*) attributes this evaluation to G. T. Bennett.

† Compare 'Theory of Sound,' § 42 *a*.

‡ Amsterdam Proceedings, vol. viii. p. 341 (1905).

§ Gray and Matthews, 'Bessel's Functions,' p. 18, equation (46).

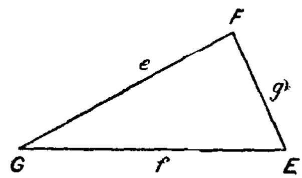
|| G. and M. p. 73.

The lower limit for θ is 0, and the upper limit is given by $\cos^2\theta = a^2/b^2$. Hence $u=1/b$, and thus

$$b\int_0^\infty J_1(bx)J_0(ax)dx = 1, \quad (b^2>a^2) \\ \text{or} \qquad\qquad = 0, \quad (a^2>b^2) \Bigg\} \quad . \quad . \quad (24)$$

A second lemma required is included in Neumann's theorem, and may be very simply arrived at. In fig. 1,

Fig. 1.



G and E being fixed points, the function at F denoted by

$$J_0(g), \quad \text{or} \quad J_0\sqrt{(e^2+f^2-2ef\cos G)},$$

is a potential satisfying everywhere the equation $\nabla^2+1=0$, and accordingly may be expanded round G in the Fourier series

$$A_0J_0(e) + A_1J_1(e)\cos G + A_2J_2(e)\cos 2G + \ldots,$$

the coefficients A being independent of e and G. Thus

$$\frac{1}{2\pi}\int_0^{2\pi} J_0\sqrt{(e^2+f^2-2ef\cos G)}dG = A_0J_0(e).$$

By parity of reasoning when E and F are interchanged, the same integral is proportional to $J_0(f)$, and may therefore be equated to $A_0'J_0(e)J_0(f)$, where A_0' is now an absolute constant, whose value is at once determined to be unity by making e, or f, vanish. The lemma

$$\int_0^{2\pi} J_0\sqrt{(e^2+f^2-2ef\cos G)}dG = 2\pi J_0(e)J_0(f), \qquad (25)$$

is thus established *.

* Similar reasoning shows that if $D_0(g)$ represent a symmetrical purely divergent wave,

$$\int_0^{2\pi} D_0\sqrt{(e^2+f^2-2ef\cos G)}dG = 2\pi J_0(e)D_0(f),$$

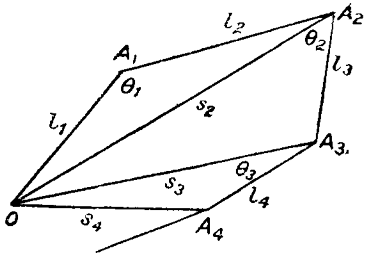
provided that $f>e$.

We are now prepared to investigate the probability

$$P_n(r\,;\ l_1,\ l_2,\ \ldots\ l_n)$$

that after n stretches $l_1, l_2, \ldots l_n$ taken in directions at random the distance from the starting-point O (fig. 2), shall be less

Fig. 2.



than an assigned magnitude r. The direction of the first stretch l_1 is plainly a matter of indifference. On the other hand the probability that the angles θ lie within the limits θ_1 and $\theta_1+d\theta_1$, θ_2 and $\theta_2+d\theta_2$, ... θ_{n-1} and $\theta_{n-1}+d\theta_{n-1}$ is

$$\frac{1}{(2\pi)^{n-1}}d\theta_1 d\theta_2 \ldots\ldots d\theta_{n-1}, \quad . \quad . \quad . \quad (26)$$

which is now to be integrated under the condition that the nth radius vector s_n shall be less than r.

Let us commence with the case of two stretches l_1 and l_2. Then

$$P_2(r\,;\ l_1,\ l_2)=\frac{1}{2\pi}\int d\theta_1,$$

the integration being taken within such limits that $s_2<r$, where

$$s_2^2=l_1^2+l_2^2-2l_1l_2\cos\theta_1.$$

The required condition as to the limits can be secured by the introduction of the discontinuous function afforded by Weber's integral. For

$$r\int_0^\infty J_1(rx)J_0(s_2x)\,dx$$

vanishes when $s_2>r$, and is equal to unity when $s_2<r$. After the introduction of this factor, the integration with respect to θ_1 may be taken over the complete range from 0 to 2π. Thus

$$P_2(r\,;\ l_1,\ l_2)=\frac{r}{2\pi}\int_0^{2\pi}d\theta_1\int_0^\infty dx\,J_1(rx)J_0(s_2x).$$

Taking first the integration with respect to θ_1, we have by (25)

$$\frac{1}{2\pi}\int_0^{2\pi} d\theta_1 J_0(s_2x) = J_0(l_1x)J_0(l_2x),$$

and thus

$$P_2(r\,;\, l_1, l_2) = r\int_0^{\infty} dx\, J_1(rx)J_0(l_1x)J_0(l_2x). \quad . \quad (27)$$

The method can be extended to any number (n) of stretches. Beginning with the integration with respect to θ_{n-1} in (26), we have as before

$$\frac{1}{2\pi}\int d\theta_{n-1} = \frac{r}{2\pi}\int_0^{2\pi} d\theta_{n-1}\int_0^{\infty} dx\, J_1(rx)J_0(s_nx)$$

$$= r\int_0^{\infty} dx\, J_1(rx)J_0(l_nx)J_0(s_{n-1}x).$$

The next integration gives

$$\frac{1}{(2\pi)^2}\iint d\theta_{n-2}d\theta_{n-1} = r\int_0^{\infty} J_1(rx)J_0(l_nx)J_0(l_{n-1}x)J_0(s_{n-2}x)dx,$$

and so on. Finally

$$P_n(r\,;\, l_1, l_2, \ldots l_n) = \frac{1}{(2\pi)^{n-1}}\iint\ldots d\theta_1 d\theta_2\ldots d\theta_{n-1}$$

$$= r\int_0^{\infty} J_1(rx)J_0(l_1x)J_0(l_2x)\ldots J_0(l_nx)dx, \quad . \quad (28)$$

—the expression for P_n discovered by Kluyver.

