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The Group of Linear Homogeneous Substitutions on mq Variables which is defined by the Invariant

$$\phi \equiv \sum_{i=1}^{m} \xi_{i1}\xi_{i2}\ldots\xi_{iq}.$$

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1. For the case $q=2$, the continuous group leaving ϕ invariant has been fully studied by Sophus Lie; also the discontinuous group of linear substitutions in a Galois field leaving ϕ invariant has been recently studied by the writer in the *Proc. Lond. Math. Soc.* We suppose $q>2$ in the present paper.

2. For the sake of clearness, the method of investigation is first illustrated in the simple case $m=1$, $q=4$. The conditions imposed upon the substitution

$$S: \quad \xi_i' = \sum_{k=1}^{4} a_{ik}\xi_k \quad (i=1, \ldots, 4),$$

in order that it leave $\xi_1\xi_2\xi_3\xi_4$ invariant, are as follows:—

(1) $$a_{1j}a_{2j}a_{3j}a_{4j} = 0 \quad (j=1, \ldots, 4),$$

(2) $$a_{1j}a_{2j}a_{3j}a_{4k} + a_{1j}a_{2j}a_{4j}a_{3k} + a_{1j}a_{3j}a_{4j}a_{2k} + a_{2j}a_{3j}a_{4j}a_{1k} = 0 \quad (j, k = 1, \ldots, 4;\ j \neq k),$$

(3) $$a_{1j}a_{2j}a_{3k}a_{4k} + a_{1j}a_{3j}a_{2k}a_{4k} + a_{1j}a_{4j}a_{2k}a_{3k} + a_{2j}a_{3j}a_{1k}a_{4k} + a_{2j}a_{4j}a_{1k}a_{3k} + a_{3j}a_{4j}a_{1k}a_{2k} = 0,$$

(4) $$\Sigma\, a_{1j}a_{2j}a_{3k}a_{4l} = 0 \quad (j, k, l = 1, \ldots, 4;\ j \neq k \neq l),$$

the sum extending over 12 terms given by permuting 1, 2, 3, 4.

(5) $$\Sigma a_{i1} a_{j2} a_{k3} a_{l4} = 1,$$

the sum extending over 24 terms obtained by giving the set (i, j, k, l) every permutation of (1, 2, 3, 4).

Multiplying equation (3) by 2, we obtain an equation of the form (4) for $k = l$; multiplying (2) by 3, we obtain an equation of the form (4) for $l = j$. Hence, if j and k be any two distinct integers chosen from 1, 2, 3, 4, we have the set of four equations given by $l = 1, 2, 3, 4$:

$$\begin{aligned}&(a_{1j} a_{2j} a_{3k} + a_{1j} a_{3j} a_{2k} + a_{2j} a_{3j} a_{1k})\, a_{4l} \\ &+ (a_{1j} a_{2j} a_{4k} + a_{1j} a_{4j} a_{2k} + a_{2j} a_{4j} a_{1k})\, a_{3l} \\ &+ (a_{1j} a_{3j} a_{4k} + a_{1j} a_{4j} a_{3k} + a_{3j} a_{4j} a_{1k})\, a_{2l} \\ &+ (a_{2j} a_{3j} a_{4k} + a_{2j} a_{4j} a_{3k} + a_{3j} a_{4j} a_{2k})\, a_{1l} = 0.\end{aligned}$$

