

ART. X.—*On the Algebraic Expression of the Diurnal Variation of Temperature*; by B. A. GOULD.\*

ALTHOUGH the general principles, which I am about to consider, apply to the cyclical mean variations of all the meteorological elements, we will restrict ourselves at present to the question of the diurnal changes of temperature; not merely for greater brevity, and because these fluctuations have a wider range and are relatively less affected by abnormal influences than those of the atmospheric pressure, moisture, etc., but because they have been made the subject of more extensive investigation, and because it is more especially regarding these that serious and harmful misapprehensions appear to exist.

It can scarcely be questioned by any one, who will give the subject his unbiased reflection, that a phenomenon due to the rotation and revolution of the earth,—however its direct manifestations may be modified by indirect and collateral influences, also purely natural,—must be governed by general laws which cannot fail to become manifest in the mean of observations sufficiently numerous. For if the indirect perturbations of the fundamental law belong to the class which may properly be regarded as casual, they will be eliminated from the mean values; and if they are not of this class they must and should be included in the general law which it is desired to discover.

Nor can it be reasonably denied, that if the diurnal variation be in fact governed by a general law, this must be of a cyclical nature, and capable of expression by some formula which completes its period with the solar day.

There are various formulas of this class, many of them interconvertible by simple analytic modifications; but none, of a thoroughly general character, which is simpler than a series progressing according to sines and cosines of successive multiples of the time elapsed since an adopted epoch. For phenom-

\* Translated from Vol. II. Chapter vi, of the "Anales de la Oficina Meteorologica Argentina," by the author.

ena of the class in question, where the curve is continuous, this is of absolute generality; and, although other forms of expression may claim equal accuracy, none surpasses, or probably equals, it in the ease with which its constants may be computed, or the facility with which its value may be deduced for any value of the variable. It is unlikely that these advantages would escape the attention of any mathematician who might desire to give algebraic expression to the law of a periodic phenomenon. But it is generally known among meteorologists by the name of Bessel's formula, not so much because it was first employed by that great astronomer in studying the diurnal oscillations of the barometer, as on account of the elaborate articles in which he long afterward pointed out its generality, its convergence, and the special circumstances which facilitate its employment and application; and earnestly commended its employment to meteorologists in determining the diurnal and annual variations of atmospheric phenomena, the influence of winds from different directions, etc.

In fact a general algebraic formula gives, as Bessel said, the result of observations expressed in the concisest form. To translate his words: "by deducing a formula which represents a phenomenon, we meet its theory half-way; thus the observed longitudes of the sun, if thus expressed as functions of the annual period would have given a formula from which, were Kepler's discovery yet unmade, the elliptic motion would be far more easily recognized than from the observations directly. . . . The periodic phenomena of meteorology belong to the same class, and the determination of their formulas, by means of observation, enables us to subject them to the readiest and most complete examination; so that whatever is to be explained by theory may be more directly deduced from the formulas than from the observations themselves or from the curves by which it is customary to represent the results of unknown laws."

The only true object of scientific investigation is the determination of principles and laws: and it may well be questioned what high aim can be attained by accumulating special results without such generalization.

It is therefore with deep regret, as well as surprise, that we have seen the eminent director of the Physical Central-Observatory of Russia, under whose immediate supervision stand all the physical observations officially made in that vast realm, express himself in terms like the following, which I quote from among very many other statements of the same sort:

"How much time has been uselessly squandered in observations laboriously carried on through day and night, and in extended and far more useless calculations of the same by the Lambert-Bessel interpolation formula."

"In the lately published volume of the physical observations by the American expedition of the *Polaris*, Mr. — has performed the most disheartening task of utterly useless calculation of all the meteorological elements according to Bessel's formula. What a pity that the time was thus lost which might have been employed in making better use of these excellent observations!"

"If then, under the erroneous idea that a certain number of terms of the Lambert-Bessel interpolation-formula can give the law of the diurnal variation of temperature, it has in many cases been sought to deduce therefrom the values for intermediate hours, or for the maxima or minima, . . . we cannot wonder at the failure of such calculations."

"The application of Bessel's formula to represent the daily course of the temperature has, thus far, much more hindered than advanced our knowledge of the same."

These citations are perhaps more than sufficient for my purposes; which are: first, to show the facility with which a backward step may be taken, even by one whose services to his department of science have been and are distinguished and undeniable, when a dominant idea is allowed to distort his judgment; and secondly, to oppose the influences of argument and demonstration to these attempts at discouraging what I believe to be the only mode by which meteorology can become a science, and by which it bids fair soon to attain that rank.

