Some problems of Diophantine approximation: The lattice-points of a right-angled triangle.

(Second memoir.)

By G. H. HARDY in Oxford and J. E. LITTLEWOOD in Cambridge.

1. Introduction.

1.1. This memoir is a sequel to one published recently in the *Proceedings of the London Mathematical Society*[1]). It contains the proofs of a number of theorems enunciated in an appendix to our former memoir, together with a considerable amount of additional matter.

The problems which we consider have occupied us at intervals since 1912, when we referred to them briefly in a communication to the Cambridge Congress[2]), and indicated certain questions which we were then unable to answer. In the meantime they have attracted the attention of Herr HECKE[3]) and Herr OSTROWSKI[4]), who have dealt with them in two very beautiful memoirs published recently in this journal, and to whom we are indebted for this opportunity of publishing our own.

The very remarkable analysis of HECKE is mainly transcendental, while OSTROWSKI's is entirely elementary, and we use both elementary and transcendental methods. Our transcendental method is entirely unlike HECKE's, and little need be said as regards the relations between his results and ours. The relations of our elementary work to OSTROWSKI's are a good deal closer. Our method, depending as it does on formulae like those of SYLVESTER and LERCH[5]), is fundamentally different, but the results are to a considerable extent the same. A detailed analysis

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1) G. H. HARDY and J. E. LITTLEWOOD, "Some problems of Diophantine approximation: The lattice-points of a right-angled triangle", *Proc. London Math. Soc.* (2), 20 (1921), 15—36. We refer to this memoir as *1.*

2) G. H. HARDY and J. E. LITTLEWOOD, "Some problems of Diophantine approximation", *Proceedings of the fifth international congress of mathematicians,* 1912, 1, 223—229.

3) E. HECKE, "Über analytische Funktionen und die Verteilung von Zahlen mod. Eins", *Hamburg. Math. Abh.* 1 (1921), 54—76. We refer to this as *H.*

4) A. OSTROWSKI, "Bemerkungen zur Theorie der Diophantischen Approximationen", *ibid.*, 77—98. We refer to this as *O.*

5) See 3.1.

of the points of resemblance and difference would occupy a good deal of space and seems to us unnecessary, though we indicate the theorems which have been proved by OSTROWSKI as they occur. We should add one word, however, as to the relative advantages of OSTROWSKI's method and our own. In some parts of the theory the advantage of OSTROWSKI's method seems to us incontestable; in others there is little between them; and in others the advantage seems to lie with ours. It seems to us desirable to develop the whole theory systematically from our own point of view; but where OSTROWSKI's method is clearly simpler, we content ourselves with an outline of our demonstrations, suppressing the algebraical details of our work and condensing our argument to the limit of intelligibility. In particular we have followed this course in 3.4.

1.2. All our theorems involve an irrational number θ, which we generally suppose positive, less than 1, and expressed as a simple continued fraction

(1.21) $$\theta = \frac{1}{a_1+}\frac{1}{a_2+}\frac{1}{a_3+}\cdots.$$

We write

(1.22) $$\theta = \frac{1}{a_1+\theta_1}, \quad \theta_1 = \frac{1}{a_2+\theta_2}, \quad \ldots,$$

and denote the convergents to (1.21) by

$$\frac{p_1}{q_1} = \frac{1}{a_1}, \quad \frac{p_2}{q_2} = \frac{a_2}{a_1 a_2+1}, \quad \ldots.$$

We shall make continual use of the two lemmas which follow, which are trivial, but very useful, and which seem to have escaped attention.

Lemma 1. *We have*

(1.23) $$\theta_r\theta_{r+1} < \frac{1}{2}.$$

More generally

(1.231) $$\theta_r\theta_{r+1}\ldots\theta_{r+s-1} < \frac{1}{u_s},$$

where u_s is the s-th term of FIBONACCI's series 1, 2, 3, 5, 8, 13, ...

We deduce this from

Lemma 2. *We have*

(1.24.) $$\frac{1}{2\theta\theta_1\ldots\theta_{r-1}} < q_r < \frac{1}{\theta\theta_1\ldots\theta_{r-1}}.$$

For

$$q_r+\theta_r q_{r-1} = (a_r+\theta_r)q_{r-1}+q_{r-2} = \frac{q_{r-1}+\theta_{r-1}q_{r-2}}{\theta_{r-1}},$$

and so

$$q_r + \theta_r q_{r-1} = \frac{1}{\theta\,\theta_1 \ldots . \theta_{r-1}},$$

which proves the lemma.

To deduce Lemma 1 we observe that (1.24) gives

$$\theta\,\theta_1 \ldots . \theta_{s-1} < \frac{1}{q_s},$$

and that, for given s, q_s is a minimum when $a_1 = a_2 = \ldots = a_s = 1$, in which case $q_s = u_s$. Taking now θ_r for θ we obtain the desired result.

1.3. We write, as usual, $[x]$ for the integral part of x, and

(1.31) $$(x) = x - [x], \quad \{x\} = x - [x] - \frac{1}{2}.$$

Thus $\{x\}$ is the arithmetical function denoted, by HECKE and OSTROWSKI, by $R(x) - \frac{1}{2}$. Further we write

(1.32) $$\overline{x} = x - X$$

where X is the integer nearest to x. If x is of the form $n + \frac{1}{2}$, we take $\overline{x} = \frac{1}{2}$.

Throughout our argument the letter A (or occasionally B, C, ...) denotes a positive constant. This constant may be absolute, or may depend upon the parameters involved in the theorem in question; it will not generally be the *same* constant in successive inequalities. The O's and o's which occur involve constants implicitly. It will generally be obvious on what, if any, parameters these constants depend.

We shall frequently be concerned with conditions of the type

(1.331) $$n^h \,|\sin n\theta\pi\,| > A \qquad (n \geqq 1),$$

or

(1.332) $$n^h \,|\sin n\theta\pi\,| < A \qquad (n = n_j),$$

where $h \geqq 1$, and the notation implies that the second inequality is satisfied for an infinite sequence n_1, n_2, ..., n_j of values of n. These conditions are obviously equivalent to the corresponding conditions in which $\sin n\theta\pi$ is replaced by $\overline{n\theta}$. Further, (1.331) and (1.332) are equivalent to

(1.341) $$q_{r+1} < A\,q_r^h \qquad (r \geqq 1),$$

(1.342) $$q_{r+1} > A\,q_r^h \qquad (r = r_j),$$

and these again, by Lemma 2, to

$$\frac{1}{\theta_r} < \frac{A}{(\theta\,\theta_1 \ldots . \theta_{r-1})^{h-1}} \qquad (r \geqq 1), \tag{1.351}$$

$$\frac{1}{\theta_r} > \frac{A}{2^h(\theta\,\theta_1 \ldots . \theta_{r-1})^{h-1}} \qquad (r = r_j). \tag{1.352}$$

It is well-known that a condition of the type (1.331) is satisfied by every algebraical θ.

The A's of these inequalities may be absolute or may depend on θ and h. If absolute in (1.331) and (1.332), they are absolute in the other inequalities.

2. The analytic treatment of the triangle problem.

2.1. In this section we continue the study of the "triangle" problem (Problem A of *1*) by analytic methods. We denote by $N(\eta)$ the number of lattice-points inside the triangle whose sides are

$$x = 0, \quad y = 0, \quad wx + w'y = \eta > 0,$$

where w and w' are two positive numbers whose ratio $\theta = \frac{w}{w'}$ is irrational. We proved in *1* that

$$N(\eta) = R(\eta) + U(\eta) = R(\eta) + \Phi(\eta) + S(\eta) \tag{2.11}$$

where

$$R(\eta) = \frac{\eta^2}{2\,w\,w'} - \frac{\eta}{2\,w} - \frac{\eta}{2\,w'}, \tag{2.111}$$

$$\frac{\eta}{w} = \left[\frac{\eta}{w}\right] + f, \quad \frac{\eta}{w'} = \left[\frac{\eta}{w'}\right] + f', \tag{2.112}$$

$$\Phi(\eta) = \frac{1}{2} f + \frac{1}{2}\,\theta f(1-f), \tag{2.113}$$

$$S(\eta) = \sum_{1 \leqq \mu \leqq \frac{\eta}{w}} \{\mu\,\theta - f'\}. \tag{2.114}$$

Our problem is the study of $U(\eta)$, or, since $\Phi(\eta) = O(1)$, of $S(\eta)$. We proved in *1*, (*a*) that

$$U(\eta) = o(\eta) \tag{2.12}$$

for every irrational θ, (*b*) that this result is the most that is universally true, (*c*) that

$$U(\eta) = O(\log \eta) \tag{2.13}$$

when θ has bounded quotients, (*d*) that there are θ's, with bounded quotients, for which each of the inequalities

(2.14) $$U(\eta) > A \log \eta, \quad U(\eta) < -A \log \eta$$

is satisfied for arbitrarily large values of η, and (*e*) that

(2.15) $$U(\eta) = O(\eta^{\alpha}),$$

where $\alpha = \alpha(\theta) < 1$, whenever θ satisfies an inequality of the type (1.331), and in particular whenever θ is algebraic. Of these results (*a*)—(*d*) were proved by elementary reasoning, and (*e*) analytically. Our immediate object is to prove more precise results in place of (*e*).

We denote by $\zeta_2(s) = \zeta_2(s, a, w, w')$ the analytic function defined, when the real part σ of $s = \sigma + it$ is greater than 2, by the series

$$\zeta_2(s) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{1}{(a + mw + nw')^s} = \sum_p l_p^{-s}$$

where a is positive and l_p^{-s} has its principal value. This function is a degenerate case of the "double Zeta-function" of BARNES[1]). Its principal properties, so far as they are relevant to our investigations, are summarised in *1*.

In this section the A's, B's, ... are in general not absolute but functions of the parameters θ, h, ...

2.2. Lemma 3. *If* $h \geqq k \geqq 1$ *and*

(2.21) $$n^h |\sin n\theta\pi|^k > A$$

then

(2.22) $$S_m = \sum_1^m \frac{1}{n^h |\sin n\theta\pi|^k} = O(\log m)^2.$$

It is plain that neither hypothesis nor conclusion is affected if we replace $\sin n\theta\pi$ by $\overline{n\theta}$. We have therefore

(2.231) $$n^h |\overline{n\theta}|^k > B\text{[2]}),$$

and, if we define h_n by

(2.232) $$n^{h_n} |\overline{n\theta}|^k = B,$$

we have $h_n < h$. Consider now the sum

(2.224) $$T_m = \sum_m^{2m} \frac{1}{n^h |\overline{n\theta}|^k} = \sum_m^{2m} u_n.$$

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[1]) E. W. BARNES, "A memoir on the double Gamma-function", *Phil. Trans. Roy. Soc.* (A), 196 (1901), 265—387.

[2]) B is the same constant throughout this sub-section.

The terms of T_m for which $h_n \leqq h-1$ contribute

$$O\left(\sum_m^{2m}\frac{1}{n}\right)=O(1).$$

We classify the remaining terms as follows. We choose a positive integer η, write

$$\mathfrak{h}_r = h-1+\frac{r}{\eta} \qquad (r=0, 1, 2, \ldots, \eta-1)$$

and call a typical term u_n of T_m a *term of class* r if $\mathfrak{h}_r \leqq h_n < \mathfrak{h}_{r+1}$. If u_n is of class r,

$$|\overline{n\theta}|^k \leqq B n^{-\mathfrak{h}_r}.$$

But

$$|\overline{s\theta}|^k > B s^{-h} > 2^k B n^{-\mathfrak{h}_r}$$

if

$$0 < s < C n^{\frac{\mathfrak{h}_r}{h}} \tag{2.25}$$

and then

$$|\overline{(n+s)\theta}|^k > B n^{-\mathfrak{h}_r} > B(n+s)^{-\mathfrak{h}_r}$$

for all values of s which satisfy (2.25). Hence no term u_{n+s} corresponding to such a value of s is a term of class as high as r.