Regarding the quantities in the parentheses as the unknowns, the determinant of the coefficients is seen to equal the determinant

$$| a_{ik} | \quad (i, k = 1, \ldots, 4)$$

of the substitution S, and is therefore assumed to be not zero. Hence the quantities in parentheses are all zero. But the above equations hold true if $j = k$. Indeed, for $l = j$, it is obtained by multiplying (1) by 12; for $l \neq j$, it is given by multiplying (2) by 3. We have therefore the result

$$a_{1j} a_{2j} a_{3k} + a_{1j} a_{3j} a_{2k} + a_{2j} a_{3j} a_{1k} = 0 \quad (j, k = 1, \ldots, 4),$$

with three similar equations obtained from it by replacing the index 1 by 4, or 2 by 4, or, finally, 3 by 4. It follows that the products $a_{1j} a_{2j}$, $a_{1j} a_{3j}$, $a_{2j} a_{3j}$ are all zero; indeed not every determinant of the matrix

$$\left| \begin{matrix} a_{31} & a_{21} & a_{11} \\ a_{32} & a_{22} & a_{12} \\ a_{33} & a_{23} & a_{13} \\ a_{34} & a_{24} & a_{14} \end{matrix} \right|$$

can be zero, since the determinant of S does not vanish.

Replacing the indices 1, 2, 3 in turn by 4, we find that also the products $a_{4j} a_{2j}$, $a_{4j} a_{3j}$, $a_{4j} a_{1j}$ vanish. Hence three of the quantities

$a_{1j}, a_{2j}, a_{3j}, a_{4j}$ vanish for every $j = 1, \ldots, 4$. In order that the determinant of S be not zero, the four non-vanishing coefficients must lie in distinct columns (as well as in distinct rows). We have therefore the result:

THEOREM.—*Every quaternary linear homogeneous substitution leaving $\xi_1 \xi_2 \xi_3 \xi_4$ invariant can be generated by the substitutions $(\xi_i \xi_j)$ together with the following:—*

$$\xi_i' = a_{ii} \xi_i \quad (i = 1, \ldots, 4),$$

$$a_{11} a_{22} a_{33} a_{44} = 1.$$

The result holds for continuous groups, collineation groups, or for groups in any Galois field.

3. We will explain for the case $m = 1$, $q = 4$ the use of a symbol employed for brevity in the general case. The 24 terms of the left member of (5) are given by the expansion of the determinant

$$| a_{rs} | \quad (r, s = 1, \ldots, 4),$$

if all signs be taken positive. We therefore write (5) thus:

$$(5) \quad \left\{ \begin{matrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{matrix} \right\} = 1.$$

In the same notation relations (4), (3), (2), (1) become respectively

$$\tfrac{1}{2} \left\{ \begin{matrix} a_{1j} & a_{1j} & a_{1k} & a_{1l} \\ a_{2j} & a_{2j} & a_{2k} & a_{2l} \\ a_{3j} & a_{3j} & a_{3k} & a_{3l} \\ a_{4j} & a_{4j} & a_{4k} & a_{4l} \end{matrix} \right\} = 0, \qquad \tfrac{1}{4} \left\{ \begin{matrix} a_{1j} & a_{1j} & a_{1k} & a_{1k} \\ a_{2j} & a_{2j} & a_{2k} & a_{2k} \\ a_{3j} & a_{3j} & a_{3k} & a_{3k} \\ a_{4j} & a_{4j} & a_{4k} & a_{4k} \end{matrix} \right\} = 0,$$

$$\tfrac{1}{6} \left\{ \begin{matrix} a_{1j} & a_{1j} & a_{1j} & a_{1k} \\ a_{2j} & a_{2j} & a_{2j} & a_{2k} \\ a_{3j} & a_{3j} & a_{3j} & a_{3k} \\ a_{4j} & a_{4j} & a_{4j} & a_{4k} \end{matrix} \right\} = 0, \qquad \tfrac{1}{24} \left\{ \begin{matrix} a_{1j} & a_{1j} & a_{1j} & a_{1j} \\ a_{2j} & a_{2j} & a_{2j} & a_{2j} \\ a_{3j} & a_{3j} & a_{3j} & a_{3j} \\ a_{4j} & a_{4j} & a_{4j} & a_{4j} \end{matrix} \right\} = 0.$$