It will be observed that (28) is symmetrical with respect to the l's ; the order in which they are taken is immaterial.

When all the l's are equal,

$$P_n(r\,;\, l) = r\int_0^{\infty} J_1(rx)\{J_0(lx)\}^n dx. \quad . \quad . \quad (29)$$

If in (29) we suppose $r=l$,

$$P_n(l\,;\, l) = -\int_0^{\infty} \{J_0(lx)\}^n d\,J_0(lx)$$

$$= -\frac{\{J_0(lx)\}^{n+1}}{n+1}\Big|_0^{\infty} = \frac{1}{n+1}; \quad . \quad . \quad . \quad (30)$$

so that after n equal components have been combined the chance that the resultant shall be less than one of the components is $1/(n+1)$, an interesting result due to Kluyver. The same author notices some of the discontinuities which present themselves, but it will be more convenient to consider this in a modified form of the problem.

The modification consists in dealing, not with the chance of a resultant less than r, but with the chance that it lies between r and $r+dr$. It may seem easy to pass from the one to the other, as it involves merely a differentiation with respect to r. We have

$$\frac{d}{dr}\{r\mathrm{J}_1(rx)\} = -\frac{d}{dr}\{r\mathrm{J}_0'(rx)\}$$

$$= -\mathrm{J}_0'(rx) - rx\,\mathrm{J}_0''(rx) = rx\,\mathrm{J}_0(rx),$$

in virtue of the differential equation satisfied by J_0. Thus, if the differentiation under the integral sign is legitimate,

$$\frac{d\mathrm{P}_n}{dr} = 2\pi r\,\phi_n(r^2) = r\int_0^\infty x\,dx\,\mathrm{J}_0(rx)\mathrm{J}_0(l_1x)\mathrm{J}_0(l_2x)\ldots\mathrm{J}_0(l_nx)\ldots,\quad(31)$$

and, if all the l's are equal,

$$\phi_n(r^2) = \frac{1}{2\pi}\int_0^\infty x\,dx\,\mathrm{J}_0(rx)\{\mathrm{J}_0(lx)\}^n, \quad . \quad . \quad (32)$$

the form employed by Pearson, whose investigation is by a different method. If we put $n=1$ in (32),

$$\phi_1(r^2) = \frac{1}{2\pi}\int_0^\infty x\,dx\,\mathrm{J}_0(rx)\mathrm{J}_0(lx), \quad . \quad . \quad . \quad (33)$$

and this is in fact the equation from which Pearson starts. But it should be remarked that the integral (33), as it stands, is *not convergent*. For when z is very great,

$$\mathrm{J}_0(z) = \sqrt{\left(\frac{2}{\pi z}\right)}\cos\left(\frac{1}{4}\pi - z\right), \quad . \quad . \quad . \quad (34)$$

so that $(r \neq 0)$

$$\frac{1}{2\pi}\int^\infty x\,dx\,\mathrm{J}_0(rx)\mathrm{J}_0(lx) = \frac{1}{2\pi^2\sqrt{(rl)}}\int^\infty dx$$

$$\{\sin(r+l)x + \cos(r-l)x\},$$

and this is not convergent when $x=\infty$.

The criticism does not apply to (29) itself when $n=1$, but it leads back to the question of differentiation under the sign of integration. It appears at any rate that any number of such operations can be justified, provided that the integrals, resulting from these and the next following operation, are finite for the values of r in question. But this condition is not satisfied in the differentiation under the integral sign of (29) when $n=1$. For the next operation upon (32) then yields

$$\int_0 x^2\,dx\,\mathrm{J}_1(rx)\mathrm{J}_0(lx).$$

When we substitute for $J_0(lx)$ from (34) and for $J_1(rx)$ from

$$J_1(z)=\sqrt{\left(\frac{2}{\pi z}\right)}\cos\left(\frac{3\pi}{4}-z\right),$$

we get

$$\int^x x\,dx\cos\left(\frac{3\pi}{4}-rx\right)\cos\left(\frac{\pi}{4}-lx\right),$$

which becomes infinite with x, even for general values of r and l.

So much by way of explanation ; but of course we do not really need to discuss the cases $n=1$, $n=2$, or even $n=3$, for which exact solutions can be expressed in terms of functions which may be regarded as known.

For higher values of n it would be of interest to know how many differentiations with respect to r may be made under the sign of integration. It may be remarked that since all J's and their derivatives to any order are less than unity, the integral can become infinite only in virtue of that part of the range where x is very great, and that there we may introduce the asymptotic values.