The number of terms of class r is therefore $O\left(m^{1-\frac{\mathfrak{h}_r}{h}}\right)$, and their contribution to T_m is

$$O\left(m^{1-\frac{\mathfrak{h}_r}{h}-h+\mathfrak{h}_{r+1}}\right) = O\left(m^{\frac{1}{\eta}}\right),$$

since

$$1-\frac{\mathfrak{h}_r}{h}-h+\mathfrak{h}_{r+1} = \frac{1}{\eta}-\left(1-\frac{1}{h}\right)\left(1-\frac{r}{\eta}\right) < \frac{1}{\eta}.$$

It follows that

$$T_m = O\left(\eta\, m^{\frac{1}{\eta}}\right) = O(\log m), \tag{2.26}$$

since we may take $\eta = [\log m]$.

If now we define ν by $D \leqq \frac{m}{2^\nu} \leqq 2D$, we have

$$U_m = \sum_1^m \frac{1}{n^h |\overline{n\theta}|^k} = O(1) + O\left(\sum_{\mu=1}^{\nu} T_{2^{-\mu}m}\right)$$
$$= O(1) + O\left(\sum_{\mu=1}^{\nu} \log\frac{m}{2^\mu}\right) = O(\log m)^2,$$

which is equivalent to (2.22).

As a corollary we have

Lemma 4. *If* (2.21) *is satisfied, the series*

$$\sum \frac{1}{n^{h+\varepsilon}|\sin n\theta\pi|^k}$$

is convergent for every positive ε.

The case of most importance is that in which $k=1$, when (2.21) reduces to (1.331).

2.31. We proceed to establish an analytical formula for the sum

$$W_k(\xi) = \sum_{l_p \leq \xi} (\xi - l_p)^k. \tag{2.311}$$

The k here has no connection with that of 2.2.

In order to abbreviate our formulae we adopt the following convention. We are often concerned with associated pairs of series, of the forms $\sum \Phi(m, w, w')$ and $\sum \Phi(m, w', w)$; and we write generally

$$X(w, w')\sum \Phi(m, w, w') + X(w', w)\sum \Phi(m, w', w) = (X(w, w')\sum \Phi(m, w, w'))^*. \tag{2.312}$$

Such associated pairs of series have been considered by various writers, and in particular by LERCH[1]). We shall also sometimes use a similar notation when there is no summation.

We recall the formula[2])

$$\frac{\zeta_2(s, a, w, w')}{(2\pi)^{s-1}\Gamma(1-s)} = \left(\frac{1}{w^s}\sum_{m=1}^{\infty} \frac{\sin\left(\frac{2m\pi}{w}\left(\frac{1}{2}w' - a\right) + \frac{1}{2}(1-s)\pi\right)}{m^{1-s}\sin\frac{mw'\pi}{w}}\right)^*. \tag{2.313}$$

This formula is valid whenever $0 < a \leqq w + w'$ and the two series on the right are absolutely convergent.

2.32. Theorem 1. *Suppose that* (1.331) *is satisfied and that* $k > h - 1$. *Then*

$$W_k(\xi) = V_k(\xi) - (2\pi)^{-k-1}\Gamma(k+1)\left(w^k \sum_{m=1}^{\infty} \frac{\cos\left(\frac{2m\pi}{w}\left(\frac{1}{2}w' + \xi - a\right) - \frac{1}{2}k\pi\right)}{m^{k+1}\sin\frac{mw'\pi}{w}}\right)^* + O(\xi^{k+1-q}), \tag{2.321}$$

where

$$V_k(\xi) = \sum_{\mu=0}^{q} C_\mu \xi^{k+2-\mu}, \tag{2.3211}$$

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[1]) See, for example, M. LERCH, "Sur une série analogue aux fonctions modulaires", *Comptes Rendus*, 18 April 1904; G. H. HARDY, "On certain series of discontinuous functions connected with the modular functions", *Quarterly Journal* (1904), 93—123; and writings of RIEMANN and H. J. S. SMITH there referred to.

[2]) (6.221) of *1.*

the C's being constants, of which

(2.3212) $$C_0 = \frac{1}{(k+1)(k+2)ww'}, \quad C_1 = \frac{w+w'-2a}{2(k+1)ww'},$$

and q is the integer such that $k+1 < q \leqq k+2$.

We suppose for the present that $k > h - \frac{1}{2}$. Since $k > 0$, we have[1])

(2.322) $$W_k(\xi) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \zeta_2(s) \frac{\Gamma(k+1)\Gamma(s)}{\Gamma(k+1+s)} \xi^{k+s} ds,$$

if $c > 2$. We choose δ so that $\frac{1}{2} < \frac{1}{2} + \delta < 1$, $k > h - \frac{1}{2} + \delta$, and $\gamma = \frac{1}{2} - k + \delta$ is not an integer. We have then

(2.323) $$\frac{1}{2} - k < \gamma < 1 - h$$

and

(2.324) $$\zeta_2(s) = O\left(|t|^{\frac{1}{2}-\gamma}\right) = O(|t|^k)$$

uniformly for $\sigma \geqq \gamma$[2]).

We may therefore apply CAUCHY's Theorem to the strip $\gamma \leqq \sigma \leqq c$, and we obtain

(2.325) $$W_k(\xi) = U_k(\xi) + \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \zeta_2(s) \frac{\Gamma(k+1)\Gamma(s)}{\Gamma(k+1+s)} \xi^{k+s} ds,$$

where

(2.326) $$U_k(\xi) = C_0 \xi^{k+2} + C_1 \xi^{k+1} + \sum_{\nu=0}^{p} \frac{(-1)^\nu \zeta_2(-\nu)}{\nu!} \frac{\Gamma(k+1)}{\Gamma(k+1-\nu)} \xi^{k-\nu},$$

p being the largest integer such that $-p > \gamma$. We may write

(2.327) $$U_k(\xi) = \sum_{\mu=0}^{r} C_\mu \xi^{k+2-\mu},$$

where $r = p+2$ is the integer such that $k + \frac{1}{2} - \delta < r < k + \frac{3}{2} - \delta$. The index of the last power in $U_k(\xi)$ lies between $\frac{1}{2} + \delta$ and $\frac{3}{2} + \delta$, whereas that in $V_k(\xi)$ lies between 0 (inclusive) and 1; the form of the two sums is otherwise the same.

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[1]) G. H. HARDY and M. RIESZ, "The general theory of Dirichlet's series", *Camb. Math. Tracts*, 18 (1915), 51 (Theorem 40).

[2]) See *1*, Lemmas β, δ. The series which occur in (2.313) (or (6.221) of *1*) are absolutely convergent for $\sigma = \gamma$, in virtue of (2.323) and Lemma 4; and the conclusion then follows from Lemma δ.

2.33. In (2.325) we substitute for $\zeta_2(s)$ from the formula (2.313), valid since $1-\gamma>h$, and integrate term-by-term. This term-by-term integration is legitimate because

$$(2\pi)^{s-1}\Gamma(1-s)\left(\frac{1}{w^s}\;\frac{\sin\left(\frac{2m\pi}{w}\left(\frac{1}{2}w'-a\right)+\frac{1}{2}(1-s)\pi\right)}{m^{1-s}\sin\frac{mw'\pi}{w}}\right)^*$$

$$=O\left(|t|^{\frac{1}{2}-\gamma}\left(\frac{1}{m^{1-\gamma}\left|\sin\frac{mw'\pi}{w}\right|}\right)^*\right)$$

and

$$\int^{\infty}O\left(|t|^{-k-\frac{1}{2}-\gamma}\right)dt,\quad \sum\left(\frac{1}{m^{1-\gamma}\left|\sin\frac{mw'\pi}{w}\right|}\right)^*$$

are convergent. We thus obtain

(2.331) $$W_k(\xi)=U_k(\xi)+J_1+J_2+J_1'+J_2',$$

where

(2.3311) $$J_1=\frac{\xi^k\Gamma(k+1)}{4\pi}\sum_1^\infty\frac{e^{\frac{2m\pi i}{w}\left(\frac{1}{2}w'-a\right)}}{m\sin\frac{mw'\pi}{w}}\,\Phi\left(\frac{2m\pi i\xi}{w}\right),$$

J_2 is conjugate to J_1, J_1' and J_2' are obtained from J_1 and J_2 by exchanging w and w',

(2.3312) $$\Phi(u)=\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{\pi}{\sin s\pi}\;\frac{(-u)^s}{\Gamma(k+1+s)}\,ds,$$

and $(-u)^s$ has its principal value, real when u is negative.

The function $\Phi(u)$ is one of a type whose asymptotic expansions have been considered by various writers. We have

(2.332) $$\Phi(u)=\frac{u^{-p-1}}{\Gamma(k-p)}+O(|u|^{-p-2})-u^{-k}e^u+O\left(|u^{-k-1}e^u|\right)$$

$$=\Phi_1+\Phi_2+\Phi_3+\Phi_4,$$

say, with similar formulae for the derivatives of $\Phi(u)$, which may be written down by formal differentiation[1]).

[1]) The function may be expressed in the form

$$-\sum_{m=0}^{\infty}\frac{u^{m-p}}{\Gamma(k+1+m-p)};$$

here p is the integer such that $-p-1<\gamma<-p$. As regards the asymptotic expansions of such functions, see, for example, E. W. Barnes, "On functions defined by simple types of hypergeometric series", *Trans. Camb. Phil. Soc.*, 20 (1906), 253—279. The actual result required here is easily proved in a variety of ways.

We denote by $J_{1,1}, \ldots$ the results of replacing Φ, in $J_1, \ldots$ by $\Phi_1 + \Phi_2$; by $J_{1,2}, \ldots$ the results of replacing Φ by $\Phi_3 + \Phi_4$: so that we have four equations of the type

(2.333) $$J_1 = J_{1,1} + J_{1,2}.$$

Consider first the sums $J_{1,1}, \ldots$ with 1 as second suffix. If we substitute Φ_1 for Φ in $J_1, \ldots$ and combine the results, we obtain, after a straightforward calculation,

$$(-1)^{p+1}(2\pi)^{-p-2}\frac{\Gamma(k+1)}{\Gamma(k-p)}\xi^{k-p-1}$$
$$\left(w^{p+1}\sum\frac{\sin\left(\frac{2m\pi}{w}\left(\frac{1}{2}w'-a\right)+\frac{1}{2}(p+2)\pi\right)}{m^{p+2}\sin\frac{mw'\pi}{w}}\right)^*,$$

which is equal, by (2.313), to

$$\frac{(-1)^{p+1}\zeta_2(-p-1)}{(p+1)!}\,\frac{\Gamma(k+1)}{\Gamma(k-p)}\,\xi^{k-p-1}.$$

This is of the same form as the general term in (2.326), and the contribution of Φ_1 may be accounted for by replacing p, in (2.326), by $p+1$. If we do this, the last index in $U_k(\xi)$ will lie between $-\frac{1}{2}+\delta$ and $\frac{1}{2}+\delta$, and $U_k(\xi)$ may become identical with $V_k(\xi)$ or may contain one extra term; it is in any case of the form $V_k(\xi) + O(\xi^{k+1-q})$.

There is also Φ_2 to be considered, but Φ_2 is of lower order than Φ_1 to the extent of a factor $\frac{1}{m\xi}$, and its contribution is accordingly trivial. We therefore obtain

(2.334) $$U_k(\xi) + J_{1,1} + J_{2,1} + J'_{1,1} + J'_{2,1} = V_k(\xi) + O(\xi^{k+1-q}).$$

Next we consider the sums $J_{1,2}, \ldots$. Substituting first Φ_3 for Φ, we obtain, after reduction,

$$-(2\pi)^{-k-1}\Gamma(k+1)\left(w^k\sum\frac{\cos\left(\frac{2m\pi}{w}\left(\frac{1}{2}w'+\xi-a\right)-\frac{1}{2}k\pi\right)}{m^{k+1}\sin\frac{mw'\pi}{w}}\right)^*.$$

And as Φ_4 is of lower order than Φ_3, by a factor $\frac{1}{m\xi}$, its contribution is $O\left(\frac{1}{\xi}\right) = O(\xi^{k+1-q})$. Thus

(2.335) $$J_{1,2}+J_{2,2}+J'_{1,2}+J'_{2,2}=S+O(\xi^{k+1-q}),$$

where S is the second term on the right of (2.321).