4. Consider a general substitution on mq indices,

$$S: \quad \xi'_{ij} = \sum_{\substack{k=1\dots m\\ l=1\dots q}} a^{ij}_{kl}\xi_{kl} \quad \begin{pmatrix} i = 1 \dots m \\ j = 1 \dots q \end{pmatrix}.$$

It transforms ϕ into the function

$$\phi' \equiv \sum_{i=1}^{m} \left\{ \left(\sum_{k_1, l_1} a^{i\,1}_{k_1 l_1} \xi_{k_1 l_1} \right) \left(\sum_{k_2, l_2} a^{i\,2}_{k_2 l_2} \xi_{k_2 l_2} \right) \dots \left(\sum_{k_q, l_q} a^{i\,q}_{k_q l_q} \xi_{k_q l_q} \right) \right\},$$

where $k_1, \dots, k_q = 1, \dots, m$; $l_1, \dots, l_q = 1, \dots, q$ independently.

Employing the symbol explained above, ϕ' takes the form

$$\sum \left[\frac{1}{C} \sum_{i=1}^{m} \begin{Bmatrix} a^{i\,1}_{k_1 l_1} & a^{i\,1}_{k_2 l_2} & \dots & a^{i\,1}_{k_q l_q} \\ a^{i\,2}_{k_1 l_1} & a^{i\,2}_{k_2 l_2} & \dots & a^{i\,2}_{k_q l_q} \\ \dots & \dots & \dots & \dots \\ a^{i\,q}_{k_1 l_1} & a^{i\,q}_{k_2 l_2} & \dots & a^{i\,q}_{k_q l_q} \end{Bmatrix} \xi_{k_1 l_1} \xi_{k_2 l_2} \dots \xi_{k_q l_q} \right].$$

summed for every *combination* of q pairs of integers (k_1, l_1), (k_2, l_2), ..., (k_q, l_q), the k's being selected from the integers $1, \dots, m$, and the l's from $1, \dots, q$, repetitions allowed. To determine the value of C for a given combination of q pairs (k_i, l_i), we separate them into τ sets

$$(6) \quad \begin{cases} (k_1, l_1) = (k_2, l_2) = \dots = (k_{t_1}, l_{t_1}), \\ (k_{t_1+1}, l_{t_1+1}) = \dots = (k_{t_1+t_2}, l_{t_1+t_2}), \\ \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \\ (k_{t_1+t_2+\dots+t_{\tau-1}+1}, l_{t_1+t_2+\dots+t_{\tau-1}+1}) = \dots = (k_{t_1+t_2+\dots+t_\tau}, l_{t_1+t_2+\dots+t_\tau}), \end{cases}$$

those in different sets being distinct pairs. We readily see that

$$C = \frac{(t_1+t_2+\dots+t_\tau)!}{t_1!\, t_2! \dots t_\tau!}, \quad t_1+t_2+\dots+t_\tau = q.$$

The identity $\phi' = \phi$ therefore requires the following relations :—*

$$(7) \quad \sum_{i=1}^{m} \begin{Bmatrix} a^{i1}_{j1} & a^{i1}_{j2} & \dots & a^{i1}_{jq} \\ a^{i2}_{j1} & a^{i2}_{j2} & \dots & a^{i2}_{jq} \\ \dots & \dots & \dots \\ a^{iq}_{j1} & a^{iq}_{j2} & \dots & a^{iq}_{jq} \end{Bmatrix} = 1 \quad (j = 1, \dots, m),$$

* The numerical factor in (8) must be retained when working with linear groups in certain Galois fields; indeed, the reciprocal of this factor might be zero, in which case it could not be omitted.

$$(8)\quad \frac{t_1!\,t_2!\,\dots\,t_r!}{q!}\sum_{i=1}^{m}\begin{Bmatrix} a^{i\,1}_{k_1 l_1} & a^{i\,1}_{k_2 l_2} & \dots & a^{i\,1}_{k_q l_q} \\ \dots & \dots & \dots & \dots \\ a^{i\,q}_{k_1 l_1} & a^{i\,q}_{k_2 l_2} & \dots & a^{i\,q}_{k_q l_q} \end{Bmatrix} = 0,$$

holding for every combination of q pairs $(k_1, l_1), \dots, (k_q, l_q)$, except the combinations $(k_1, 1), (k_1, 2), \dots, (k_1, q)$, that can be formed from $(1, 1), \dots, (1, q), \dots, (m, 1), \dots, (m, q)$. The numerical factor is determined by (6).