We have thus to consider

$$\frac{d^p}{dr^p}\phi_n(r^2)=\frac{1}{2\pi}\int_0^\infty dx\,x^{p+1}J_0{}^p(rx)\{J_0(lx)\}^n. \quad . \quad (35)$$

For the leading term when z is very great, we have

$$J_0{}^p(z)=\frac{d^p}{dz^p}\left\{\sqrt{\left(\frac{2}{\pi z}\right)}\cos\left(\frac{1}{4}\pi-z\right)\right\}$$

$$=\sqrt{\left(\frac{2}{\pi z}\right)}\cos\left(\frac{1}{4}\pi-z-\frac{1}{2}p\pi\right), \quad . \quad . \quad (36)$$

$$\{J_0(z)\}^n=\left(\frac{2}{\pi z}\right)^{\frac{1}{2}n}\cos^n\left(\frac{1}{4}\pi-z\right), \quad . \quad . \quad . \quad . \quad . \quad (37)$$

so that with omission of constant factors our integral becomes

$$\int^\infty dx\,x^{p+\frac{1}{2}-\frac{1}{2}n}\cos\left(\frac{1}{4}\pi-rx-\frac{1}{2}p\pi\right)\cos^n\left(\frac{1}{4}\pi-lx\right). \quad (38)$$

In this $\cos^n(\frac{1}{4}\pi-lx)$ can be expanded in a series of cosines of multiples of $(\frac{1}{4}\pi-lx)$, commencing with $\cos n(\frac{1}{4}\pi-lx)$ and ending when n is odd with $\cos(\frac{1}{4}\pi-lx)$, and when n is even with a constant term. The various products of cosines are then to be replaced by cosines of sums and differences. The most unfavourable case occurs when this operation leaves a constant term, which can happen only for values of r which are multiples of l. We are then left with

$$\int^{\infty} dx\, x^{p+\frac{1}{2}-\frac{1}{2}n} = \left. \frac{x^{p+\frac{3}{2}-\frac{1}{2}n}}{p+\frac{3}{2}-\frac{1}{2}n} \right|^{\infty}.$$

The integral is thus finite or infinite according as

$$p < \text{ or } > \tfrac{1}{2}(n-3).$$

If, however, there arise no constant term, we have to consider

$$\int^{\infty} dx\, x^s \cos mx = \left. \frac{x^s}{m} \sin mx \right|^{\infty} - \frac{s}{m^2}\int^{\infty} dx\, x^{s-1} \sin mx,$$

where m is finite; and this is finite if s, that is $p+\frac{1}{2}-\frac{1}{2}n$, be negative. The differentiations are then valid, if

$$p < \tfrac{1}{2}(n-1).$$

We may now consider more especially the cases $n=4$, &c. When $n=4$, $s=p+\frac{1}{2}-\frac{1}{2}n=p-\frac{3}{2}$.

If $p=1$, $s=-\frac{1}{2}$, and the cosine factors in (38) become

$$\cos\left(\tfrac{1}{4}\pi + rx\right) \cos^4\left(\tfrac{1}{4}\pi - lx\right),$$

yielding finally

$$\cos\left(\frac{5\pi}{4} + rx - 4lx\right),\ \cos\left(\frac{3\pi}{4} - rx - 4lx\right),$$

$$\cos\left(\frac{3\pi}{4} + rx - 2lx\right),\ \cos\left(\frac{\pi}{4} - rx - 2lx\right),\ \cos\left(\frac{\pi}{4} + rx\right),$$

so that there is no constant term unless $r=4l$, or $2l$. With these exceptions, the original differentiation under the integral sign is justified.

We fall back upon ϕ_4 itself by putting $p=0$, making $s=-\frac{3}{2}$. The integral is then finite in all cases $(r \neq 0)$, in agreement with Pearson's curve.

Next for $n=5$, $s=p-2$.

When $p=1$, $s=-1$, and we find that the cosine factors yield a constant term only when $r=3l$. Pearson's curve does not suggest anything special at $r=3l$; it may be remarked that the integral with $p=1$ is there only logarithmically infinite.

If $n=5$, $p=0$, $s=-2$; and the integral for ϕ_5 is finite for all values of r.

When $n=6$, $s=p-2\frac{1}{2}$. In this case, whether $p=1$, or 0, no question can arise. The integrals are finite for all values of r.

A fortiori is this so, when $n>6$.

If we suppose $p=2$, $s=\frac{1}{2}(5-n)$. Thus $n=7$ makes $s=-1$, and infinities might occur for special values of r. But if $n>7$, $s<\frac{3}{2}$, and infinities are excluded whatever may be the value of r.

Similarly if $p=3$, infinities are excluded if $n>9$, and so on.

Our discussion has not yielded all that could be wished; the subject may be commended to those better versed in pure mathematics. Probably what is required is a better criterion as to the differentiation under the integral sign.

We may now pass on to consider what becomes of Kluyver's integral when n is made infinite. As already remarked, Pearson has developed for it a series proceeding by powers of $1/n$, and it may be convenient to give a version of his derivation, without, however, carrying the process so far.

The evaluation of the principal term depends upon a formula due, I think, to Weber *, viz.

$$u=\int_0^\infty J_0(rx)e^{-p^2x^2}x\,dx=\frac{1}{2p^2}e^{-r^2/4p^2}, \quad . \quad . \quad . \quad (39)$$

making †

$$\frac{du}{dr}=\int_0^\infty J_0'(rx)e^{-p^2x^2}x^2\,dx=-\frac{1}{2p^2}\int_0^\infty J_0'(rx)x\,de^{-p^2x^2}$$

$$=\frac{1}{2p^2}\int_0^\infty e^{-p^2x^2}\{J_0'(rx)+rx\,J_0''(rx)\}dx$$

$$=-\frac{r}{2p^2}\int_0^\infty J_0(rx)e^{-p^2x^2}x\,dx=-\frac{r}{2p^2}u.$$

Hence $$u=Ce^{-r^2/4p^2}.$$

To determine C we have merely to make $r=0$. Thus

$$C=u_{r=0}=\int_0^\infty e^{-p^2x^2}x\,dx=\frac{1}{2p^2},$$

by which (39) is established.

Unless lx is small, the factor $\{J_0(lx)\}^n$ in (32) diminishes rapidly as n increases, inasmuch as $J_0(lx)$ is less than unity for any finite lx. Thus when n is very great, the important part of the range of integration corresponds to a small lx.

* Gray and Matthews, *loc. cit.* p. 77.