Had the author perceived that the sine-formula is not simply a means of interpolation, but a concise mode of general, complete and definite expression of a cyclical law, he would have aimed his denunciations at its frequent misapplication, but not at its use. The immediate origin of his strongly expressed opinions seems to have been his recognition of a relatively sharp bend in the curve of mean temperature at many stations, near the time of sunrise, and of sundry discordances between the epochs of maxima and minima as deduced under his direction by graphical methods, on the one hand, and by computation on the other. But it would be strange if the upward bend near sunrise should not be essentially balanced by a downward flexure after sunset, although the effect of this might be less conspicuous in the curve when graphically delineated, and the sharpness with which the moments of maxima and minima may be determined depends upon the form of the curve in their vicinity. Inasmuch as the minimum usually precedes sunrise by only a small interval, while the maximum generally precedes sunset by a very considerable one, a want of symmetry in the shape of the curve near the two daily extremes ought to be expected. Using identical data, it is a remarkable assertion that the free hand and judgment of a draughtsman can afford a more correct representation of the true curve than would be

obtained from accurate numerical calculation by means of the formula.

I am not aware that it has ever been considered desirable to base extended numerical calculations upon observations made more frequently than once an hour, for the determination of the diurnal curve; nor do I understand that the employment of such an increased number is now advocated. Were this desired, the influence of the additional observations could, if perceptible, be introduced into the formula quite as well as into the graphical sketch. It may therefore be assumed that the form of the diurnal curve, as afforded by hourly observations is amply sufficient for all the present purposes of scientific inquiry.

So much being premised, it may be well briefly to consider, even though only in an elementary manner, how and to what extent the formula in question may find legitimate application.

Denoting by  $h$  the number of hours elapsed since any given moment (for which we will, as usual, adopt that of midnight), the observed temperature will be represented with absolute precision for each one of the twenty-four hours, by the formula which, in its most compact and elegant form, may be written

$$T_h = M + a \sin(h + A) + b \sin(2h + B) + c \sin(3h + C) + d \sin(4h + D) \\ + \dots + l \sin(12h + L)$$

in which  $a, A, b, B, c, C$ , etc., are constants which can only be determined by observation, and  $M$  is the mean of the twenty-four equidistant observations, this being the "daily mean" as defined by the established usage of physicists. It is perhaps needless to add that  $h$  should in strictness be made to denote hours of true and not of mean time; but this is a point which need not be considered if the length of the interval, for which the formula is determined, be either such as to eliminate the equation of time from the means of the daily observations, or so short that its inclusion may be recognized and allowed for, if desired.

An absolute representation of each of the twenty-four hourly observations, although afforded by the formula if its full number of twelve terms be employed,\* is more than the demands of meteorology require; for all the errors of observation, not eliminated from the data, would thereby be reproduced, whereas by

\* The curious numerical values upon p. 60 of Dr. Wild's "*Temperatur-Verhältnisse des Russischen Reiches*" from which it is inferred that the legitimate results from twenty-four equidistant observations cannot be fully represented by a formula containing twenty-four constants, are vitiated by the joint effect of an error of computation and the employment of an insufficient number of decimals. The effect of these errors is quite insignificant, it is true, and amounts to only a very few hundredths of a degree; but it is precisely upon these small quantities, much inferior in magnitude to the unavoidable errors of observation, that the whole argument of the author depends.

the omission of the latest terms these errors will be equated out to any desired extent in the same way that this end could be attained by graphical methods, but with far more accuracy. And a general expression, which sufficiently represents the known mean temperature at intervals of an hour throughout the day, cannot fail to give that which corresponds to any intermediate moment with all the exactness needful in the present state of science. But since each variable term of the equation as above written contains two unknown quantities, its determination demands, in the absence of other data, a knowledge of the temperature corresponding to two different hours at proper intervals.