5. Consider the totality of relations (8) in which $(k_1, l_1) = (k_2, l_2)$. Multiply each by the factor $\frac{1}{2}\frac{q!}{t_1!\dots t_r!}$, which will be an integer since the pair (k_1, l_1) is of multiplicity at least 2. Group together the mq relations in which $k_1, k_2, \dots, k_{q-1}$; $l_1, l_2, \dots, l_{q-1}$ have arbitrarily fixed values, while k_q runs from 1 to m, l_q from 1 to q. Note that we tacitly assume that $q > 2$. We may expand the general one of these mq relations into the form

$$\sum_{i=1}^{m}\left[\tfrac{1}{2}\begin{Bmatrix} a^{i\,1}_{k_1 l_1} & \dots\dots & a^{i\,1}_{k_{q-1} l_{q-1}} \\ \dots & \dots & \dots \\ a^{i\,q-1}_{k_{q-1} l_{q-1}} & \dots & a^{i\,q-1}_{k_{q-1} l_{q-1}} \end{Bmatrix} a^{i\,q}_{k_q l_q} + \dots + \tfrac{1}{2}\begin{Bmatrix} a^{i\,2}_{k_1 l_1} & \dots & a^{i\,2}_{k_{q-1} l_{q-1}} \\ \dots & \dots & \dots \\ a^{i\,q}_{k_1 l_1} & \dots & a^{i\,q}_{k_{q-1} l_{q-1}} \end{Bmatrix} a^{i\,1}_{k_q l_q}\right] = 0.$$

In these mq equations, given by $k_q = 1, \dots, m$; $l_q = 1, \dots, q$, the mq quantities in brackets are the same throughout, and may be regarded as the unknown quantities. The determinant of their coefficients

$$\left(a^{i\,j}_{k_q l_q}\right) \quad \begin{pmatrix} i, k_q = 1, \dots, m \\ j, l_q = 1, \dots, q \end{pmatrix}$$

is not zero, being the determinant of the substitution S. Hence the mq unknowns are all zero, viz.,

$$(9)\quad \tfrac{1}{2}\begin{Bmatrix} a^{i\,\alpha_1}_{k_1 l_1} & a^{i\,\alpha_1}_{k_2 l_2} & \dots & a^{i\,\alpha_1}_{k_{q-1} l_{q-1}} \\ \dots & \dots & \dots & \dots \\ a^{i\,\alpha_{q-1}}_{k_1 l_1} & a^{i\,\alpha_{q-1}}_{k_2 l_2} & \dots & a^{i\,\alpha_{q-1}}_{k_{q-1} l_{q-1}} \end{Bmatrix} = 0,$$

where $\alpha_1, \alpha_2, \dots, \alpha_{q-1}$ are distinct integers chosen arbitrarily from $1, 2, \dots, q$, and $i, k_1, k_3, \dots, k_{q-1} = 1, \dots, m$; $l_1, l_3, \dots, l_{q-1} = 1, \dots, q$ independently. If $q-1 = 2$, we have the result (11′) below.