† I apprehend that there can be no difficulty here as to the differentiation, the situation being dominated by the exponential factor.

Writing s for $\frac{1}{2}nl^2$, we have

$$\log J_0(lx) = \log\left(1 - \frac{sx^2}{2n} + \frac{s^2x^4}{16n^2} - \frac{s^3x^6}{288n^3} + \dots\right)$$

$$= -\frac{sx^2}{2n} - \frac{s^2x^4}{16n^2} - \frac{s^3x^6}{72n^3} + \dots;$$

so that

$$\{J_0(lx)\}^n = e^{-\frac{1}{2}sx^2}\left(1 - \frac{s^2x^4}{16n} - \frac{s^3x^6}{72n^2} + \frac{s^4x^8}{512n^2}\right),$$

making

$$2\pi\phi_n(r^2) = \int_0^\infty x\,dx\,J_0(rx)e^{-\frac{1}{2}sx^2}\left(1 - \frac{s^2x^4}{16n} - \frac{s^3x^6}{72n^2} + \frac{s^4x^8}{512n^2}\right). \quad (40)$$

Calling the four integrals on the right I_1, I_2, I_3, and I_4, we have by (39)

$$I_1 = \int_0^\infty x\,dx\,J_0(rx)e^{-\frac{1}{2}sx^2} = \frac{1}{s}e^{-r^2/2s}, \quad . \quad . \quad (41)$$

$$-I_2 = \frac{s^2}{4n}\frac{d^2I_1}{ds^2} = \frac{s^2}{4n}\frac{d^2}{ds^2}\left(\frac{1}{s}e^{-r^2/2s}\right), \quad . \quad . \quad . \quad (42)$$

$$I_3 = \frac{s^3}{9n^2}\frac{d^3I_1}{ds^3} = \frac{s^3}{9n^2}\frac{d^3}{ds^3}\left(\frac{1}{s}e^{-r^2/2s}\right), \quad . \quad . \quad (43)$$

$$I_4 = \frac{s^4}{32n^2}\frac{d^4I_1}{ds^4} = \frac{s^4}{32n^2}\frac{d^4}{ds^4}\left(\frac{1}{s}e^{-r^2/2s}\right). \quad . \quad (44)$$

Thus

$$2\pi\phi_n(r^2) = \frac{e^{-r^2/2s}}{s}\left\{1 - \frac{1}{4n}\left(2 - \frac{2r^2}{s} + \frac{r^4}{4s^2}\right)\right.$$

$$- \frac{1}{9n^2}\left(6 - \frac{9r^2}{s} + \frac{9r^4}{4s^2} - \frac{r^6}{8s^3}\right)$$

$$\left. + \frac{1}{32n^2}\left(24 - \frac{48r^2}{s} + \frac{18r^4}{s^2} - \frac{2r^6}{s^3} + \frac{r^8}{16s^4}\right)\right\}$$

$$= \frac{e^{-r^2/2s}}{s}\left\{1 - \frac{1}{4n}\left(2 - \frac{2r^2}{s} + \frac{r^4}{4s^2}\right) + \frac{1}{12n^2}\left(1 - \frac{6r^2}{s}\right.\right.$$

$$\left.\left. + \frac{15r^4}{4s^2} - \frac{7r^6}{12s^3} + \frac{3r^8}{128s^4}\right)\right\}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (45)$$

in agreement (so far as it goes) with Pearson, whose σ^2 is equal to our s. The leading term is that given in 1880.

Three Dimensions.

We may now pass on to the corresponding problem when flights take place in three dimensions, where we shall find, as might have been expected, that the mathematics are simpler. And first for *two* flights of length l_1 and l_2. If μ be the cosine of the angle between l_1 and l_2 and r the resultant,

$$r^2 = l_1^2 + l_2^2 - 2l_1l_2\mu,$$

giving

$$r\,dr = -l_1l_2\,d\mu. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (46)$$

The chance of r lying between r and $r+dr$ is the same as the chance of μ lying between μ and $\mu+d\mu$, that is $-\frac{1}{2}d\mu$, since all directions in space are to be treated as equally probable. Accordingly the chance of a resultant between r and $r+dr$ is

$$\frac{r\,dr}{2l_1l_2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (47)$$

The corresponding volume is $4\pi r^2dr$, so that in the former notation

$$\phi_2(r\,;\,l) = \frac{1}{8\pi l^2 r}, \quad . \quad . \quad . \quad . \quad . \quad (48)$$

l_1 and l_2 being supposed equal. It will be seen that this is simpler than (21). It applies, of course, only when $r<2l$. When $r>2l$, $\phi_2=0$.

In like manner when l_1 and l_2 differ, the chance of a resultant less than r is zero, when r falls short of the difference between l_2 and l_1, say l_2-l_1. Between l_2-l_1 and l_2+l_1 the chance is

$$\int_{l_2-l_1}^{r} \frac{r\,dr}{2l_1l_2} = \frac{r^2-(l_2-l_1)^2}{4l_1l_2}. \quad . \quad . \quad . \quad . \quad . \quad (49)$$

When r has its greatest value (l_2+l_1), (49) becomes

$$\frac{(l_2+l_1)^2-(l_2-l_1)^2}{4l_1l_2} = 1. \quad . \quad . \quad . \quad . \quad . \quad (50)$$

The "chance" is then a certainty, as also when $r>l_1+l_2$.