As regards the degree of minuteness with which it is desirable that the formula should represent the original observations, there is much to say: but it will suffice in this place to call attention to the fact that the mean of values derived from a lifetime of observations would probably be inadequate to represent the true typical value, even within the limits of appreciable error. Most of the so-called casual influences, which so greatly disturb the normal values that the result of a single year's observations affords but a first approximation to the fundamental law,—are it is true, themselves subject to the law of probabilities, and are eliminated from the mean of the observations made during a large number of successive years. But in general the number of years requisite for such elimination will be far greater than is available; so that any minute and scrupulous attention to the computation of quantities smaller than the probable error of the data is misapplied, and even absurd, unless such quantities are demonstrably constant for the various individual values whose mean is taken. To devote punctilious attention to hundredths of a thermometric degree in results derived from data which are uncertain by ten or twenty times that amount is to “strain out a gnat and swallow a camel.” From this point of view, also, it may be maintained that the last terms of the formula might be advantageously omitted.

The problem, which most frequently presents itself in practice, is to determine with sufficient approximation and from a relatively small number of daily observations, the most characteristic points of the curve and the daily mean. This is easy in proportion to the rapidity of convergence of the formula; and the degree of this rapidity affords a criterion of the extent to which the true law is represented by the earlier terms. If, by a few terms of the formula, containing no more than six or eight constants instead of twenty-four, the whole series of hourly temperatures can be represented within the limits of their probable errors, no further demonstration is requisite to show that the true law of daily variation may thereby be represented to an equal degree.

The general relations between the curves of daily variation at different places happily provide a means of deducing very approximate values of the constants from a smaller number of daily observations than would otherwise be requisite. Were the true moment known for either the maximum, or the minimum, or the true value of the daily mean, any one of these would supply the place of an additional daily observation. And the simple self-evident fact that, in the absence of very abnormal topographical conditions, the diurnal curve has but one point of contrary flexure, affords yet additional facilities for attaining the desired end.

The most important term of the formula, for various reasons, is the constant  $M$ , the daily mean temperature, which may be deduced with very considerable approximation from a relatively small number of observations. Thus in the mean of two values, corresponding to moments separated by an interval of half the day, all terms containing uneven multiples of  $h$  are eliminated, so that

$$\frac{1}{2} (T_h + T_{h+12}) = M + b \sin (2h + B) + d \sin (4h + D) + f \sin (6h + F) + \text{etc.},$$

while from the half difference of the same values we obtain the total amount of the omitted terms,

$$\frac{1}{2} (T_h - T_{h+12}) = a (\sin h + A) + c \sin (3h + C) + e \sin (5h + E) + \text{etc.},$$

a value which the, frequently very rapid, convergence of the series often renders extremely useful.

Thus from the mean of three equidistant values, all terms are eliminated in which the coefficient of  $h$  is not divisible by 3; so that

$$\frac{1}{3} (T_h + T_{h+8} + T_{h+16}) = M + c \sin (3h + C) + f \sin (6h + F) + \text{etc.}$$

Similarly the mean of four observations, at intervals of six hours, gives us

$$\frac{1}{4} (T_h + T_{h+6} + T_{h+12} + T_{h+18}) = M + d \sin (4h + D) + \text{etc.},$$

in which the first variable term is generally of small importance, and the remaining two absolutely insignificant.

And, in general, all those terms which do not depend upon multiples of  $nh$  are eliminated from the mean of  $n$  equidistant values so that the mean of observations made at intervals of three hours will differ from the daily mean only by the amount of the eighth term of the general formula; and that deduced from two hourly observations will differ from it only by the twelfth term. How small the coefficients of these high terms really are, will soon be seen.

From considerations analogous to the foregoing, we may easily perceive what is the amount by which the mean of determinations, made at any given hours of the day, will differ from the true daily mean. For example, if there be three daily determinations at the hours  $h+m$ ,  $h+n$ ,  $h+p$ , their mean will be

$$M + aq_1 \sin (h + A + Q_1) + bq_2 \sin (2h + B + Q_2) + cq_3 \sin (3h + C + Q_3) + \text{etc.}$$

in which

$$q_1 \sin Q_1 = \frac{1}{3} (\sin m + \sin n + \sin p); \quad q_2 \sin Q_2 = \frac{1}{3} (\sin 2m + \sin 2n + \sin 2p)$$

$$q_1 \cos Q_1 = \frac{1}{3} (\cos m + \cos n + \cos p); \quad q_2 \cos Q_2 = \frac{1}{3} (\cos 2m + \cos 2n + \cos 2p) \text{ etc.}$$

Supposing the hours of observation to be  $7^h$ ,  $14^h$ , and  $21^h$ , as we have arranged them for this country, the corresponding quantities become

$$\begin{aligned} q_1 &= 0.161, & q_4 &= 0.667; & Q_1 &= 14^h, & Q_4 &= 8^h \\ q_2 &= 0.244, & q_6 &= 0.311; & Q_2 &= 16^h, & Q_6 &= 10^h \\ q_3 &= 0.805, & q_8 &= 0.333; & Q_3 &= 18^h, & Q_8 &= 12^h \text{ etc.} \end{aligned}$$

