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XI. *Remarks on Prof. Henrici's Paper made by Prof. PERRY, F.R.S., in which he describes a Simple Machine which may be used to develop any Arbitrary Function in Series of Functions of any Normal Forms*.*

I CONGRATULATE Prof. Henrici, first upon his success in these Analysers, with which I shall presently form a practical acquaintance when the latest of them yet constructed reaches me from Zurich, second on the admirably clear way in which he described them to us.

I have had no experience with the hatchet, that simplest of all planimeters; but with regard to the Robertson-Hyne instrument, which comes to us from America, and which in the size here exhibited is well suited to Indicator-diagram work, I can say that some of my students instituted a careful comparison between it and the Amsler which they use for Indicator diagrams, and they found that the average error with it was about one third of that with the Amsler.

We know that in mathematical physics generally the development of an arbitrary function in a Fourier's Series is often of great importance; but I wish to say that this subject is becoming of greater and greater importance to the practical man—the engineer.

Thus in alternating-electric-current work, all the disturbing, distracting, dangerous troubles are considerably increased when the currents are not simple harmonic functions of the time. With two-phase or three-phase currents, if the amplitudes are not equal, the rotating magnetic field neither remains of constant strength nor has it constant angular velocity; and if there are overtones we have extra fields of changing magnitude, which rotate irregularly at two or more times the speed of the fundamental.

I have long thought that mechanical engineers need such instruments as Prof. Henrici has designed, if only to familiarize them with the ideas of Fourier. It has for some time been my habit, when studying with students any kind of reciprocating motion of a piece of machinery, to resolve the motion into its fundamental harmonic motion and overtones.

* Read April 13, 1894.

For example, if one is studying the forces causing the motion, one ought to keep in mind that the reciprocating motion which (speaking rather vaguely) requires the smallest forces or moments of forces to produce it, is the simple harmonic motion. The accelerating forces due to an octave are four times as great as for a fundamental of the same amplitude. The motion of the piston of a steam-engine is, with sufficient exactness for practical calculations, a fundamental of amplitude r and an octave of amplitude $\frac{r^2}{4l}$, where r is the length of the crank and l the length of connecting-rod.

A special graphical method of study may be discovered and employed for any special motion; but for applicability to reciprocating motions in general I know of nothing to compare with the method of study which is based on finding the fundamental motion and one or more overtones.

Again, the difference between one kind of slide-valve motion and another may be exceedingly great, practically, and yet the theories found in books show no difference at all. Indeed, the complete mathematical methods of study are too troublesome, but the mathematics of link motions and radial valve-gears become very simple when we consider, not merely the fundamental simple harmonic motion, which is all that is usually studied, but the octave, which is found to help or hurt in the various forms.

I was first attracted to this subject when studying the beautiful but little-known valve-motion invented long ago by Sir F. Bramwell, in which the only overtone is three times the fundamental.

Given any function completely, we can by a numerical method, and with as much accuracy as we please, develop it in Fourier's Series. In the 'Electrician' of Feb. 5th, 1892, I published the numerical work of one example calculating from 23 ordinates. In the sheet which I here exhibit one of my students, Mr. Fox, has done the same work by a graphical method. Probably he is the very first to carry out the idea of the late Prof. Clifford by descriptive geometry*. That is,

* *Note added May 29th.*—The descriptive geometry method is fairly quick, and may be made as accurate as one pleases, but of course it cannot compare in quickness with the Henrici Analyser.

It is obvious that by properly shaping one's cylinder, wrapping the

we have imagined the curve to be wrapped round the cylinder, and it was surprising to find how rapidly its projections could be drawn upon the two planes and their areas obtained by the planimeter. We then imagined the curve to be wrapped twice round and the projections drawn and their areas taken. I wish I had time to dwell upon the interesting problems that arose during the work, for example as to whether the area was to be taken as positive or negative. However many loops such a figure may possess, the well-known rule for autotomic plane circuits (Thomson and Tait's 'Elements,' §445) is really attended to by the planimeter. The direction of motion of the tracer must be that in which x increases on the real curve. I here give the results :—

The values of the arbitrary function to be analysed were really calculated from

$$y = 10 + 5 \sin \left(\frac{2\pi}{c}x + 30^\circ \right) - \sin \left(\frac{4\pi}{c}x - 60^\circ \right).$$

The result obtained numerically and published in the 'Electrician,' using 23 ordinates, was

$$y = 9.966 + 5.039 \sin \left(\frac{2\pi}{c}x + 29^\circ.9 \right) - 1.053 \sin \left(\frac{4\pi}{c}x - 55^\circ.3 \right).$$