In proceeding to the general value of n, we may conveniently follow the analogy of the two-dimensional investigation of Kluyver, for which purpose we require a function that shall be unity when $s<r$, and zero when $s>r$. Such a function is

$$\frac{2}{\pi}\int_0^\infty dx \frac{\sin sx}{sx} \frac{\sin rx - rx\cos rx}{x}; \qquad (51)$$

for it may be written

$$-\frac{2r}{\pi s}\int_0^\infty \sin sx\, d\left(\frac{\sin rx}{rx}\right) = \frac{2}{\pi}\int_0^\infty \frac{\sin rx}{x}\cos sx\, dx$$

$$= \frac{1}{\pi}\int_0^\infty \frac{\sin(s+r)x - \sin(s-r)x}{x}\, dx = 1 \text{ or } 0,$$

according as s is less or greater than r.

In like manner for a second lemma, corresponding with (25), we may reason again from the triangle GFE (fig. 1). $J_0(g)$ is replaced by $\sin g/g$, a potential function symmetrical in three dimensions about E and satisfying *everywhere* $\nabla^2+1=0$. It may be expanded about G in Legendre's series *

$$A_0\frac{\sin e}{e} + A_1\mu\left(\frac{\sin e}{e^2} - \frac{\cos e}{e}\right) + \ldots,$$

μ being written for $\cos G$, and accordingly

$$\tfrac{1}{2}\int_{-1}^{+1} d\mu \frac{\sin\sqrt{(e^2+f^2-2ef\mu)}}{\sqrt{(e^2+f^2-2ef\mu)}} = A_0\frac{\sin e}{e}.$$

When E and F are interchanged, the same integral is seen to be proportional to $\sin f/f$, and may therefore be equated to

$$A_0'\frac{\sin e}{e}\frac{\sin f}{f},$$

where A_0' is now an absolute constant, whose value is determined to be unity by putting e, or f, equal to zero. We may therefore write

$$\tfrac{1}{2}\int_{-1}^{+1} d\mu \frac{\sin\sqrt{(e^2+f^2-2ef\mu)}}{\sqrt{(e^2+f^2-2ef\mu)}} = \frac{\sin e}{e}\,\frac{\sin f}{f}. \quad . \quad (52)$$

As in the case of two dimensions, similar reasoning shows that

$$\tfrac{1}{2}\int_{-1}^{+1} d\mu \frac{\cos\sqrt{(e^2+f^2-2ef\mu)}}{\sqrt{(e^2+f^2-2ef\mu)}} = \frac{\sin e}{e}\,\frac{\cos f}{f}, \quad . \quad (53)$$

provided $e < f$.

With appropriate changes, we may now follow Kluyver's argument for two dimensions. The same diagram (fig. 2) will serve, only the successive triangles are no longer limited to lie in one plane. Instead of the angles θ, we have now to deal with their cosines, of which all values are to be

* 'Theory of Sound,' § 330.

regarded as equally probable. The probability that these cosines shall lie within the interval μ_1 and $\mu_1+d\mu_1$, μ_2 and $\mu_2+d\mu_2$, ... μ_{n-1} and $\mu_{n-1}+d\mu_{n-1}$ is

$$\frac{1}{2^{n-1}}d\mu_1 d\mu_2 \ldots . d\mu_{n-1}, \quad . \quad . \quad . \quad . \quad (54)$$

which is now to be integrated under the condition that the nth radius s_n shall be less than r.

We begin with two stretches l_1 and l_2. Then, in the same notation as before, we have

$$P_2(r\,;\,l_1,\,l_2)=\tfrac{1}{2}\int d\mu,$$

the integration being within such limits as make $s_2>r$, where

$$s_2{}^2=l_1{}^2+l_2{}^2-2l_1l_2\mu.$$

Hence, by introduction of the discontinuous function (51),

$$P_2(r\,;\,l_1,\,l_2)=\frac{1}{\pi}\int_{-1}^{+1}d\mu\int_0^\infty dx\frac{\sin s_2x}{s_2x}\frac{\sin rx-rx\cos rx}{x}.$$

But by (52)

$$\tfrac{1}{2}\int_{-1}^{+1}d\mu\frac{\sin s_2x}{s_2x}=\frac{\sin l_1x}{l_1x}\frac{\sin l_2x}{l_2x},$$

and thus

$$P_2(r\,;\,l_1,\,l_2)=\frac{2}{\pi}\int_0^\infty dx\frac{\sin rx-rx\cos rx}{x}\frac{\sin l_1x}{l_1x}\frac{\sin l_2x}{l_2x}. \quad (55)$$

A simpler form is available for dP_2/dr, since

$$\frac{d}{dr}(\sin rx-rx\cos rx)=rx^2\sin rx.$$

Thus

$$\frac{dP_2}{dr}=\frac{2r}{\pi l_1l_2}\int_0^\infty\frac{dx}{x}\sin rx\sin l_1x\sin l_2x, \quad . \quad . \quad (56)$$

in which we replace the product of sines by means of

$$4\sin rx\sin l_1x\sin l_2x=\sin(r+l_2-l_1)x$$
$$+\sin(r-l_2+l_1)x-\sin(r+l_2+l_1)x-\sin(r-l_2-l_1)x.$$

If r, l_2, l_1 are sides of a real triangle, any two of them together are in general greater than the third, and thus when the integration is effected by the formula

$$\int_0^\infty\frac{\sin u}{u}du=\tfrac{1}{2}\pi,$$

we obtain three positive and one negative term. Finally

$$\frac{d\mathrm{P}_2}{dr}=\frac{r}{2l_1l_2},$$

in agreement with (47). The expression is applicable only when the triangle is possible. In the contrary case we find $d\mathrm{P}/dr$ equal to zero when r is less than the difference and greater than the sum of l_1 and l_2.