Collecting our results from (2.331), (2.333), (2.334), and (2.335), we obtain the result of Theorem 1. At present, however, the theorem is proved only when $k>h-\frac{1}{2}$, and it is necessary to extend this range to $k>h-1$.

2.34. Suppose then that $k=h+\eta>h>h-\frac{1}{2}$, so that (2.321) is proved; and let us differentiate formally with respect to ξ, and divide by k. We have

$$\frac{1}{k}\frac{dW_k}{d\xi}=W_{k-1},$$

and

$$\frac{1}{k}\frac{dV_k}{d\xi}=V_{k-1}+O(\xi^{k+1-q})=V_{k-1}+O(\xi^{k-q'}),$$

where $q'=q-1$ is the integer such that $k<q'\leqq k+1$. Finally the same process, applied to the infinite series, yields the corresponding series for $k-1$, a series which is, by Lemma 4, absolutely and uniformly convergent. It appears then that we are led back to our original formula, with $k-1$ in place of k, and this is just what we require. The proof is however insufficient, since we are not entitled to differentiate the error term $O(\xi^{k+1-q})$.

There is no difficulty of principle in completing the proof, but it is necessary to go back to (2.331). We differentiate this equation, and substitute for $\Phi(u)$ and its derivative $\Phi'(u)$ the approximations given by (2.332) and the corresponding derived equation. The result is an absolutely and uniformly convergent series, and the term-by-term differentiation is thereby justified. We have then only to repeat our previous calculations, in a slightly more complicated form, the formulae which we use being in substance the formal derivatives of those which we have used already. The final result is the same as before, except that k is replaced by $k-1$, and that the result holds whenever $k-1=h-1+\eta>h-1$. When we restore k in the place of $k-1$, the proof of Theorem 1 is completed.

2.4. From Theorem 1 we can deduce a proof of the equation numbered (2) in the appendix to our memoir *1*. This equation is not quite so precise as one which we shall obtain later in an elementary manner, but it is of some interest to show how it follows from the analytic theory.

Theorem 2. *If* (1.331) *is satisfied then*

(2.41) $$U(\eta)=O\left(\eta^{1-\frac{1}{h}+\varepsilon}\right)$$

for every positive ε.

Suppose, in (2.321), that k is the integer such that $k \leq h < k+1$. The q of Theorem 1 is now $k+2$, and

(2.42)
$$W_k(\xi) = V_k(\xi) - A\left(w^k \sum_1^\infty \frac{\cos\left(\frac{2m\pi}{w}\left(\frac{1}{2}w' + \xi - a\right) - \frac{1}{2}k\pi\right)}{m^{k+1}\sin\frac{mw'\pi}{w}}\right)^* + O\left(\frac{1}{\xi}\right)$$
$$= V_k(\xi) + S(\xi) + O\left(\frac{1}{\xi}\right)$$

say. If now we suppose $0 < \mathfrak{z} < 1$, and write generally.

$$\Delta f(\xi) = \Delta_{\mathfrak{z}}^k f(\xi) = f(\xi + k\mathfrak{z}) - \binom{k}{1} f(\xi + (k-1)\mathfrak{z}) + \binom{k}{2} f(\xi + (k-2)\mathfrak{z}) - \dots,$$

we have

(2.43)
$$\Delta W_k = \Delta V_k + \Delta S + O\left(\frac{1}{\xi}\right).$$

We consider first ΔS. Since

$$\Delta \cos\left(\frac{2m\pi}{w}\left(\frac{1}{2}w' + \xi - a\right) - \frac{1}{2}k\pi\right) = O(\mathrm{Min}(m^k \mathfrak{z}^k, 1)),$$

we have

(2.441)
$$\Delta S = O\left(\mathfrak{z}^k \sum_{m\mathfrak{z} \leq 1} \frac{1}{m\left|\sin\frac{mw'\pi}{w}\right|}\right)^* + O\left(\sum_{m\mathfrak{z} > 1} \frac{1}{m^{k+1}\left|\sin\frac{mw'\pi}{w}\right|}\right)^*$$
$$= O\left(\mathfrak{z}^k \left(\frac{1}{\mathfrak{z}}\right)^{h-1+\varepsilon}\right) + O\left(\left(\frac{1}{\mathfrak{z}}\right)^{-k-1+h+\varepsilon}\right) = O(\mathfrak{z}^{k+1-h-\varepsilon}).$$

Next

$$\Delta W_k(\xi) = k! \int_\xi^{\xi+\mathfrak{z}} d\xi_1 \int_{\xi_1}^{\xi_1+\mathfrak{z}} d\xi_2 \dots \int_{\xi_{k-1}}^{\xi_{k-1}+\mathfrak{z}} W(\xi_k)\, d\xi_k,$$

and so

(2.442)
$$k!\,\mathfrak{z}^k W(\xi) \leq \Delta W_k(\xi) \leq k!\,\mathfrak{z}^k W(\xi + k\mathfrak{z}).$$

Finally

(2.443)
$$\Delta V_k(\xi) = \mathfrak{z}^k V_k^{(k)}(\xi) + O(\mathfrak{z}^{k+1}\xi) = k!\,\mathfrak{z}^k V(\xi) + O(\mathfrak{z}^{k+1}\xi),$$

where

(2.4431)
$$V(\xi) = \frac{\xi^2}{2ww'} + \frac{w + w' - 2a}{2ww'}\xi + M,$$

M being a constant.

From (2.43), (2.441), (2.442), (2.443), and (2.4431) we deduce, in the first place,

(2.45)
$$W(\xi) \leq V(\xi) + O(\mathfrak{z}\xi) + O\left(\frac{1}{\mathfrak{z}^k \xi}\right) + O(\mathfrak{z}^{1-h-\varepsilon}).$$

In (2.45) we take $\mathfrak{z} = \xi^{-\frac{1}{h}}$, and we obtain

$$W(\xi) < V(\xi) + O\left(\xi^{1-\frac{1}{h}+\varepsilon}\right). \tag{2.46}$$

Similarly we have

$$W(\xi + k\mathfrak{z}) > V(\xi) + O\left(\xi^{1-\frac{1}{h}+\varepsilon}\right), \tag{2.47}$$

or, on replacing $\xi + k\mathfrak{z}$ by ξ

$$W(\xi) > V(\xi) + O\left(\xi^{1-\frac{1}{h}+\varepsilon}\right). \tag{2.48}$$

Finally, from (2.46) and (2.48) we deduce

$$W(\xi) = V(\xi) + O\left(\xi^{1-\frac{1}{h}+\varepsilon}\right). \tag{2.49}$$

Attributing to a the special value $w + w'$ and replacing ξ by η, we obtain (2.41)[1].

2.51. We next prove a theorem which shows that the index $1 - \dfrac{1}{h}$ of Theorem 2 is the "correct" one.

Theorem 3. *If $h > 1$ and (1.332) is satisfied for an infinity of values of n, then each of the inequalities*

$$U(\eta) > A\eta^{1-\frac{1}{h}}, \quad U(\eta) < -A\eta^{1-\frac{1}{h}} \tag{2.511}$$

is satisfied for arbitrarily large values of η.

Let $f(x)$ be the function defined, when $\Re(x) > 0$, by the equations

$$f(x) = \sum_{m,n=1}^{\infty} e^{-x(mw+nw')} = \sum_{p=1}^{\infty} e^{-xl_p} = \frac{e^{-x(w+w')}}{(1-e^{-xw})(1-e^{-xw'})}. \tag{2.512}$$

We have

$$f(x) = \frac{1}{ww'x^2} - \frac{w+w'}{2ww'x} + \frac{w^2+3ww'+w'^2}{12ww'} + O(x), \tag{2.513}$$

when x is small.

Suppose next that

$$x = \frac{2n\pi i}{w'} + \delta, \tag{2.514}$$

where δ is small and positive, and n has one of the values for which (1.332) is true. Then

$$\left|e^{-(w+w')x}\right| > A, \quad \left|1 - e^{-w'x}\right| < A\delta,$$

[1] The proof of the theorem is modelled on the argument used by LANDAU, „Über Dirichlets Teiler-Problem", *Münchener Sitzungsberichte*, 1915, 317—328.

and

$$|1-e^{-wx}| = \sqrt{(1-e^{-w\delta})^2+4e^{-w\delta}\sin^2 n\pi\theta} < A\sqrt{\delta^2+n^{-2h}},$$

so that

(2.515) $$|f(x)| > \frac{A}{\delta\sqrt{\delta^2+n^{-2h}}}.$$

On the other hand, we have

(2.516) $$f(x) = \sum_1^\infty e^{-xl_p} = \sum_1^\infty p\int_{l_p}^{l_{p+1}} xe^{-xu}\,du = x\int_0^\infty N(u)e^{-xu}\,du,$$

since $N(u)$ is the number of l's which do not exceed u; or

(2.517) $$f(x) = \frac{1}{ww'x^2} - \frac{w+w'}{2ww'x} + \Phi(x),$$

where

(2.5171) $$\Phi(x) = x\int_0^\infty U(u)e^{-xu}\,du.$$

Comparing with (2.513), we see that

(2.518) $$\Phi(x) \to M = \frac{w^2+3ww'+w'^2}{12ww'}$$

when $x \to 0$, and in particular if $x = \delta$ and $\delta \to 0$.

2.52. Now suppose that $\alpha > 0$ and

(2.521) $$\chi(u) = U(u) + Bu^\alpha \geqq 0 \text{ [1]}$$

for all sufficiently large values of u, say for $u \geqq u_0$. It follows from (2.521), (2.5171), and (2.518) that

(2.522) $$\int_0^\infty \chi(u)e^{-\delta u}\,du \sim B\int_0^\infty u^\alpha e^{-\delta u}\,du = B\Gamma(1+\alpha)\delta^{-1-\alpha} = C\delta^{-1-\alpha},$$

say, when $\delta \to 0$. On the other hand, if x is given by (2.514), we have, by (2.5171) and (2.521),

$$\Phi(x) = x\int_0^\infty \chi(u)e^{-xu}\,du - Bx\int_0^\infty u^\alpha e^{-xu}\,du = x\int_0^\infty \chi(u)e^{-xu}\,du + O(1),$$

[1]) B and C (unlike A) retain the same values throughout the argument which follows.

(2.523) $$|\Phi(x)| < |x|\left(\left|\int_0^{u_0}\chi(u)e^{-xu}\,du\right| + \int_{u_0}^{\infty}\chi(u)e^{-\delta u}\,du\right) + O(1)$$
$$< |x|\left(\left|\int_0^{\infty}\chi(u)e^{-\delta u}\,du\right| + O(1)\right) + O(1)$$
$$< 2\cdot\frac{2n\pi}{w'}\cdot C\delta^{-1-\alpha} < ACn\delta^{-1-\alpha}$$

by (2.522). Comparing (2.515), (2.517) and (2.523), we see that

$$ACn\delta^{-1-\alpha} > \frac{A}{\delta\sqrt{\delta^2+n^{-2h}}},\quad C^2n^2(\delta^2+n^{-2h}) > A\delta^{2\alpha}.$$

Taking in particular $\delta = n^{-h}$, we have

(2.524) $$C > An^{(1-\alpha)h-1}.$$

From (2.524) it follows that $\alpha \geqq 1 - \frac{1}{h}$; and if we take $\alpha = 1 - \frac{1}{h}$, then $C > A$ or $B > A$. Unless these conditions are satisfied, (2.521) cannot be true for all sufficiently large values of u; and therefore the second of the inequalities (2.511) must be true for arbitrarily large values of η and *some* value of B. The first inequality can naturally be proved in a similar manner.

We have supposed $h > 1$, so that the critical value of α is positive. In this case a less precise form of (2.518), *viz.* $\Phi(x) = o(x^{-\alpha})$, would have been sufficient for our argument. When $h = 1$, the critical value of α is zero. In this case the value of M becomes relevant to the argument. We must take $\chi(u) = U(u) + M + B$, and the final conclusion is that $B > A$, *i. e.* that *each of*

$$U(\eta) > M + A,\qquad U(\eta) < M - A$$

is true for arbitrarily large values of η. The conclusion is not entirely trivial, but it is certainly much less interesting, and is no longer in any sense a best possible result.