If $q-1>2$, we consider the relations (9) in which $i, k_1, k_3, \ldots, k_{q-2}$; $l_1, l_3, \ldots, l_{q-2}$ have arbitrarily fixed values, while k_{q-1} runs from 1 to m, l_{q-1} from 1 to q. The general one of these mq relations may be expanded into the form

$$\tfrac{1}{2}\begin{Bmatrix} a_{k_1 l_1}^{i\ a_1} & \cdots & a_{k_{q-2} l_{q-2}}^{i\ a_1} \\ \cdots & \cdots & \cdots \;\; \cdots \\ a_{k_1 l_1}^{i\ a_{q-2}} & \cdots & a_{k_{q-2} l_{q-2}}^{i\ a_{q-2}} \end{Bmatrix} a_{k_{q-1} l_{q-1}}^{i\ a_{q-1}} + \ldots + \tfrac{1}{2}\begin{Bmatrix} a_{k_1 l_1}^{i\ a_2} & \cdots & a_{k_{q-2} l_{q-2}}^{i\ a_2} \\ \cdots & \cdots & \cdots \;\; \cdots \\ a_{k_1 l_1}^{i\ a_{q-1}} & \cdots & a_{k_{q-2} l_{q-2}}^{i\ a_{q-1}} \end{Bmatrix} a_{k_{q-1} l_{q-1}}^{i\ a_1} = 0.$$

Consider as unknowns the $q-1$ quantities in brackets. The matrix of the coefficients is composed of $q-1$ rows of the determinant of S. Hence not every determinant of order $q-1$ in the matrix is zero. Hence the $q-1$ unknowns are all zero, viz.,

$$\text{(10)} \qquad \tfrac{1}{2}\begin{Bmatrix} a_{k_1 l_1}^{i\ b_1} & a_{k_2 l_2}^{i\ b_1} & \cdots & a_{k_{q-2} l_{q-2}}^{i\ b_1} \\ \cdots & \cdots & \cdots & \cdots \;\; \cdots \\ a_{k_1 l_1}^{i\ b_{q-2}} & a_{k_2 l_2}^{i\ b_{q-2}} & \cdots & a_{k_{q-2} l_{q-2}}^{i\ b_{q-2}} \end{Bmatrix} = 0,$$

where $b_1, b_2, \ldots, b_{q-2}$ are distinct integers chosen arbitrarily from $1, 2, \ldots, q$, while $i, k_1, k_3, \ldots, k_{q-2} = 1, 2, \ldots, m$; $l_1, l_3, \ldots, l_{q-2} = 1, 2, \ldots, q$ independently. If $q-2=2$, we have the equations (11′) below.

If $q-2>2$, we proceed as before. Finally, we reach the result

$$\text{(11′)} \qquad \tfrac{1}{2}\begin{Bmatrix} a_{k_1 l_1}^{i\ c} & a_{k_2 l_2}^{i\ c} \\ a_{k_1 l_1}^{i\ d} & a_{k_2 l_2}^{i\ d} \end{Bmatrix} = 0,$$

where c, d are distinct integers chosen arbitrarily from $1, 2, \ldots, q$. Since $(k_1, l_1) = (k_2, l_2)$, we have the result

$$\text{(11)} \qquad a_{kl}^{ic}\, a_{kl}^{id} = 0 \quad \begin{pmatrix} i, k = 1, \ldots, m \\ l, c, d = 1, \ldots, q;\ c \neq d \end{pmatrix}.$$

6. To derive the result (11′), the only use made of the hypothesis $(k_1, l_1) = (k_2, l_2)$ was to distinguish between the relations (7) and (8). But relations (8), and not (7), are always defined if we take $k_2 \neq k_1$. Hence, if we drop the factor $\frac{1}{2}$ throughout, the investigation in § 5 leads at once to the result

$$\text{(12)} \qquad \begin{Bmatrix} a_{k_1 l_1}^{i\ c} & a_{k_2 l_2}^{i\ c} \\ a_{k_1 l_1}^{i\ d} & a_{k_2 l_2}^{i\ d} \end{Bmatrix} = 0 \quad \begin{pmatrix} i, k_1, k_2 = 1, \ldots, m;\ k_1 \neq k_2 \\ c, d = 1, \ldots, q;\ c \neq d \\ l_1, l_2 = 1, \ldots, q \end{pmatrix}.$$

In virtue of the relations (11) and (12), the relations (8) are all satisfied identically.