This argument must appear very roundabout, if the object were merely to obtain the result for $n=2$. The advantage is that it admits of easy extension to the general value of n. To this end we take the last stretch l_n and the immediately preceding radius s_{n-1} in place of l_2 and l_1 respectively, and then repeat the operation with l_{n-1}, s_{n-2}, and so on, until we reach l_2 and s_1 $(=l_1)$. The result is evidently

$$\mathrm{P}_n(r\,;\,l_1,l_2,\ldots l_n)=\frac{2}{\pi}\int_0^\infty dx\frac{\sin rx-rx\cos rx}{x}\frac{\sin l_1x}{l_1x}\frac{\sin l_2x}{l_2x}\cdots\frac{\sin l_nx}{l_nx}, \quad . \quad . \quad . \quad . \quad (57)$$

or if we suppose, as for the future we shall do, that the l's are all equal,

$$\mathrm{P}_n(r\,;\,l)=\frac{2}{\pi}\int_0^\infty dx\frac{\sin rx-rx\cos rx}{x}\left(\frac{\sin lx}{lx}\right)^n. \quad (58)$$

This is the chance that the resultant is less than r. For the chance that the resultant lies between r and $r+dr$, we have, as the coefficient of dr,

$$\frac{d\mathrm{P}_n}{dr}=\frac{2r}{\pi l^n}\int_0^\infty\frac{dx}{x^{n-1}}\sin rx\sin^n lx \quad . \quad . \quad . \quad (59)$$

Let us now consider the particular case of $n=3$, when

$$\frac{d\mathrm{P}_3}{dr}=\frac{2r}{\pi l^3}\int_0^\infty\frac{dx}{x^2}\sin rx\sin^3 lx. \quad . \quad . \quad . \quad . \quad (60)$$

In this we have

$$\sin rx\sin^3 lx=\tfrac{1}{8}\{3\cos(r-l)x-3\cos(r+l)x-\cos(r-3l)x+\cos(r+3l)x\}.$$

And

$$\int_0^\infty\frac{dx}{x^2}\{\cos(r-l)x-\cos(r+l)x\}$$

$$=2\int_0^\infty\frac{dx}{x^2}\left\{\sin^2\frac{(r+l)x}{2}-\sin^2\frac{(r-l)x}{2}\right\}$$

$$=\tfrac{1}{2}\pi\{r+l-|r-l|\}\,;$$

and in like manner for the second pair of cosines.

Thus

$$\frac{dP_3}{dr}=\frac{r}{8l^3}\{2r-3\,|r-l|+|r-3l|\}. \quad . \quad . \quad (61)$$

expresses the complete solution. When

$$r<l, \quad dP_3/dr=r^2/2l^3,$$

$$3l>r>l, \quad dP_3/dr=(3lr-r^2)/4l^3,$$

$$r>3l, \quad dP_3/dr=0.$$

It will be observed that dP_3/dr is itself continuous; but the next derivative changes suddenly at $r=l$ and $r=3l$ from one finite value to another.

Next take $n=4$. From (59)

$$\frac{dP_4}{dr}=\frac{2r}{\pi l^4}\int_0^\infty\frac{dx}{x^3}\sin rx\sin^4 lx,$$

and

$$-\frac{d^2}{dr^2}\left(\frac{1}{r}\frac{dP_4}{dr}\right)=\frac{2}{\pi l^4}\int_0^\infty\frac{dx}{x}\sin rx\sin^4 lx$$

$$=\frac{1}{8\pi l^4}\int_0^\infty\frac{dx}{x}\{\sin(r+4l)x+\sin(r-4l)x$$

$$-4\sin(r+2l)x-4\sin(r-2l)x+6\sin rx\}$$

$$=\frac{1}{16l^4}\{1\pm1-4\mp4+6\}=\frac{1}{16l^4}\{3\pm1\mp4\},$$

the alternatives depending upon the signs of $r-4l$ and $r-2l$.

When $$r<2l, \quad -16l^4\frac{d^2}{dr^2}\left(\frac{1}{r}\frac{dP_4}{dr}\right)=6,$$

$$4l>r>2l, \quad -16l^4\frac{d^2}{dr^2}\left(\frac{1}{r}\frac{dP_4}{dr}\right)=-2,$$

and when $r>4l$, the value is zero. In no case can the value be infinite, from which we may infer that

$$\frac{d}{dr}\left(\frac{1}{r}\frac{dP_4}{dr}\right) \quad \text{and} \quad \frac{1}{r}\frac{dP_4}{dr}$$

must be continuous throughout.

From these data we can determine the form of dP_4/dr, working backwards from the large value of r, where all derivatives vanish.

$$(4l>r>2l) \quad -16l^4\frac{d}{dr}\left(\frac{1}{r}\frac{dP_4}{dr}\right)=-2(r-4l),$$

$$(2l>r) \quad -16l^4\frac{d}{dr}\left(\frac{1}{r}\frac{dP_4}{dr}\right)=6(r-2l)+4l=6r-8l,$$

giving continuity at $r=4l$ and $r=2l$. Again

$$(4l>r>2l) \quad -16l^4\frac{1}{r}\frac{dP_4}{dr}=-(r^2-16l^2)+8l(r-4l)$$
$$=-(r-4l)^2,$$
$$(2l>r) \quad -16l^4\frac{1}{r}\frac{dP_4}{dr}=3(r^2-4l^2)-8l(r-2l)-4l^2$$
$$=3r^2-8rl.$$

Finally

$$\left.\begin{aligned}\frac{dP_4}{dr}=4\pi r^2\phi_4(r\,;\,l)&=\frac{r^2(8l-3r)}{16l^4} \quad (r<2l)\\ \text{or} \qquad &=\frac{r(4l-r)^2}{16l^4} \quad (4l>r>2l)\end{aligned}\right\}, \quad (62)$$

and vanishes, of course, when $r>4l$.