The result now obtained graphically is

$$y = 10.01 + 5.0096 \sin \left(\frac{2\pi}{c}x + 30^\circ.38 \right) - 1.0099 \sin \left(\frac{4\pi}{c}x - 59^\circ.22 \right).$$

curve round it, and then finding the area of it, projected on a plane parallel to the axis, one may develop an arbitrary function in a series of any normal forms. Thus if $Q(x)$ is any tabulated function of x , and y is the arbitrary function of x , and we wish to find the integral $\int_0^a y \cdot Q(x) \cdot dx$, the shape of the curve which must be used instead of a circle in the Clifford construction is easy to find. It must be such that the cosine of the angle which the short length δx of the curve makes with the trace of the plane on which the projection is to take place shall be proportional to $Q(x)$, and several easy methods of drawing the curve or a series of such curves may be found. Once found, there is no more difficulty in developing any new arbitrary function in any series of normal forms than Mr. Fox found with his Fourier Series. A series of curves will be needed for a development in Zonal Harmonics, but only one curve will be needed for the Zeroth Bessels. These curves, or shapes of sections of cylinders, I am now proceeding to draw on a sufficiently large scale for exact work.

It is curious that Prof. Henrici should have based the construction of his first or 1889 instrument on the beautiful idea of the late Prof. Clifford, and not on what I call the Henrici principle. He gives the Henrici principle to explain the later instruments, and does not seem to see that his first instrument is the most beautiful example of its application. I take the Henrici principle to be that

$$\int y \cdot \sin \theta \cdot d\theta = \int \cos \theta \cdot dy,$$

the integrations being for a whole period. Well, in his first instrument, whilst its tracer moves through the distance dy , the ordinarily fixed part of the planimeter now has a displacement $\cos \theta$, and this is the same as if in the ordinary use of the instrument a curve is being traced whose ordinate is $\cos \theta$.

It is only on the assumption that the Henrici principle applies to his first instrument, that I venture to say that the following analyser is on the Henrici principle. We have at present to develop functions in sines and cosines, spherical harmonics and Bessel functions, because we know that when we have effected such developments we can convert them at once into the solutions of certain physical problems. As time goes on we shall require developments in many other normal forms. I propose to describe a machine which will effect any such development. I mean, that my machine will

evaluate the integral $\int_0^a f(x) \cdot Q(x) \cdot dx$, where $y=f(x)$ is an arbitrary function of x and $Q(x)$ is any tabulated function. Following Henrici, we convert the required integral into

$$\left[\int_0^a f(x) \cdot H(x) \right] - \int H(x) \cdot dy,$$

where $H(x)$ is the integral of $Q(x)$ and may be tabulated as $Q(x)$ is tabulated. Now in many cases the part between the square brackets is zero, but this is of little consequence in comparison with the fact that the part $\int H(x) \cdot dy$ may be evaluated by a machine somewhat like Prof. Henrici's first or 1889 instrument. I have worked with this 1889 instrument, and I am not disposed to think it so inaccurate as its inventor thinks it. Its defects are really defects of mechanical construction; for example, the amplitude of the simple harmonic motion of its table is very much too small.

I have already put my machine in hand and hoped to exhibit it here to-day in working order, but unfortunately the Easter holidays have prevented it being finished in time. It is arranged to develop an arbitrary function in Bessels of the zeroth order, or rather Fourier cylindric functions. Thus it is required to determine the constants $A_1, A_2, \&c.$, in

$$f(r) = A_1 J_0(\mu_1 r) + A_2 J_0(\mu_2 r) + \&c., \quad . \quad . \quad . \quad (1)$$

where $\mu_1, \mu_2, \&c.$ are the successive roots of some such equation as

$$J_0(\mu a) = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

or

$$\mu a J_1(\mu a) - \lambda J_0(\mu a) = 0, \quad . \quad . \quad . \quad . \quad (3)$$

where λ has a given value.

It is well known that

$$A_s = M \int_0^a r f(r) \cdot J_0(\mu_s r) \cdot dr,$$

where $M = 2/a^2 [J_1(\mu_s a)]^2$, if $\mu_1, \mu_2, \&c.$ are the roots of (2), and

$M = 2\mu_s^2/(\lambda^2 + \mu_s^2 a^2) [J_0(\mu_s a)]^2$, if $\mu_1, \mu_2, \&c.$ are the roots of (3).

In every case the practical difficulty consists in finding the integral. I exhibit to the Society an easy example of such an analysis worked out numerically (I suppose that such a thing has never been done before) by two of my students, Mr. H. F. Hunt and Mr. W. Fennell.