2.61. We proceed next to the proof of the exact formula for $N(\eta)$ enunciated at the end of our former memoir[1]). This is the analogue of VORONOÏ's formula for the number of lattice-points in the area $x > 0$, $y > 0$, $xy \leqq \eta$.

It is now necessary to consider the exact definition of $N(\eta)$ when η is of the form $pw + qw'$ and there is a lattice-point on the boundary of the triangle. We agree that such a point is to be counted as one-half.

[1]) *1*, p. 35, formula (5).

Lemma 5. *If λ is positive and ν real, then the integral*

(2.611)
$$\int_0^\infty \frac{\sin \nu x}{(\cosh w\lambda - \cos wx)(\cosh w'\lambda - \cos w'x)} \frac{dx}{x}$$

is convergent; and it may be evaluated by expanding the subject of integration as a double power-series in $e^{-w\lambda}$ and $e^{-w'\lambda}$, and integrating term-by-term.

The formal result of this process is

(2.612)
$$\frac{4}{\sinh w\lambda \sinh w'\lambda} \sum_{p,q=0}^{\infty} \varepsilon_p \varepsilon_q e^{-\lambda(pw+qw')} \int_0^\infty \frac{\sin \nu x \cos pwx \cos qw'x}{x} dx,$$

where $\varepsilon_p = \frac{1}{2}\ (p=0)$, $\varepsilon_p = 1\ (p>0)$. There is at most one term for which $\nu \pm pw \pm qw' = 0$. This term, if it exists, we remove from the double series and consider independently, and we denote the modified series by $\sum'$. Since term-by-term integration is certainly permissible over any finite range $(0, X)$, it is sufficient to show that

(2.613)
$$\sum{}' \varepsilon_p \varepsilon_q e^{-\lambda(pw+qw')} \int_X^\infty$$

is convergent and tends to zero when $X \to \infty$; and this will be so if

(2.614)
$$\sigma = \sum{}' \varepsilon_p \varepsilon_q e^{-\lambda(pw+qw')} \left| \int_X^\infty \frac{\sin \varrho x}{x} dx \right| < \varepsilon,$$

where

(2.615)
$$\varrho = \nu \pm pw \pm qw',$$

is less than ε for every positive ε and sufficiently large values of X.

We write

(2.616)
$$\sigma = \sum{}' = \sum_{|\rho| X \geqq H} + \sum_{|\rho| X < H} = \sigma_1 + \sigma_2,$$

say, where $H > 0$. Since

$$\left| \int_X^\infty \frac{\sin \varrho x}{x} dx \right| = \left| \int_{|\rho| X}^\infty \frac{\sin y}{y} dy \right| < \operatorname{Min}\left(\frac{A}{|\varrho| X}, A\right),$$

we have

(2.617)
$$\sigma_1 < \frac{A}{H} \sum_{p,q=0}^{\infty} e^{-\lambda(pw+qw')} < \frac{A}{H} < \frac{1}{2}\varepsilon$$

if H is sufficiently large. It is therefore sufficient for our purpose to prove that *when H is fixed we can so choose X_0 that*

(2.618) $$A\sum{}' e^{-\lambda(pw+qw')} < \frac{1}{2}\epsilon \qquad (X \geqq X_0),$$

the summation extending over those values of p and q for which

(2.619) $$-\frac{H}{X} < \nu \pm pw \pm qw' < \frac{H}{X}.$$

2.62. We divide the terms in question into four blocks corresponding to the four choices of sign, and establish a corresponding conclusion for each of them, with $\frac{1}{8}\epsilon$ in place of $\frac{1}{2}\epsilon$. If the signs attached to p and q are the same, there is nothing to prove; for, as ν is not of the form $pw+qw'$ or $-pw-qw'$, there is no term which satisfies (2.619) when X is sufficiently large. It is therefore sufficient to consider the case in which, for example, $\varrho = \nu - pw + qw'$. But the inequalities

$$-\frac{H}{X} < \nu - pw + qw' < \frac{H}{X},$$

where H is fixed, are only possible when $p > \xi$, $q > \xi$, where $\xi \to \infty$ when $X \to \infty$, and the number of values of q, corresponding to a given p, is 1 at most. Hence the sum extended over such values of p and q does not exceed

$$\sum_{p>\xi} e^{-Ap} < Ae^{-A\xi},$$

which tends to zero when $X \to \infty$. This establishes our conclusion and completes the proof of the lemma.

2.63. Lemma 6. *If λ is positive and ζ real, then the integrals*

(2.631) $$\int_{i\lambda}^{i\lambda+\infty} \Im\left(\frac{e^{\zeta i x}}{\sin\frac{1}{2}wx\sin\frac{1}{2}w'x}\right)\frac{dx}{x}, \quad \int_{-i\lambda}^{-i\lambda+\infty} \Im\left(\frac{e^{\zeta i x}}{\sin\frac{1}{2}wx\sin\frac{1}{2}w'x}\right)\frac{dx}{x},$$

in which the path of integration is a line parallel to the real axis, are convergent; and their values may be calculated by expanding the cosecants in powers of $e^{-w\lambda}$ and $e^{-w'\lambda}$, and integrating term by term.

The series to be used are different in the two integrals; for

$$\frac{1}{\sin\frac{1}{2}wx} = -2i\sum_0^\infty e^{\left(p+\frac{1}{2}\right)wix} = -2i\sum_0^\infty e^{\left(p+\frac{1}{2}\right)wi\xi-\left(p+\frac{1}{2}\right)w\lambda}$$

or

$$\frac{1}{\sin\frac{1}{2}wx} = 2i\sum_0^\infty e^{-\left(p+\frac{1}{2}\right)wix} = 2i\sum_0^\infty e^{-\left(p+\frac{1}{2}\right)wi\xi-\left(p+\frac{1}{2}\right)w\lambda},$$

according as $x = \xi + i\lambda$ or $x = \xi - i\lambda$. Apart from this, the argument is the same for the two integrals, and it will be sufficient to consider the second.

Consider first the analogous integral in which $\frac{1}{x}$ is replaced by $\frac{1}{x} - \frac{1}{x+i\lambda}$. The method of evaluation contemplated is certainly legitimate for *this* integral, since

$$\int^{\infty} O\left(\frac{1}{x^2}\right) \sum_{p,q} O\left(e^{-(pw+qw')\lambda}\right) dx$$

is convergent. We may therefore replace $\frac{1}{x}$ by $\frac{1}{x+i\lambda}$ and then $\frac{dx}{x+i\lambda} = \frac{d\xi}{\xi}$ is real.

Next

$$\Im\left(\frac{e^{\zeta i x}}{\sin\frac{1}{2}wx \sin\frac{1}{2}w'x}\right) = 4e^{\zeta\lambda}\Im\frac{e^{\zeta i\xi}\sin\frac{1}{2}w(\xi+i\lambda)\sin\frac{1}{2}w'(\xi+i\lambda)}{(\cosh w\lambda - \cos w\xi)(\cosh w'\lambda - \cos w'\xi)}.$$

Working out the imaginary part of the numerator, we find that it is a sum of constant multiples of terms of the type $\sin\left(\zeta \pm \frac{1}{2}w + \frac{1}{2}w'\right)\xi$. Thus Lemma 6 is reduced to Lemma 5.

2.64. We define a sequence (R_j) as a sequence of values R_j of R which tends to infinity and all of whose members differ, by more than A, from any of the numbers $\frac{2m\pi}{w}$, $\frac{2n\pi}{w'}$ $(m, n = 0, 1, 2, \ldots)$.

Theorem 4. *If* $\eta > w + w'$ (*so that* $N(\eta) > 0$), *then*

(2.641)
$$N(\eta) = R(\eta) + \frac{w^2 + 3ww' + w'^2}{12ww'} - \frac{1}{2\pi}\sum\left(\frac{\cos\frac{2m\pi}{w}\left(\eta - \frac{1}{2}w'\right)}{m\sin\frac{mw'\pi}{w}}\right)^{*}.$$

The sign of summation is to be interpreted as follows: we form the sum of all those terms of the two associated series for which $m < \frac{wR}{2\pi}$, $n < \frac{w'R}{2\pi}$ *respectively, and then make* R *tend to infinity through a sequence* (R_j).

We write $\zeta = \eta - \frac{1}{2}(w + w')$ and

(2.642)
$$J(R_j) = \Re\left(\frac{1}{2\pi i}\int \frac{e^{\zeta i x} - 1}{\sin\frac{1}{2}wx \sin\frac{1}{2}w'x}\frac{dx}{x}\right)$$
$$= \frac{1}{2\pi}\int \Im\left(\frac{e^{\zeta i x} - 1}{\sin\frac{1}{2}wx \sin\frac{1}{2}w'x}\frac{dx}{x}\right),$$

the contour of integration being the rectangle $(-i\lambda,\ R_j-i\lambda,\ R_j+i\lambda,\ i\lambda)$, where $\lambda>0$, except that the origin is excluded by a small semi-circle of radius ϱ, described about it as centre. We make $j\to\infty$, $\varrho\to 0$.

The integrals along the imaginary axis vanish identically. The integral along the side parallel to the imaginary axis tends to zero. That along the semi-circle tends to the limit

$$\frac{\zeta^2}{ww'}.$$

The integrals along the sides parallel to the real axis also tend to limits, by Lemma 6. It follows that the sum of the real parts of the residues of the integrand, at poles within the contour, tends to a limit S; and we have

$$S=\Re\int_{-i\lambda}^{\infty-i\lambda}-\Re\int_{i\lambda}^{\infty+i\lambda}+\frac{\zeta^2}{ww'}=J_1-J_2+\frac{\zeta^2}{ww'}, \tag{2.643}$$

say.

We evaluate J_1 by integration term-by-term, which is shown to be legitimate by Lemma 6; and we obtain

$$J_1=\sum_{p,q=1}^{\infty} j_{p,q}, \tag{2.644}$$

where

$$j_{p,q}=\Re\left(-\frac{2}{\pi i}\int_{-i\lambda}^{-i\lambda+\infty}(e^{\zeta ix}-1)\,e^{-\left(\Omega-\frac{1}{2}(w+w')\right)ix}\frac{dx}{x}\right) \tag{2.6441}$$

and $\Omega=pw+qw'$. We may add to $j_{p,q}$ the corresponding integral along the line $(0,-i\lambda)$, since this vanishes identically, and we may then deform the path of integration into the real axis. This gives

$$j_{p,q}=-\frac{2}{\pi}\int_0^\infty\left(\sin(\eta-\Omega)x+\sin\left(\Omega-\frac{w+w'}{2}\right)x\right)\frac{dx}{x},$$

which is zero if $\Omega>\eta$ and -2 if $\Omega<\eta$. Thus we obtain from (2.644)

$$J_1=-2\sum_{\Omega<\eta}1=-2N(\eta). \tag{2.645}$$

This equation still holds when η is of the form $pw+qw'$, if we adopt the convention stated in 2.61.

Similarly we obtain a series for J_2, in which the typical term involves the integral

$$\frac{2}{\pi}\int_0^\infty \left(\sin(\eta+\Omega-w-w')x-\sin\left(\Omega-\frac{w+w'}{2}\right)x\right)\frac{dx}{x}=0.$$

Thus

(2.646) $$J_2=0,$$

as is evident *a priori*, since the value of the integral is independent of λ and tends to 0 when $\lambda\to\infty$.

From (2.643), (2.645), and (2.646) we deduce

(2.647) $$N(\eta)=\frac{\left(\eta-\frac{1}{2}w-\frac{1}{2}w'\right)^2}{2ww'}-\frac{1}{2}S.$$

A straightforward calculation shows that

(2.648) $$S=\frac{1}{\pi}\sum\left(\frac{\cos\frac{2m\pi}{w}\left(\eta-\frac{1}{2}w'\right)}{m\sin\frac{mw'\pi}{w}}-\frac{(-1)^m}{m\sin\frac{mw'\pi}{w}}\right)^*.$$

But[1])

$$\frac{1}{2\pi}\sum\left(\frac{(-1)^m}{m\sin\frac{mw'\pi}{w}}\right)^*=-\frac{w^2+w'^2}{24ww'}=\frac{w^2+3ww'+w'^2}{12ww'}-\frac{(w+w')^2}{8ww'}.$$

Thus we obtain the result of the theorem.