From (61), (64) we may verify Pearson's relation, $\phi_4(0)=\phi_3(l)$.

From these examples the procedure will be understood. When n is even, we differentiate (59) $(n-2)$ times, thus obtaining

$$\frac{d^{n-2}}{dr^{n-2}}\left(\frac{1}{r}\frac{dP_n}{dr}\right)=\frac{2}{\pi l^n}\int_0^\infty\frac{dx}{x}\sin rx\sin^n lx, \qquad (63)$$

in which $\sin^n lx$ is replaced by the series containing $\cos nlx$, $\cos(n-2)lx, \ldots$ and ending with a constant term. When this is multiplied by $\sin rx$, we get sines of $(r\pm nl)x$, $\{r\pm(n-2)l\}x, \ldots \sin rx$, and the integration can be effected. Over the various ranges of $2l$ the values are constant, but they change discontinuously when r is an *even* multiple of l. The actual forms for dP_n/dr can then be found, as already exemplified, by working backwards from $r>nl$, where all derivatives vanish, and so determining the constants of integration as to maintain continuity throughout. These forms are in all cases algebraic.

When n is odd, we differentiate $(n-3)$ times, thus obtaining a form similar to (60) where $n=3$. A similar procedure then shows that the result assumes constant values over finite ranges with discontinuities when r is an *odd* multiple of l. On integration the forms for dP_n/dr are again algebraic.

I have carried out the detailed calculation for $n=6$. It will suffice to record the principal results. For the values of

$$-2^6l^6\frac{d^4}{dr^4}\left(\frac{1}{r}\frac{dP_6}{dr}\right)$$

we find for the various ranges :

$$(r<2l),\ -20;\quad (2l<r<4l),\ +10;$$
$$(4l<r<6l),\ -2;\quad (6l<r),\ 0.$$

And on integration for

$$-2^6 l^6\left(\frac{1}{r}\frac{dP_6}{dr}\right), \quad . \quad . \quad . \quad . \quad . \quad (64)$$

$$(0-2l)\quad -\frac{5r^4}{6}+4lr^3-16l^3r,$$

$$(2l-4l)\quad +\frac{5r^4}{12}-6lr^3+30l^2r^2-56l^3r+20l^4,$$

$$(4l-6l)\quad -\frac{r^4}{12}+2lr^3-18l^2r^2+72l^3r-108l^4,$$

$$(r>6l)\quad 0.$$

We may now seek the form approximated to when n is very great. Setting for brevity $l=1$ in (59), we have

$$\log\left(\frac{\sin x}{x}\right)^n = n\left\{-\frac{x^2}{6}+h_4x^4+h_6x^6+\ldots\right\},$$

where

$$h_4=-\frac{1}{180},\quad h_6=-\frac{1}{35.81}, \quad . \quad . \quad . \quad (65)$$

and

$$\left(\frac{\sin x}{x}\right)^n = e^{-nx^2/6}\{1+nh_4x^4+nh_6x^6+\tfrac{1}{2}n^2h_4{}^2x^8+\ldots\},$$

so that

$$\frac{1}{r}\frac{dP_n}{dr}=\frac{2}{\pi}\int_0^\infty x\,dx\sin rx\,e^{-nx^2/6}\{1+nh_4x^4+nh_6x^6$$
$$+\tfrac{1}{2}n^2h_4{}^2x^8+\ldots\}. \quad (66)$$

The expression for the principal term is a known definite integral, and we obtain for it

$$\frac{dP_n}{dr}=\frac{3\sqrt{6}\,.\,r^2}{\sqrt{\pi}\,.\,n^{3/2}}e^{-3r^2/2n}, \quad . \quad . \quad . \quad . \quad . \quad (67)$$

which may be regarded as the approximate value when n is very large. To restore l, we have merely to write r/l for r throughout.

In pursuing the approximation we have to consider the relative order of the various terms. Taking nx^2 as standard, so that x^2 is regarded as of the order $1/n$, nx^8 is of order n^{-3} and is omitted. But n^2x^8 is of order n^{-2} and is retained. The terms written down in (66) thus suffice for an approximation to the order n^{-2} inclusive.

The evaluation of the auxiliary terms in (66) can be effected by differentiating the principal term with respect to n. Each such differentiation brings in $-x^2/6$ as a factor, and thus four operations suffice for the inclusion of the term containing x^8. We get

$$\frac{dP_n}{dr}=\frac{3\sqrt{6}\,.\,r^2}{\sqrt{\pi}\,.\,l^3}\left[N+nh_4\,.\,6^2\frac{d^2N}{dn^2}\right.$$
$$\left.-nh_6\,.\,6^3\frac{d^3N}{dn^3}+\tfrac{1}{2}n^2h^4\,.\,6^4\frac{d^4N}{dn^4}\right], \quad (68)$$

where
$$N=n^{-3/2}e^{-3r^2/2nl^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (69)$$

Finally

$$\frac{dP_n}{dr}=\frac{3\sqrt{6}\,.\,r^2e^{-3r^2/2nl^2}}{\sqrt{\pi}\,.\,l^3\,.\,n^{3/2}}\left\{1-\frac{3}{20n}\left(5-\frac{10r^2}{nl^2}+\frac{3r^4}{n^2l^4}\right)\right.$$
$$\left.+\frac{1}{40n^2}\left(\frac{29}{4}-\frac{69r^2}{nl^2}+\frac{981r^4}{10n^2l^4}-\frac{1341r^6}{35n^3l^6}+\frac{81r^8}{20n^4l^8}\right)\right\}. \quad (70)$$

Here $dP_n/dr\,.\,dr$ is the chance that the resultant of a large number n of flights shall lie between r and $r+dr$. In Pearson's notation,

$$4\pi r^2\phi_n=dP_n/dr.$$

The maximum value of the principal term (67) occurs when $r/l=\sqrt{(2n/3)}$.