It will be seen that the work is rather tedious. It was made more tedious by their having found it necessary to calculate numbers and tabulate them in a handy form, interpolating between the numbers given in Lommel by the use of his formula. Before this work was finished we discovered Dr. Meissel's elaborate tables, from which the remainder of our handy four-figure tables is merely copied. These handy tables of $J_0(x)$ and $J_1(x)$ are at the service of the Society; they would occupy just four pages of the Journal. We have found them of practical value, but I do not know whether they are of such general value that they ought to be printed.

It is well known that $\int_0^x x J_0(x) \cdot dx = x J_1(x)$, and hence if

we write $\phi(r)$ for $\mu r J_1(\mu r)$, and y for our arbitrary function $f(r)$, the required integral $\int_0^a r f(r) \cdot J_0(\mu r) \cdot dr$ is

$$\frac{1}{\mu^2} \int_0^a y \cdot \frac{d\phi(r)}{dr} dr = \left[\frac{1}{\mu^2} f(r) \phi(r) \right] - \frac{1}{\mu^2} \int \phi(r) \cdot dy.$$

The part between the square brackets is usually 0, but however that may be, we see that we can evaluate the integral by the machine. A curve is drawn representing $f(r)$ from $r=0$ to $r=a$ on a sheet of paper which is wrapped round a roller. a need not be equal to the whole circumference of the roller and the scale of r is unimportant. Of course the ordinate y or $f(r)$ lies parallel to the axis of the roller. It is the measurement of y in inches which my instrument analyses, as my planimeter is graduated in square inches. A table whose upper surface is in a plane tangential to the roller carries the usually fixed part and rolling wheel of an Amsler planimeter. On turning through an angle θ a handle which drives a shaft on which a properly shaped cam is keyed, the table is displaced in its own plane towards the roller, through the distance $x J_1(x)$, θ being proportional to x , and at the same time the roller is driven so that the paper moves circumferentially through a distance proportional to x . For this particular kind of problem a few different but definite trains of gearing might be used to connect the handle and the roller, but for general purposes I would prefer variable friction gearing to give any relative speeds that may be necessary. In my model now being constructed I am using two disks, one of which rests on the other at a point which may be altered, radially. As in Prof. Henrici's instrument, the tracing-point of the planimeter is held against a straight edge, so that it can only move along the tangent-line of roller and table whilst following the curve on the roller.

If μ is a root of $J_0(\mu a) = 0$ as in the well-known drum-head problem, a being the radius of the drum-head,—in the first operation to find A_1 , the gearing must be adjusted so that when the whole curve on the roller passes under the tracing-point of the planimeter, a graduated circle on the

shaft turned by the handle indicates that it has turned through an angle proportional to $2\cdot405$, which is the first value of x which satisfies $J_0(x)=0$.

It is advisable to have a pointer and a scale to indicate exactly the displacement of the table, so as to test the accuracy with which the cam performs its duties. Of course, when the graduated-circle indication is $2\cdot405$, the displacement of the table is to be $2\cdot405 J_1(2\cdot405)$ or $-1\cdot249$ inches. The area recorded on the planimeter in square inches must now be multiplied by $2(2\cdot405)^2/a^4[J_1(2\cdot405)]^2$, and the answer is A_1 .

To find A_2 : change the gearing so that when the whole roller-curve passes under the tracing-point of the planimeter, the graduated circle indicates $5\cdot5201$, and check the error of the cam by noting that the displacement indication ought now to be $5\cdot5201 J_1(5\cdot5201)$ or $1\cdot878$ inches. The area recorded by the planimeter in square inches must now be multiplied by $2(5\cdot5201)^2/a^4[J_1(5\cdot5201)]^2$. If variable frictional gearing is used, it is important that the roller should be placed on roller bearings of small resistance.

To develop an arbitrary function in Bessels of any other order, or in Fourier's Series, or in zonal harmonics, or in series of functions of any other normal forms, we have only to replace the cam by one of another shape; so that this one simple machine is suited to quite general analytical use.

XII. *On the Mechanism of Electrical Conduction.*—Part I.

Conduction in Metals. By CHARLES V. BURTON, D.Sc.*

1. THE view of electrical conduction which it is here my object to explain receives general support from more than one consideration; for it leads to the conclusion that deviations from Ohm's Law must be quite inappreciable in the case of metallic conductors, and it goes far to explain, I think, why metals are so much less opaque than their ordinary conductivities would lead us to infer. But it is not

* Read April 27, 1894.