7. The coefficients of S being not all zero, we may take

$$a_{k_1 l_1}^{i_1 j_1} \neq 0.$$

Then, by (11) and (12) respectively,

$$a_{k_1 l_1}^{i_1 s} = 0 \quad (s = 1, \ldots, q;\ s \neq j_1),$$

$$a_{k_2 l_2}^{i_1 s} = 0 \quad \begin{pmatrix} k_2 = 1, \ldots, m;\ k_2 \neq k_1 \\ l_2 = 1, \ldots, q \end{pmatrix}.$$

Hence the substitution S affects $q-1$ of the indices as follows :—

$$\xi'_{i_1 s} = \sum_{l \neq l_1}^{l=1 \ldots q} a_{k_1 l}^{i_1 s} \xi_{k_1 l} \quad (s = 1, \ldots, q;\ s \neq j_1).$$

Since the determinant of S is not zero, not all of these coefficients are zero, for example,

$$a_{k_1 l_2}^{i_1 j_2} \neq 0 \quad (j_2 \neq j_1,\ l \neq l_1).$$

Then, by (11), $$a_{k_1 l_2}^{i_1 r} = 0 \quad (r = 1, \ldots, q;\ r \neq j_2).$$

Hence S affects $q-2$ of the indices as follows :—

$$\xi'_{i_1 r} = \sum_{l \neq l_1, l_2}^{l=1 \ldots q} a_{k_1 l}^{i_1 r} \xi_{k_1 l} \quad (r = 1, \ldots, q;\ r \neq j_1, j_2).$$

Not all of these coefficients are zero, for example,

$$a_{k_1 l_3}^{i_1 j_3} \quad (j_3 \neq j_1, j_2;\ l_3 \neq l_1, l_2).$$

Proceeding in like manner, we reach, after $q-1$ steps, an index $\xi_{i_1 j_q}$ which S replaces by

$$a_{k_1 l_q}^{i_1 j_q} \xi_{k_1 l_q}.$$

Besides, we have proven the existence of q coefficients

$$(13) \quad a_{k_1 l_1}^{i_1 j_1},\ a_{k_1 l_2}^{i_1 j_2},\ a_{k_1 l_3}^{i_1 j_3},\ \ldots,\ a_{k_1 l_q}^{i_1 j_q},$$

all different from zero, in which $l_1, l_2, \ldots, l_q$ are all distinct, and $j_1, j_2, \ldots, j_q$ all distinct, and consequently each set a permutation of the integers $1, 2, \ldots, q$.

The above process may therefore be repeated, starting with any one of the set (13). We conclude that S affects q of the indices as follows :—

$$\xi'_{i_1 j_t} = a^{i_1\, j_t}_{k_1\, l_t}\, \xi_{k_1 l_t} \quad (t = 1, \ldots, q).$$

8. Since the determinant of S is not zero, we may take

$$a^{i_2 j}_{k_2 l} \neq 0 \quad (i_2, k_2) \neq (i_1, k_1).$$

By the argument of § 7, S affects the q indices $\xi_{i_2 1}, \xi_{i_2 2}, \ldots, \xi_{i_2 q}$ as follows :—

$$\xi'_{i_2 j} = a^{i_2 j}_{k_2 l}\, \xi_{k_2 l} \quad (j = 1, \ldots, q).$$

Applying the process m times, we see that S has the form

$$\xi'_{ij} = a^{i\,j}_{k\,l}\, \xi_{k\,l} \quad (i = 1, m;\; j = 1, \ldots, q),$$

where k and l are such functions of i and j that, in the determinant of the coefficients of S, no two non-vanishing coefficients lie in the same column or in the same row.