It is some check upon the formulæ to compare the exact results for $n=6$ in (64) with those derived for the case of n great in (70), although with such a moderate value of n no precise agreement could be expected. The following Table gives the numerical results for $l\,dP_6/dr$ in the two cases :—

r/l.	From (64).	From (70).
0	·2500 r^2/l^2	·2483 r^2/l^2
·5	·05900	·05886
1	·2005	·2007
2	·4167	·4169
3	·2930	·2922
4	·0833	·1055
5	·00652	·00716
6	·00000	......

So far as the principal term in (70) is concerned, the maximum value occurs when $r/l=2$.

It will be seen that the agreement of the two formulæ is in fact very good, so long as r/l does not much exceed $\sqrt{n}$. As the maximum value of r/l for which the true result differs from zero, is approached, the agreement necessarily falls off. Beyond $r/l=n$, when the true value is zero, (70) yields finite, though small, values.

Terling Place, Witham,
January 24th, 1919.

P.S. *March 3rd.*

In (45) we have the expression for the probability of a resultant (r) when a large number (n) of isoperiodic vibrations are combined, whose representative points are distributed at random along the circumference of a circle of radius l, so that the component amplitudes are all equal. It is of interest to extend the investigation to cover the case of a number of groups in which the amplitudes are different, say a group of p_1 components of amplitude l_1, a group containing p_2 of amplitude l_2, and so on to any number of groups, but always under the restriction that every p is very large. The total number (Σp) may still be denoted by n. The result will be applied to a case where the number of groups is infinite, the representative points of the components being distributed at random over the *area* of a circle of radius L. We start from (31), now taking the form

$$2\pi\phi_n(r^2)=\int_0^\infty x\,dx\,\mathrm{J}_0(rx)\{\mathrm{J}_0(l_1x)\}^{p_1}\{\mathrm{J}_0(l_2x)\}^{p_2}\ldots, \quad . \quad (71)$$

The derivation of the limiting form proceeds as before, where only one l was considered. Writing $s_1=\frac{1}{2}p_1l_1^2$, $s_2=\frac{1}{2}p_2l_2^2$, &c., we have

$$\log\left[\{\mathrm{J}_0(l_1x)\}^{p_1}\{\mathrm{J}_0(l_2x)\}^{p_2}\ldots\ldots\right]$$
$$=-\frac{x^2}{2}\Sigma(s)-\frac{x^4}{16}\Sigma\left(\frac{s^2}{p}\right)-\frac{x^6}{72}\Sigma\left(\frac{s^3}{p^2}\right),$$

and thus

$$2\pi\,\phi_n(r^2)=\int_0^\infty x\,dx\,\mathrm{J}_0(rx)e^{-\frac{1}{2}x^2\Sigma(s)}\left[1-\frac{x^4}{16}\Sigma\left(\frac{s^2}{p}\right)\right.$$
$$\left.-\frac{x^6}{72}\Sigma\left(\frac{s^3}{p^2}\right)+\frac{x^8}{512}\left\{\Sigma\left(\frac{s^2}{p}\right)\right\}^2\right] \quad . \quad . \quad . \quad (72)$$

As before, the leading term on the right is

$$I_1 = \frac{1}{\Sigma(s)} e^{-\frac{1}{2}r^2/\Sigma(s)}, \quad . \quad . \quad . \quad . \quad . \quad (73)$$

and the other integrals can be derived from it by differentiations with respect to $\Sigma(s)$. So far as the first two terms inclusive, we find

$$2\pi\,\phi_n(r^2) = \frac{e^{-\frac{1}{2}r^2/\Sigma(s)}}{\Sigma(s)} \left\{ 1 - \frac{\Sigma(s^2/p)}{4} \left(\frac{2}{\{\Sigma(s)\}^2} - \frac{2r^2}{\{\Sigma(s)\}^3} + \frac{r^4}{4\{\Sigma(s)\}^4} \right) \right\}, \quad . \quad . \quad . \quad (74)$$

from which we may fall back upon (45) by dropping the Σ and making $p=n$. In general $\Sigma(p)=n$. The approximation could be pursued.

Let us now suppose that the representative points are distributed over the area of a circle of radius L, all infinitesimal equal areas being equally probable. Of the total n the number (p) which fall between l and $l+dl$ should be $n \,.\, (2l\,dl/L^2)$, and thus

$$\Sigma(s) = \tfrac{1}{2}\Sigma(pl^2) = \frac{n}{L^2}\int_0^L l^3\,dl = \frac{nL^2}{4}, \quad . \quad . \quad . \quad (75)$$

$$\Sigma(s^2/p) = \tfrac{1}{4}\Sigma(pl^4) = \frac{n}{2L^2}\int_0^L l^5\,dl = \frac{nL^4}{12}. \quad . \quad . \quad . \quad (76)$$

Introducing these values in (74), we get

$$2\pi\,\phi_n(r^2) = \frac{4e^{-2r^2/nL^2}}{nL^2}\left\{1 - \frac{2}{3n}\left(1 - \frac{4r^2}{L^2} + \frac{2r^4}{n^2L^4}\right)\right\}. \quad (77)^*$$

A similar extension may be made in the problem where the component vectors are drawn in three dimensions.

XXXII. *On the Fundamental Law of Electrical Action.* By

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IN the present paper an attempt has been made to determine the law of attraction between two moving electrons, with the aid of the New Electrodynamics as modified by the Principle of Relativity. The problem is a

* The applicability of the second term (in $1/n$) to the case of an entirely random distribution over the area of the circle L is not over secure.

† Communicated by Prof. D. N. Mallik.