3. The sum $s(n,\theta)$.

3.1. In this section we use elementary methods. We are concerned primarily with what, in our former memoir, we called Problem B, that of the order of magnitude of the sum

(3.11) $$s(n,\theta)=\sum_{m=1}^{n}\{m\theta\},$$

though sometimes we return to Problem A.

Lemma 7. *If θ is positive and irrational, $x\geqq 0$, $y=\theta x$, and $f(0)=g(0)=0$, then*

(3.12)
$$\sum_{1\leqq m\leqq x}f([m\theta])(g(m)-g(m-1))+\sum_{1\leqq n\leqq y}g\left(\left[\frac{n}{\theta}\right]\right)(f(n)-f(n-1))=f([y])g([x])$$

[1]) This is easily proved directly by contour integration. See the second memoir quoted in 2.31.

The sum with respect to m is

$$\sum_{n=0}^{[y]} f(n) \sum_{[m\theta]=n} (g(m)-g(m-1)) - f([y]) \sum_{[m\theta]=[y],\, m>x} (g(m)-g(m-1))$$

$$= \sum_{n=0}^{[y]} f(n)\left(g\left(\left[\frac{n+1}{\theta}\right]\right) - g\left(\left[\frac{n}{\theta}\right]\right)\right) - f([y])\left(g\left(\left[\frac{[y]+1}{\theta}\right]\right) - g([x])\right)$$

and the first term here is

$$f([y])\, g\left(\left[\frac{[y]+1}{\theta}\right]\right) - \sum_{n=1}^{[y]} g\left(\left[\frac{n}{\theta}\right]\right)(f(n)-f(n-1)),$$

which gives the result.

In (3.12) take $f(u) = g(u) = u$, and write

$$\text{(3.13)} \qquad \alpha_m = \alpha_m(\theta) = \{m\theta\}, \quad \beta_n = \beta_n(\theta) = \alpha_n\left(\frac{1}{\theta}\right) = \left\{\frac{n}{\theta}\right\},$$

$$\text{(3.14)} \qquad \mu = [x], \quad \nu = [y] = [\theta x], \quad \delta = \theta_{\mu-\nu} = \theta[x] - [\theta x].$$

We obtain

$$\text{(3.15)} \qquad \sum_1^{\mu} [m\theta] + \sum_1^{\nu} \left[\frac{n}{\theta}\right] = \mu\nu,$$

or, on expressing $[m\theta]$ and $\left[\frac{n}{\theta}\right]$ in terms of α_m and β_n, and reducing,

$$\text{(3.16)} \qquad \sum_1^{x} \alpha_m + \sum_1^{\theta x} \beta_n = \frac{\delta}{2} - \frac{\delta(1-\delta)}{2\theta}.$$

Lemma 8. *If n, n_1, n_1' are positive or zero integers such that*

$$\text{(3.17)} \qquad n\theta = n_1 + d = n_1' + e, \quad 0 < d < 1, \quad -1 < e < 0,$$

then

$$\text{(3.18)} \qquad s(n,\theta) + s(n_1,\theta_1) = \frac{d}{2} - \frac{d(1-d)}{2\theta},$$

$$\text{(3.19)} \qquad s(n,\theta) + s(n_1',\theta_1) = \frac{e}{2} - \frac{e(1-e)}{2\theta} - \left[\frac{-e}{\theta}\right].$$

The first of these formulae is the special form of (3.16) obtained by supposing that x is an integer n. The second is a simple variant. Since $n_1' = n_1 + 1$, the left hand sides of (3.18) and (3.19) differ only by $\{n_1'\theta_1\}$, and (3.19) follows from (3.18) by simple algebra.

The formula (3.18) is that which, in our former memoir, we attributed to LERCH[1]). Herr OSTROWSKI has pointed out to us that it had (in substance

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[1]) See *1*, p. 20.

at any rate) been found before by SYLVESTER[1]). SYLVESTER's formula is indeed equivalent to the more general formulae (3.15) and (3.16). General formulae of the type (3.12) appear to originate with DIRICHLET, and the actual formula (3.12) was given, in the special case in which x is an integer, by HACKS[2]).

With these formulae should be associated the formula (3.24) of our memoir *1*.

3.2. Theorem 5. *If* θ *satisfies* (1.331), *and* $h > 1$, *then*

$$s(n, \theta) = O\left(n^{1-\frac{1}{h}}\right). \tag{3.21}$$

This theorem is due to OSTROWSKI[3]). In the less precise form in which $1-\frac{1}{h}$ is replaced by $1-\frac{1}{h}+\varepsilon$, it is included in Theorem 2, which was enunciated without proof in our memoir *1*[4]). The reading of OSTROWSKI's memoir suggested to us the following theorem, in which Theorem 5 is included[5]).

Theorem 6. *If* θ *satisfies* (1.331), *and* $h > 1$, *then*

$$U(\eta) = O\left(\eta^{1-\frac{1}{h}}\right). \tag{3.22}$$

This theorem includes both Theorem 5 and Theorem 2. It is easily proved by a combination of Lemma 2 with formulae taken from *1*.

If (1.331) is true, (1.341) is also true. As in *1*, we choose r so that

$$\xi\,\theta\,\theta_1\,\ldots\,\theta_{r-1}\,\theta_r < 1 \leqq \xi\,\theta\,\theta_1\,\ldots\,\theta_{r-1}, \tag{3.23}$$

where $\xi = \frac{\eta}{w'}$. We have then[6])

$$U(\eta) = O\left(\frac{1}{\theta}+\frac{1}{\theta_1}+\ldots+\frac{1}{\theta_{r-1}}\right) + O(\xi\,\theta\,\theta_1\,\ldots\,\theta_{r-1}). \tag{3.24}$$

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[1]) J. J. SYLVESTER, "Sur la fonction $E(x)$", *Comptes Rendus,* 50 (1860), 732—734 (*Collected math. papers,* 2, 179—180). See also pp. 176, 177, 179 of the same volume of the collected papers.

[2]) „Über Summen von größten Ganzen", *Acta Mathematica,* 10 (1887), 1—52. See also J. W. L. GLAISHER, "On certain transformations of Lejeune-Dirichlet's in the theory of numbers, and similar theorems", *Quarterly Journal,* 43 (1912), 123—142.

[3]) *l. c.* p. 82.

[4]) *l. c.* p. 35, equation (2).

[5]) Generally, an "O" theorem relating to Problem B is included in the corresponding theorem relating to Problem A, as is explained in *1*. An "Ω" theorem, that is to say, a theorem which, like Theorem 3, tends in the opposite direction, is on the other hand more difficult than the corresponding theorem concerning Problem A.

[6]) p. 22, equation (4.151).

Write now

(3.25) $$\theta\,\theta_1 \ldots \theta_{r-1} = \xi^{-j},$$

where $0 < j \leqq 1$. Then, by (1.351), $\xi^{-(h-1)j} < A\,\theta_r$, and so

$$\xi^{1-hj} = \xi^{1-j-(h-1)j} < A\,\xi\,\theta\,\theta_1 \ldots \theta_{r-1}\theta_r < A.$$

It follows that $j \geqq \frac{1}{h}$, and

(3.26) $$\xi\,\theta\,\theta_1 \ldots \theta_{r-1} = \xi^{1-j} = O\left(\xi^{1-\frac{1}{h}}\right) = O\left(\eta^{1-\frac{1}{h}}\right).$$

Again, if $1 \leqq s \leqq r$, we have

$$\frac{1}{\theta\,\theta_1 \ldots \theta_{s-1}} \leqq \xi\,\theta_s \ldots \theta_{r-1},$$

by (3.23). Using (1.351), we obtain

(3.27) $$\theta_{s-1}^{-1-\frac{1}{h-1}} < A\,\xi\,\theta_s \ldots \theta_{r-1}, \quad \frac{1}{\theta_{s-1}} < A(\xi\,\theta_s \ldots \theta_{r-1})^{1-\frac{1}{h}}.$$

From (3.27) it follows that

(3.28)
$$\frac{1}{\theta_{r-1}} + \frac{1}{\theta_{r-2}} + \ldots + \frac{1}{\theta} < A\,\xi^{1-\frac{1}{h}}\left(1 + \theta_{r-1}^{1-\frac{1}{h}} + (\theta_{r-2}\theta_{r-1})^{1-\frac{1}{h}} + \ldots\right)$$
$$< A\,\xi^{1-\frac{1}{h}} \sum_{k=0}^{\infty} 2^{-k\left(1-\frac{1}{h}\right)} < A\,\xi^{1-\frac{1}{h}} < A\,\eta^{1-\frac{1}{h}},$$

by Lemma 1. Finally, from (3.24), (3.26), and (3.28) the theorem follows.

The constants of the argument are not absolute: the theorem is not true uniformly in h.

3.31. Theorem 7. *If* $h > 1$ *and* (1.332) *is true, then*

(3.311) $$|s(n, \theta)| > A\,n^{1-\frac{1}{h}}$$

for an infinity of values of n.

It should be observed that this theorem has different interpretations, according as the A's of (1.332) and (3.311) depend upon θ and h or are absolute constants. It is true on either interpretation, but is a little harder to prove on the second.

Taking the second interpretation, we may restate what we have to prove as follows:—*If* $h > 1$ *and*

(3.3121) $$\varliminf n^h |\sin n\,\theta\,\pi| \leqq B$$

then

(3.3122) $$|s(n,\theta)| > Cn^{1-\frac{1}{h}},$$

where C depends only on B. In what follows A's denote absolute constants, C's constants depending only on B; the O's are absolute.

Let H be the upper bound of the numbers x for which

(3.313) $$\underline{\lim}\, n^{x} |\sin n\theta\pi| \leqq B,$$

Clearly we have $H \geqq h$. We proceed to show that *there exists an $h_1 \geqq h$ such that*

(3.3141) $$\underline{\lim}\, n^{h_1} |\sin n\theta\pi| \leqq B$$

and

(3.3142) $$\lim (\theta\theta_1 \ldots \theta_{n-1})^{h_1-1}\left(\frac{1}{\theta}+\frac{1}{\theta_1}+\ldots+\frac{1}{\theta_{n-1}}\right)=0$$

when $n \to \infty$. We must distinguish two cases.

Case (i): $H > 2$. In this case we have only to take

$$h_1 = \operatorname{Max}(h, 2).$$

For, since (3.313) holds both when $x = h$ and when $x = 2 < H$, (3.3141) is satisfied. Also

$$(\theta\theta_1 \ldots \theta_{n-1})^{h_1-1}\left(\frac{1}{\theta}+\frac{1}{\theta_1}+\ldots+\frac{1}{\theta_{n-1}}\right)$$
$$\leqq (\theta\theta_1 \ldots \theta_{n-1})\left(\frac{1}{\theta}+\frac{1}{\theta_1}+\ldots+\frac{1}{\theta_{n-1}}\right) < Ane^{-An} = o(1).$$

Case (ii): $H \leqq 2$. We begin the discussion of this case by showing that numbers h_1 and H_1 exist for which

(3.315) $$h \leqq h_1 \leqq H < H_1,$$

(3.316) $$K_1 = (h_1-1)(H_1-1)-(H_1-h_1) > 0,$$

(3.317) $$\underline{\lim}\, n^{h_1} |\sin n\theta\pi| \leqq B,$$

(3.318) $$\underline{\lim}\, n^{H_1} |\sin n\theta\pi| = \infty.$$

The last of these is an immediate consequence of $H < H_1$, and we need only consider the first three. If $h = H$ we choose $h_1 = h$, and H_1 greater than H by so little that (3.316) is satisfied. If $h < H$ we choose h_1 and H_1 on either side of H, and differing from it by so little that $h \leq h_1$ and (3.316) is satisfied. It is clear that (3.317), or (3.3141), is satisfied in either case.