9. *It follows that every linear homogeneous substitution S leaving ϕ invariant is the product of a literal substitution L on the mq letters ξ_{ij} with the systems of imprimitivity*

$$\begin{array}{l} \xi_{11},\ \xi_{12},\ \ldots,\ \xi_{1q}, \\ \xi_{21},\ \xi_{22},\ \ldots,\ \xi_{2q}, \\ \xi_{m1},\ \xi_{m2},\ \ldots,\ \xi_{mq}, \end{array}$$

by a linear substitution M of the form

$$\xi'_{ij} = a^{ij}_{ij}\, \xi_i \quad (i = 1, m;\; j = 1, \ldots, q),$$

where, by (7), $$a^{i1}_{i1}\, a^{i2}_{i2} \ldots a^{iq}_{iq} = 1 \quad (i = 1, \ldots, q).$$

The totality of linear substitutions M form a commutative group which is an invariant sub-group of the total group leaving ϕ invariant. The quotient group is the group of substitutions L. The latter group has an invariant sub-group, the direct product of m symmetric groups on q letters, the quotient group being the symmetric group on m letters, viz., the m systems of imprimitivity.

We have therefore determined completely the structure of the largest linear group leaving invariant the function

$$\phi \equiv \sum_{i=1}^{m} \xi_{i1}\, \xi_{i2} \ldots \xi_{iq},$$

whether the coefficients be taken in the field of continuous quantity, as roots of unity, or, finally, as marks in an arbitrary Galois field.

10. *Note.*—While our final result enables us to give the reciprocal of any substitution S leaving ϕ invariant, it is nevertheless interesting to verify directly by means of the relations (7) and (8) that S^{-1} has the form

$$\xi'_{ij} = \sum_{\substack{k=1\dots m \\ l=1\dots q}} A^{kl}_{ij}\,\xi_{kl} \quad (i = 1, \dots, m;\ j = 1, \dots, q),$$

where A^{kl}_{ij} denotes the "adjoint" of a^{kl}_{ij} in the symbol

$$\begin{Bmatrix} a^{k1}_{i1} & a^{k1}_{i2} & \dots & a^{k1}_{iq} \\ \dots & \dots & \dots & \dots \\ a^{kq}_{i1} & a^{kq}_{i2} & \dots & a^{kq}_{iq} \end{Bmatrix}.$$

For example,
$$A^{k1}_{i2} \equiv \begin{Bmatrix} a^{k2}_{i1} & a^{k2}_{i3} & \dots & a^{k2}_{iq} \\ \dots & \dots & \dots & \dots \\ a^{kq}_{i} & a^{kq}_{i3} & \dots & a^{kq}_{iq} \end{Bmatrix}.$$

We verify our statement by showing that $SS^{-1} = 1$. Indeed, SS^{-1} replaces the general index ξ_{ij} by

$$\sum_{\substack{k=1\dots m \\ l=1\dots q}} A^{kl}_{ij} \Big(\sum_{\substack{r=1\dots m \\ s=1\dots q}} a^{kl}_{rs}\,\xi_{rs} \Big) = \sum_{\substack{r=1\dots m \\ s=1\dots q}} \Big\{ \sum_{k=1\dots m} \Big(\sum_{l=1\dots q} A^{kl}_{ij}\, a^{kl}_{rs} \Big) \Big\}\, \xi_{rs}.$$

But the quantity in brackets is, by (7) and (8),

$$\sum_{k=1\dots m} \begin{Bmatrix} a^{k1}_{i1} & \dots & a^{k1}_{ij-1} & a^{k1}_{rs} & a^{k1}_{ij+1} & \dots & a^{k1}_{iq} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ a^{kq}_{i1} & \dots & a^{kq}_{ij-1} & a^{kq}_{rs} & a^{kq}_{ij+1} & \dots & a^{kq}_{iq} \end{Bmatrix} = \begin{matrix} 0 \text{ if } (r, s) \neq (i, j), \\ 1 \text{ if } (r, s) = (i, j). \end{matrix}$$

Hence SS^{-1} replaces ξ_{ij} by ξ_{ij}.