3.32. We denote by K's positive constants depending only on B, h_1 and H_1. It follows from (1.352) and (3.317) that

$$\frac{1}{\theta_\nu} > \frac{C}{2^{h_1}(\theta\,\theta_1 \dots \theta_{\nu-1})^{h_1-1}}, \tag{3.321}$$

for an infinity of values of ν; and from (1.351) and (3.318) that

$$\frac{1}{\theta_n} < \frac{K}{(\theta\,\theta_1 \dots \theta_{n-1})^{H_1-1}}, \tag{3.322}$$

for all values of n. Hence, observing that $h_1 \leqq 2$, and using Lemma 1, we have

(3.323)
$$\frac{(\theta\,\theta_1 \dots \theta_{n-1})^{h_1-1}}{\theta_r} < K e^{-K(n-r)} \frac{(\theta\,\theta_1 \dots \theta_r)^{h_1-1}}{\theta_r} \quad (0 \leqq r \leqq n-1),$$

(3.324)
$$\frac{(\theta\,\theta_1 \dots \theta_r)^{h_1-1}}{\theta_r} = \left(\frac{(\theta\,\theta_1 \dots \theta_{r-1})^{H_1-1}}{\theta_r}\right)^{2-h_1} (\theta\,\theta_1 \dots \theta_{r-1})^{h_1-1-(H_1-1)(2-h_1)}$$
$$< K^{2-h_1}(\theta\,\theta_1 \dots \theta_{r-1})^{K_1} < K e^{-Kr}.$$

From (3.323) and (3.324) it follows that

$$(\theta\,\theta_1 \dots \theta_{n-1})^{h_1-1} \sum_{r=0}^{n-1} \frac{1}{\theta_r} < K \sum_{r=0}^{n-1} e^{-Kn} < K n e^{-Kn} = o(1),$$

which is (3.3142). This completes the discussion of case (ii).

3.33. It is now not difficult to prove Theorem 7. We suppose ν selected so that (3.321) is true, and we write

$$n_\nu = \left[\frac{1}{3\theta_\nu}\right], \quad n_r = \left[\frac{n_{r+1}}{\theta_r}\right] \; (0 \leqq r < \nu), \quad n_0 = n, \tag{3.331}$$

so that

$$n < \frac{n_\nu}{\theta\,\theta_1 \dots \theta_{\nu-1}} < \frac{A}{\theta\,\theta_1 \dots \theta_\nu} < \left(\frac{2^{h_1}}{C}\right)^{\frac{1}{h_1-1}} \left(\frac{1}{\theta_\nu}\right)^{\frac{h_1}{h_1-1}}, \tag{3.332}$$

by (3.321), or

$$\frac{1}{\theta_\nu} > C^{\frac{1}{h_1}} n^{1-\frac{1}{h_1}} > C n^{1-\frac{1}{h_1}}. \tag{3.333}$$

On the other hand we have, by (3.19)[1],

[1] We have $n_{r+1} = n_r \theta_r + g$, where $0 < g < 1$. It is therefore the second of the transformation formulae (3.18) and (3.19) to which we appeal.

$$s(n_r, \theta_r) + s(n_{r+1}, \theta_{r+1}) = O\left(\frac{1}{\theta_r}\right)$$

for $0 \leqq r < \nu$; and so

$$s(n, \theta) = (-1)^\nu s(n_\nu, \theta_\nu) + O\left(\frac{1}{\theta} + \frac{1}{\theta_1} + \ldots + \frac{1}{\theta_{\nu-1}}\right),$$

$$|s(n, \theta)| > |s(n_\nu, \theta_\nu)| - A\left(\frac{1}{\theta} + \frac{1}{\theta_1} + \ldots + \frac{1}{\theta_{\nu-1}}\right).$$

But

$$s(n_\nu, \theta_\nu) = \sum_1^{n_\nu} \{n\theta_\nu\} < \sum_1^{n_\nu} \left(\frac{1}{3} - \frac{1}{2}\right) < -An_\nu < -\frac{A}{\theta_\nu},$$

and so

(3.334) $$|s(n, \theta)| > \frac{A}{\theta_\nu} - A\left(\frac{1}{\theta} + \frac{1}{\theta_1} + \ldots + \frac{1}{\theta_{\nu-1}}\right).$$

The ratio of the second term on the right to the first is less than

$$K(\theta\theta_1 \ldots \theta_{\nu-1})^{h_1 - 1}\left(\frac{1}{\theta} + \frac{1}{\theta_1} + \ldots + \frac{1}{\theta_{\nu-1}}\right)$$

by (3.321), and this tends to zero as $\nu \to \infty$, by (3.3142). Hence

$$|s(\theta, n)| > \frac{A}{\theta_\nu} > Cn^{1 - \frac{1}{h_1}} > Cn^{1 - \frac{1}{h}},$$

since $h \leqq h_1$.

3.34. It will be useful to observe that the n of our argument satisfies inequalities

(3.341) $$Aq_{\nu+1} < n < Aq_{\nu+1} < q_{\nu+1}$$

when ν is large. The second and third of these are immediate consequences of (3.331), (3.332), and Lemma 2. To prove the first we observe that

$$n_r \geqq n_\nu > \frac{A}{\theta_\nu} > \frac{K}{(\theta\theta_1 \ldots \theta_{\nu-1})^{h-1}} > K2^{\frac{1}{2}(h-1)\nu} > \nu \quad (0 \leqq r < \nu)$$

for all sufficiently large values of ν. Hence

$$n_r > \frac{n_{r+1}}{\theta_r} - 1 > \frac{n_{r+1}}{\theta_r}\left(1 - \frac{1}{n_{r+1}}\right) > \frac{n_{r+1}}{\theta_r}\left(1 - \frac{1}{\nu}\right),$$

and so

$$n > \left(1 - \frac{1}{\nu}\right)^\nu \frac{A}{\theta\theta_1 \ldots \theta_\nu} > Aq_{\nu+1}.$$

3.41. The proofs of our next two theorems are the most difficult in the memoir. The results were enunciated without proof in our former memoir[1]), and discovered and proved independently by OSTROWSKI[2]).

We give a proof here based on the formulae (3.18) and (3.19). We have abbreviated this proof in every possible way, and present it almost in the form of a sketch; for we recognise that the quite different proof of OSTROWSKI is simpler. It is indeed here that OSTROWSKI's method shows to the greatest relative advantage. At the same time our proof seems to us interesting in itself, and it is essential, if we are to develop the theory systematically from our own point of view, that this crucial theorem should appear in its proper place.

Theorem 8. *There is a positive A such that*

$$|s(n,\theta)| > A\log n \tag{3.411}$$

for every irrational θ and an infinity of values of n.

Theorem 9. *There is a $B = B(K)$ such that each of the inequalities*

$$s(n,\theta) > B\log n, \quad s(n,\theta) < -B\log n \tag{3.412}$$

is true for every θ for which $a_n < K$ and for an infinity of values of n.

In proving Theorem 8, we may suppose that θ satisfies (1.331) for some h: we may take, for example $h = 2$, in which case

$$\frac{1}{\theta_\nu} < \frac{A}{\theta\theta_1 \ldots \theta_{\nu-1}}. \tag{3.413}$$

For, if the condition is not satisfied for $h = 2$, we have, by Theorem 7,

$$|s(n,\theta)| > A\sqrt{n} > A\log n$$

for an infinity of values n[3]).

3.42. Let

$$\alpha_r = 1\,(a_r \leqq 3), \quad \alpha_r = \left[\frac{1}{2}a_r\right](a_r > 3), \tag{3.421}$$

$$\gamma = \delta = \alpha_4\theta\theta_1\theta_2\theta_3 + \alpha_8\theta\theta_1\ldots\theta_7 + \alpha_{12}\theta\theta_1\ldots\theta_{11} + \ldots, \tag{3.422}$$

$$\gamma_1 = -\delta_1 = \frac{\delta}{\theta}, \quad \gamma_2 = \delta_2 = -\frac{\delta_1}{\theta_1}, \quad \gamma_3 = -\delta_3 = \frac{\delta_2}{\theta_2}, \tag{3.423}$$

$$\gamma_4 = \delta_4 = -\frac{\delta_3}{\theta_3} - \alpha_4 = \alpha_8\theta_4\theta_5\theta_6\theta_7 + \alpha_{12}\theta_4\theta_5\ldots\theta_{11} + \ldots, \tag{3.424}$$

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1) p. 36. We raised the question which they answer in our note of 1912.

2) *O.*, pp. 85—92.

3) One of the inherent advantages of OSTROWSKI's method is that it enables him to avoid making this distinction.

with corresponding equations in which every suffix is increased by 4, 8, 12, ...

Further, let m be a large positive integer, and

(3.425) $\zeta_{4m+2} = \theta_{4m+2}(1-\gamma_{4m+3}), \quad \zeta_{r-1} = -\theta_{r-1}\zeta_r (0 \leqq r \leqq 4m+2),$

(3.426) $$N_r\theta_r = N_{r+1} + \delta_r + \zeta_r \quad (0 \leqq r \leqq 4m+2),$$

(3.427) $$N_{4m+3} = 0,$$

it being understood that a zero suffix may always be omitted, so that, *e.g.*, $\gamma_0 = \gamma$. The ζ's are defined by (3.425), and the equations (3.426) and (3.427) then define $N_{4m+2}, N_{4m+1}, \ldots, N_0 = N$ in turn. It is not obvious from the definitions that the N's are integers, but it follows immediately from them that $N_{4m+2} = 1$. If now N_r is an integer, and we consider congruences to modulus 1, we have

$$N_{r+1} = N_r\theta_r - \delta_r - \zeta_r \equiv N_r(a_r + \theta_r) - \delta_r - \zeta_r = \frac{N_r}{\theta_{r-1}} - \delta_r - \zeta_r$$
$$= \frac{N_{r-1}\theta_{r-1} - \delta_{r-1} - \zeta_{r-1}}{\theta_{r-1}} - \delta_r - \zeta_r = N_{r-1},$$

by (3.427) and (3.426). It follows by induction that every N_r is integral.

We write

(3.428) $$s_r = s(N_r, \theta_r).$$

3.43. We use the following properties of the numbers γ, δ, ζ, N:—

(3.431) $$0 < \gamma < \gamma_1 < \gamma_2 < \gamma_3 < 1,$$

(3.432) $$\gamma_3 > A,$$

(3.433) $$1 - \gamma_i > A\frac{a_4 - \alpha_4}{\alpha_4} \quad (0 \leqq i \leqq 3),$$

(3.434) $$\alpha_4 = \left[\frac{\gamma_3}{\theta_3}\right]$$

(with similar results in which every suffix is increased by 4, 8, 12, ...),

(3.435) $$|\zeta_r| < 1 - \gamma_r \quad (0 \leqq r \leqq 4m+2),$$

(3.436) $$\begin{aligned} 0 < \delta_{2r} + \zeta_{2r} < 1 \quad (2r \leqq 4m+2), \\ -1 < \delta_{2r+1} + \zeta_{2r+1} < 0 \quad (2r+1 \leqq 4m+1), \end{aligned}$$

(3.437) $$N_r \geqq 1 \quad (0 \leqq r \leqq 4m+2),$$

(3.438) $$N_r \to \infty \quad (m \to \infty),$$

(3.439) $$N < \frac{2^{4m+2}}{\theta\theta_1 \ldots \theta_{4m+1}}.$$

Of these results, (3.431), (3.432), (3.433) and (3.434) follow from Lemma 1 and the definitions of the γ's; (3.435) is obvious from the definitions when $r = 4m+2$, and is easily proved generally by induction; and (3.436) is an immediate consequence of (3.435).

The results (3.437) and (3.438) follow at once from (3.426) and (3.436):—we find in fact that $N_{r-1} \geqq N_r$, and that the sign of equality is impossible if r is even. Finally, (3.426) now gives $N_r < 2N_{r+1}$, which proves (3.439).

3.44. Lemma 9. *If*

(3.441) $$u_r = \delta_r - \frac{\delta_r(1-\delta_r)}{\theta_r},$$

(3.442) $$U_t = u_{4t} - u_{4t+1} + u_{4t+2} - u_{4t+3} + 2\alpha_{4t+4},$$

then

(3.443) $$2s(n,\theta) = \sum_{t=0}^{m-1} U_t + u_{4m} - u_{4m+1} + O(1).$$

Here, and in the arguments which follow, the O's and A's are absolute.

Let $\varphi_r = (N_r\theta_r) = (N_{r+1} + \delta_r + \zeta_r)$. Then, by (3.426) and (3.436), we have

$$\varphi_{2r} = \delta_{2r} + \zeta_{2r}, \quad \varphi_{2r+1} = 1 + \delta_{2r+1} + \zeta_{2r+1}.$$

By (3.18)

(3.4441) $$2(s+s_1) = \delta + \zeta - \frac{(\delta+\zeta)(1-\delta-\zeta)}{\theta}$$
$$= \delta - \frac{\delta(1-\delta)}{\theta} + O\left(\frac{\zeta}{\theta}\right) = u + O(\zeta_1),$$

and by (3.19)

(3.4442)
$$2(s_1+s_2) = \delta_1 + \zeta_1 - \frac{(\delta_1+\zeta_1)(1-\delta_1-\zeta_1)}{\theta_1} - 2\left[-\frac{\delta_1+\zeta_1}{\theta}\right] = u_1 + O(\zeta_2),$$

since

$$\left[-\frac{\delta_1+\zeta_1}{\theta_1}\right] = \left[\frac{\gamma_1}{\theta_1}\right] + O\left(\frac{\zeta_1}{\theta_1}\right) = O(\zeta_2).$$

Similarly we find

(3.4443) $$2(s_2+s_3) = u_2 + O(\zeta_3),$$

(3.4444) $$2(s_3+s_4) = u_3 - 2\alpha_4 + O(\zeta_4).$$

From (3.4441)—(3.4444) it follows that

(3.445) $$2(s_0 - s_4) = U_0 + O(\zeta_4).$$

We have similar equations in which every suffix is increased by 4, 8, ... Adding them, and using (3.425) and Lemma 1, we obtain

(3.446) $$2(s_0 - s_{4m}) = \sum_0^{m-1} U_t + O(1).$$

We have also

(3.447) $$2(s_{4m} - s_{4m+2}) = u_{4m} - u_{4m+1} + O(\zeta_{4m+2}),$$

and (3.443) follows from (3.446) and (3.447), since $s_{4m+2} = O(1)$ and $\zeta_{4m+2} = O(1)$.

3.45. Lemma 10. *We have*

(3.451) $$s(N,\theta) > -A + A\sum_{t=0}^{m-1}(a_{4t+4} - \alpha_{4t+4}) + A\sum_{t=0}^{m-1}\theta_{4t+2}.$$

An elementary reduction shows that

(3.452) $$\begin{aligned} U_t &= 2(\gamma_{4t} - \gamma_{4t+4}) + \frac{\gamma_{4t+1}}{\theta_{4t+1}}(1 - \theta_{4t}\theta_{4t+1})(1 - \gamma_{4t+1}) \\ &+ \frac{\gamma_{4t+3}}{\theta_{4t+3}}(1 - \theta_{4t+2}\theta_{4t+3})(1 - \gamma_{4t+3}) > 2(\gamma_{4t} - \gamma_{4t+4}) \\ &+ \frac{1}{2}\theta_{4t+2}\gamma_{4t+3}(1 - \theta_{4t+1}\theta_{4t+2}\gamma_{4t+3}) + \frac{1}{2}\alpha_{4t+4}(1 - \gamma_{4t+3}) \\ &> 2(\gamma_{4t} - \gamma_{4t+4}) + A\theta_{4t+2} + A(a_{4t+4} - \alpha_{4t+4}), \end{aligned}$$

by Lemma 1, (3.423), (3.432), and (3.433). Also

(3.453) $$u_{4m} - u_{4m-1} = O(1) + O\left(\frac{\gamma_{4m}}{\theta_{4m}}\right) + O\left(\frac{\gamma_{4m+1}}{\theta_{4m+1}}\right) = O(1).$$

The result follows from (3.452) and (3.453).

3.46. Theorems 8 and 9 follow easily from Lemma 10.

The number N is a function $N(m,\theta)$ of m and θ alone. We write

(3.461) $$N_i = N(m,\theta_i) \qquad (i = 0, 1, 2, 3).$$

By Lemma 10,

(3.462) $$\sum_{i=0}^{3} s(N_i,\theta_i) > -A + A\sum_{r=0}^{4m+3}(a_r - \alpha_r) + A\sum_{r=1}^{4m+1}\theta_r.$$

If $a_r \geqq 4$, $a_r - \alpha_r > A\,a_r > \dfrac{A}{\theta_{r-1}}$; and if $a_r \leqq 3$, $\theta_{r-1} > \dfrac{A}{\theta_{r-1}}$. Hence

(3.462) involves

$$(3.463)\qquad \sum_{i=0}^{3} s(N_i, \theta_i) > -A + A\sum_{r=1}^{4m+2} \frac{2}{\theta_{r-1}}$$

$$> -A + A(4m+2)\,\mathfrak{N}^{\frac{1}{4m+2}} > -A + A\log\mathfrak{N} > A\log\mathfrak{N},$$

where

$$(3.4631)\qquad \mathfrak{N} = \frac{2^{4m+2}}{\theta\theta_1\ldots\theta_{4m+1}}.$$

If now $\overline{N}$ is the greatest of N, N_1, N_2, N_3, we have, by (3.439) and (3.413),

$$(3.464)\qquad \overline{N} < \frac{2^{4m+5}}{\theta\theta_1\ldots\theta_{4m+4}} < \frac{8\,\mathfrak{N}}{\theta_{4m+2}\,\theta_{4m+3}\,\theta_{4m+4}} < \mathfrak{N}^A.$$

Hence, if $s(\overline{n}, \overline{\theta})$ is that one of the sums $s(N_i, \theta_i)$ whose modulus is greatest, we have

$$|s(\overline{n}, \overline{\theta})| > A\log\mathfrak{N} > A\log\overline{N} > A\log\overline{n};$$

and $\overline{n} \to \infty$. Theorem 8 is therefore true for one of the three numbers $\theta, \theta_1, \theta_2, \theta_3$ and therefore, by the fundamental formula (3.18), it is true for θ.

The deduction of Theorem 9 is more immediate. In this case $\theta_r > B$[1]), and so

$$s(N, \theta) > -A + A\sum_{0}^{m-1}\theta_{4t+2} > Bm,$$

while, by (3.439), $N < B^{4m+2}$. It follows that

$$s(N, \theta) > B\log N,$$

which is one of the desired inequalities. The formula (3.18) then shows at once that

$$s(N', \theta) < -B\log N'$$

for arbitrarily large values of N'.

It is not possible, in the general case, to prove two-sided inequalities of the type $s > A\log n$, $s < -A\log n$. Herr OSTROWSKI has gone further in this direction; he has shown that s may in fact be bounded on one side, and has investigated the conditions under which this is possible[2]). We have not attempted to apply our own method to this problem, as we had not considered it before the publication of OSTROWSKI's memoir, and it is clear that a proof on these lines could not be so simple as his.

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1) B denotes throughout a positive number depending only on K.

2) The question is left open in O. (S. 92).

3.51. We return for a moment to the triangle problem. In this problem the analogue of Theorem 9 holds without restriction on θ. To prove this we require the lemma which follows, which is of course included in OSTROWSKI's work, where it occupies a more central position.

Lemma 11. *We have*

$$s(q_\nu, \theta) = O(1). \tag{3.511}$$

It is evident that, if $\frac{p_\nu}{q_\nu}, \frac{p_{\nu,1}}{q_{\nu,1}}, \frac{p_{\nu,2}}{q_{\nu,2}}, \ldots$ are typical convergents to $\theta, \theta_1, \theta_2, \ldots,$ we have

$$p_\nu = q_{\nu-1,1},\ p_{\nu-1,1} = q_{\nu-2,2}, \ldots.$$

If now we take $n = q_\nu$ in (3.18) or (3.19), we have $n_1 = p_\nu$ or $n_1' = p_\nu$, while d or e, whichever is relevant, is $O\left(\frac{1}{q_{\nu+1}}\right)$. Hence

$$s(q_\nu, \theta) + s(q_{\nu-1,1}, \theta_1) = O\left(\frac{1}{q_{\nu+1}\theta}\right),$$

$$s(q_{\nu-1,1}, \theta_1) + s(q_{\nu-2,2}, \theta_2) = O\left(\frac{1}{q_{\nu,1}\theta_1}\right),$$

and so on. Consequently

$$\begin{aligned} s(q_\nu, \theta) &= O\left(\frac{1}{q_{\nu+1}\theta} + \frac{1}{q_{\nu,1}\theta_1} + \ldots + \frac{1}{q_{1,\nu}\theta_\nu}\right) \\ &= O(\theta_1\theta_2 \ldots \theta_\nu + \theta_2 \ldots \theta_\nu + \ldots + \theta_\nu + 1) = O(1), \end{aligned}$$

by Lemma 1.

3.52. Theorem 10. *There is an A such that each of the inequalities*

$$U(\eta) > A\log\eta, \quad U(\eta) < -A\log\eta \tag{3.521}$$

is satisfied for every θ and arbitrarily large values of η.

It is sufficient[1]) to prove that

$$S(\eta) = \sum_{1 \leq \mu \leq \frac{\eta}{w}} \{\mu\theta - f'\} > A\log\eta \tag{3.522}$$

for arbitrarily large values of η, together with an analogous inequality involving $-A\log\eta$. By a change in η, of magnitude $O(1)$, we can make f' anything we please between 0 and 1[2]). It is therefore enough

[1]) *1*, p. 18.

[2]) *1*, p. 17.

to establish the existence of a $g = g(n)$ such that $0 \leqq g < 1$ and

$$S_1(n) = \sum_1^n \{\mu\theta - g\} > A \log n \tag{3.523}$$

for arbitrarily large values of n, with a corresponding result for $-A \log n$.

We suppose first that (1.331) is satisfied for some particular h, say $h = 2$. Then

$$\log q_{\nu+1} < A \log q_\nu. \tag{3.524}$$

There are, by Theorem 8, large values of n for which *one* of the inequalities

$$\sum_1^n \{\mu\theta\} > A \log n, \quad \sum_1^n \{\mu\theta\} < -A \log n$$

is true, say the first. Then (3.523) is true, with $g = 0$, for such values of n. Let N be one of these values, as determined in 3.42, and let

$$q_{\nu-1} \leqq N < q_\nu; \tag{3.525}$$

then, by Lemma 11,

$$\sum_{N+1}^{q_{\nu+1}} \{\mu\theta\} = \sum_1^{q_{\nu+1}} - \sum_1^N < -A \log N,$$

or

$$\sum_1^{q_{\nu+1}-N} \{m\theta + (N\theta)\} < -A \log N. \tag{3.526}$$

If now p is the least positive integer for which

$$0 < g = p\theta - (N\theta) < 1$$

and

$$n = q_{\nu+1} - N + p, \tag{3.527}$$

then $p < \dfrac{A}{\theta}$ and

$$\sum_1^n \{\mu\theta - g\} = \sum_1^{q_{\nu+1}-N} \{m\theta + (N\theta)\} + O\left(\frac{1}{\theta}\right) < -A \log N, \tag{3.528}$$

if N is large enough. But

$$\log N \geqq \log q_{\nu-1} > A \log q_{\nu+1} > A \log n,$$

by (3.525) and (3.524), and so

$$\sum_1^n \{\mu\theta - g\} < -A \log n$$

for an infinity of values of n.

3.53. This proves the theorem when (1.331) is satisfied for $h=2$. When it is not satisfied, much more is true; for then (1.332) is satisfied for $h=2$, and in this case, by Theorem 3, we have

(3.531) $$N(\eta) > A\sqrt{\eta}, \quad N(\eta) < -A\sqrt{\eta},$$

each for arbitrarily large values of η.

The proof of Theorem 3, given in 2.5, was transcendental; and it is worth while to observe that it may also be proved by an argument like that of 3.52. We have only to suppose that N is the n of 3.3 so that (say)

$$\sum_1^N \{\mu\theta\} > AN^{1-\frac{1}{h}}.$$

Arguing just as in (3.52), and making use of (3.341), we establish the existence of an infinity of values of n for which

$$\sum_1^n \{\mu\theta - g\} < -An^{1-\frac{1}{h}},$$

and the theorem then follows in the same way.

The Cesàro means of the series $\sum\{n\theta\}$.

3.61. A good deal of additional light is thrown on the behaviour of $s(n,\theta)$ by the study of the corresponding Cesàro mean

(3.611) $$\sigma(x,\theta) = \frac{1}{x}\sum_{m=1}^{x}(x-m)\{m\theta\}.$$

The study of $\sigma(x,\theta)$ leads us naturally to consider also the sum

(3.612) $$t(x,\theta) = \sum_{m=1}^{x}\left(\{m\theta\}^2 - \frac{1}{12}\right).$$

Lemma 12. *We have, in the notation of 3.1,*

(3.613) $$\sum_1^x\left(\alpha_m^2 - \frac{1}{12}\right) - \theta\sum_1^y\left(\beta_n^2 - \frac{1}{12}\right) = \frac{\Phi(\theta,\delta)}{\theta},$$

where

(3.614) $$\Phi(\theta,\delta) = \frac{1}{6}\delta(2\delta^2 - 3(1-\theta)\delta + 1 - 3\theta + \theta^2).$$

If in (3.12) we take $f(u)=u^2$, $g(u)=u$, express the summands in terms of α_m and β_n, and reduce, we obtain

(3.615)

$$\sum_1^x\left(\alpha_m^2 - \frac{1}{12}\right) - 2\left(\theta\sum_1^x m\,\alpha_m + \sum_1^y n\,\beta_n\right) + \left(\sum_1^x \alpha_m + \sum_1^y \beta_n\right) = \frac{P(\mu,\theta,\delta)}{\theta},$$

where P is a polynomial whose coefficients are absolute constants. If on the other hand we take $f(u)=u$, $g(u)=u^2$, we obtain a similar formula in which x, y; $\theta, \frac{1}{\theta}$; α_m, β_n are interchanged. When we multiply this by θ, and subtract from (3.615), the terms in $\sum m\,\alpha_m$ and $\sum n\,\beta_n$ disappear; and when finally we substitute for $\sum\alpha_m+\sum\beta_n$ from (3.16), we obtain the result of the lemma.

3.62. We say that θ belongs to class $\Gamma(H)$ if

(3.621) $$\sum_{r=0}^{\infty}\frac{\theta\,\theta_1\ldots\theta_{r-1}}{\theta_r}\leqq H.$$

The convergence of the series is equivalent to that of the series $\sum\frac{q_{r+1}}{q_r^2}$.

Theorem 11. *If θ is of class $\Gamma(H)$ then*

(3.622) $$t(x,\theta)=O(H).$$

Write

(3.623) $$x_1=\theta x,\quad x_2=\theta_1 x_1,\ldots,\quad \alpha_m^2-\frac{1}{12}=a_m.$$

By (3.613), we have

$$\sum_1^x a_m(\theta)-\theta\sum_1^{x_1}a_m(\theta_1)=O\left(\frac{1}{\theta}\right).$$

Write x_1, θ_1 for x, θ, and multiply by θ; x_2, θ_2 for x, θ and multiply by $\theta\,\theta_1$; and so on until the second sum disappears. Adding the resulting equations, and using (3.621), we obtain the result[1]).

3.63. Lemma 13. *If θ belongs to class $\Gamma(H)$, then*

(3.631) $$\sigma(x,\theta)+\sigma(y,\theta_1)=c(\theta)+O\left(\frac{H}{\theta x}\right)+O\left(\frac{1}{\theta^2 x}\right),$$

where

(3.632) $$c(\theta)=\frac{1}{4}-\frac{1}{12}\left(\theta+\frac{1}{\theta}\right).$$

We have, from (3.611) and (3.615),

$$2y\,(\sigma(x,\theta)+\sigma(y,\theta_1))=\frac{P}{\theta}+(2y-1)\left(\sum_1^x\alpha_m+\sum_1^y\beta_n\right)-\sum_1^x\left(\alpha_m^2-\frac{1}{12}\right).$$

The last term is $O(H)$, by Theorem 11. We write $\mu\,\theta+O(1)$ for y, substitute for $\sum\alpha_m+\sum\beta_n$ from (3.16), and divide by $2y$; and a simple calculation gives the result.

3.64. We say that θ belongs to class $C(K)$ if $a_n<K$, where $K>2$. In this case all of θ, θ_1, θ_2, … belong to a class $\Gamma(H)$ for which $H<AK$.

[1]) This is the revised form of the theorem stated incorrectly on p. 229 of our Cambridge communication, and noted as incorrect on p. 36 of *1*.

Lemma 14. *If θ is of class $C(K)$, and $\nu = \nu(x)$ is defined by*

(3.641) $$x_{\nu+2} = \theta\theta_1 \ldots \theta_{\nu+1} x < 1 \leqq \theta\theta_1 \ldots \theta_\nu x = x_{\nu+1},$$

then

(3.642) $$\sigma(x, \theta) = \sum_{r=0}^{\nu-1} (-1)^r c(\theta_r) + O(K^2).$$

We have, by repeated use of Lemma 13,

(3.643) $$\sigma(x, \theta) = (-1)^\nu \sigma(x_\nu, \theta_\nu) + \sum_{r=0}^{\nu-1} (-1)^r c(\theta_r)$$
$$+ O\left(\frac{H}{x}\left(\frac{1}{\theta} + \frac{1}{\theta\theta_1} + \ldots + \frac{1}{\theta\theta_1 \ldots \theta_{\nu-1}}\right)\right)$$
$$+ O\left(\frac{1}{x}\left(\frac{1}{\theta^2} + \frac{1}{\theta\theta_1^2} + \ldots + \frac{1}{\theta\theta_1 \ldots \theta_{\nu-2}\theta_{\nu-1}^2}\right)\right).$$

But, since $a_n < K$, we have $\dfrac{1}{\theta_r} = O(K)$, and the second line of (3.643) is

$$O\left(\frac{K}{\theta\theta_1 \ldots \theta_{\nu-1} x}(1 + \theta_{\nu-1} + \theta_{\nu-1}\theta_{\nu-2} + \ldots)\right) = O\left(\frac{K}{\theta_\nu}\right) = O(K^2),$$

by (3.641) and Lemma 1. Similarly the third line of (3.643) is $O(K^2)$. Finally

$$\sigma(x_\nu, \theta_\nu) = O(x_\nu) = O\left(\frac{x_{\nu+2}}{\theta_\nu \theta_{\nu+1}}\right) = O(K^2),$$

whence the result.

3.65. Theorem 12. *There are θ's of class $C(K)$ for which*

(3.651) $$\sigma(x, \theta) = O(1),$$

and others for which

(3.652) $$\sigma(x, \theta) \sim L \log x,$$

where $L > \dfrac{AK}{\log K}$ or $L < -\dfrac{AK}{\log K}$.

(i) If

$$\theta = \frac{1}{1+}\,\frac{1}{1+\ldots}$$

then $\theta_r = \theta$ for every r, and (3.651) follows directly from (3.642).

(ii) If $k = [K] \geqq 2$ and

$$\theta = \frac{1}{k+}\,\frac{1}{1+}\,\frac{1}{k+}\,\frac{1}{1+\ldots},$$

that so

$$\theta + \frac{1}{\theta} - \theta_1 - \frac{1}{\theta_1} = k - 1,$$

then it follows from (3.642) that

(3.653) $$\sigma(x,\theta) = (k-1)\left[\frac{1}{2}\nu\right] + O(K^2) \sim \frac{1}{2}(k-1)\nu.$$

Also

$$\log x = \sum_{r=0}^{\nu} \log\frac{1}{\theta_r} + O(\log K) = \sum_{r=0}^{\left[\frac{1}{2}\nu\right]} \log\frac{1}{\theta_{2r}\theta_{2r+1}} + O(\log K)$$
$$= \left(\left[\frac{1}{2}\nu\right]+1\right)\log\frac{1}{\theta\,\theta_1} + O(\log K) \sim \frac{1}{2}\nu\log\frac{k+2+\sqrt{k^2+4k}}{2}.$$

Thus we obtain (3.652) with

$$L = \frac{k-1}{\log\frac{1}{2}\left(k+2+\sqrt{k^2+4k}\right)} > \frac{AK}{\log K}.$$

If we exchange θ and θ_1, we obtain an example in which L has the opposite sign.

4. Conclusion.

4.1. The proof of Lemma 13 indicates clearly that, if we were to attempt the construction of a *complete* theory of the series $\sum \alpha_n$, it would be necessary to construct at the same time a theory of the series $\sum\left(\alpha_n^2 - \frac{1}{12}\right)$. A little further investigation shows that we must also consider the series $\frac{1}{12}\sum(4\alpha_n^3 - \alpha_n)$, ..., the n^{th} term of the p^{th} series being substantially the p^{th} Bernoullian function of $\alpha_n + \frac{1}{2}$. There are many curious theorems connected with these series: we content ourselves with mentioning one.

Theorem 11. *If θ belongs to a class $\Gamma(H)$ (and in particular if a_n is bounded) then the series $\sum\left(\alpha_n^2 - \frac{1}{12}\right)$ is summable $(C, 1)$, or by any Cesàro mean of positive order, to sum $-\frac{1}{12}$.*

4.2. We conclude by a brief reference to a different matter. It is of considerable interest to determine the largest half-planes in which the functions

(4.21) $$f_1(s) = \sum\frac{\alpha_n}{n^s},\quad f_2(s) = \sum\frac{\alpha_n^2 - \frac{1}{12}}{n^s},\quad f_3(s) = \frac{1}{12}\sum\frac{4\alpha_n^3 - \alpha_n}{n^s},\ \dots$$

are regular. HECKE has shown that, when θ is a quadratic irrational, $f_1(s)$ is *meromorphic* all over the plane, and has at most a doubly infinite system of simple poles, at the points

$$-2k \pm 2m\gamma\pi i \qquad (k,\ m = 0, 1, 2, \dots)$$

where γ is a constant depending on θ.

Our more elementary methods are applicable to problems of this kind also. Let λ be defined as the least number for which

$$\frac{(\theta\,\theta_1 \ldots \theta_{n-1})^{\lambda+\varepsilon}}{\theta_n} \to 0$$

for every positive ε, so that (1.331) is satisfied with $h = 1+\lambda+\varepsilon$ but not with $h = 1+\lambda-\varepsilon$. Then we can prove that

(a) *$f_p(s)$ is regular for*

$$\sigma > \sigma_p = 1 - \frac{p}{1+\lambda};$$

(b) *$\sigma = \sigma_p$ is a barrier of singularities for $f_p(s)$, except possibly when $\lambda = 0$, so that*

(c) *$f_p(s)$ is either regular for $\sigma > 1-p$, or has a barrier of singularities to the right of $\sigma = 1-p$, and in particular*

(d) *$f_1(s)$ is either regular for $\sigma > 0$ or has a barrier to the right of $\sigma = 0$;*

(e) *the series for $f_p(s)$ is summable by Cesàro's means for $\sigma > \sigma_p$ and in particular*

(f) *the series $\sum \alpha_n n^{-s}$ is convergent, when $\lambda > 0$, throughout the region of existence of the function $f_1(s)$.*

The propositions (a) and (e) have been proved independently, and in a different manner, by Herr BEHNKE[1]).

The case in which $\lambda = 0$ is exceptional and more difficult. It would seem that $\sigma = \sigma_p = 1-p$ is still a barrier in all cases *except that of a quadratic θ*, but this we have not been able to establish rigorously.

In this last case, finally, our method reveals the existence of HECKE's poles, though it does not render a complete account of them so readily as that of HECKE himself. This is only natural, as HECKE's method is so much more special and so much deeper than ours.

In view of the length of the memoir, we confine ourselves here to the statement of these results, reserving a fuller discussion for publication elsewhere.

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[1]) H. BEHNKE, Über die Verteilung von Irrationalitäten mod. 1. (Diese Abhandlungen, Bd. I, vorliegendes Heft.)