Were the hours of observation the somewhat more symmetrical, although in the present case less advantageous, ones  $7^h$ ,  $13^h$ , and  $21^h$ , we should have

$$\begin{aligned} q_1 &= 0.173, & q_4 &= 0.577; & Q_1 &= 12^h, & Q_4 &= 6^h \\ q_2 &= 0.333, & q_6 &= 0.644; & Q_2 &= 18^h, & Q_6 &= 12^h \\ q_3 &= 0.745, & q_8 &= 0.333; & Q_3 &= 16^h, & Q_8 &= 6^h \end{aligned}$$

The smallness of the coefficients for the earlier variable terms shows how near is the approach to the desired value of  $M$ . In fact the approximate value thus obtained, is usually sufficient, when considered in connection with the general form of the diurnal curve (also approximately known), to afford valuable aid in attaining a knowledge of the principal terms of the formula, as will hereafter be seen.

Let us now consider the problem in the form in which it practically offers itself for those places in the Argentine Republic at which we have been able to secure three daily observations. From these it is desired to infer the mean daily curve with the greatest degree of approximation which they can be made to yield; availing ourselves of every circumstance which can help to bring us nearer to the truth, yet avoiding illusory results which although derived from the data may have no foundation in the true law.

There are but three points in all the national territory at which more than three daily observations have been made

during a sufficient period to permit the form of the daily curve of temperature to be deduced independently. These are Buenos Ayres, where there were but two additional hours of observation; Bahia Blanca, where the different hours, though comparatively numerous, were at different dates; and Cordoba, where the series of hourly observations comprises but a small number of years. And moreover, those hours, which it has been found necessary to adopt for the regular observations made for this office in the various parts of the Argentine territory, are not those which permit an exact determination of the daily mean, even so far as the accuracy of the determination is uninfluenced by the third variable term of the general formula.

But notwithstanding these adverse circumstances, there are favorable considerations of yet more importance. In the first place, the configuration of the country is such as to make it probable that throughout the vast extent of the pampa regions, at least, the general form of the daily curve is not dissimilar; secondly, we have, in the periodic sine-formula arranged according to multiples of the time, a series of terms, each of which is independent of the rest, and susceptible of separate investigation, being in fact so many epicycles; thirdly, the mean daily curve cannot have more than one point of contrary flexure unless the local topographical conditions be so exceedingly abnormal that the investigation of the daily curve would have no general scientific value; and in addition to all these, the curvature in the vicinity of the maxima and minima is so gentle that a very considerable error as to the moment of their occurrence is productive of but comparatively slight influence upon the form of the computed curve, or upon other values which are determined through their agency.

If we neglect the variable terms beyond the third, we shall have from the mean of observations made at any of our stations at 7 A. M., 2 P. M., and 9 P. M. during a given interval, the three equations

$$(1) \quad T_7 = M + a \sin (7^h + A) + b \sin (14^h + B) + c \sin (21^h + C)$$

$$(2) \quad T_{14} = M + a \sin (14^h + A) + b \sin (4^h + B) + c \sin (18^h + C)$$

$$(3) \quad T_{21} = M + a \sin (21^h + A) + b \sin (18^h + B) + c \sin (15^h + C)$$

and if the epochs  $H_1$ ,  $H_2$ , of maximum and minimum together with the corresponding temperatures  $m_1$ ,  $m_2$ , were known, we should have the four additional equations

$$(4) \quad 0 = a \cos (H_1 + A) + 2b \cos (2H_1 + B) + 3c \cos (3H_1 + C)$$

$$(5) \quad 0 = a \cos (H_2 + A) + 2b \cos (2H_2 + B) + 3c \cos (3H_2 + C)$$

$$(6) \quad m_1 - M = a \sin (H_1 + A) + b \sin (2H_1 + B) + c \sin (3H_1 + C)$$

$$(7) \quad m_2 - M = a \sin (H_2 + A) + b \sin (2H_2 + B) + c \sin (3H_2 + C)$$



and by the seven independent equations thus available, the seven unknown quantities which they contain could be fully determined; so that the form of the daily curve would be absolutely known, excepting so far as it might be affected by epicyclical terms which complete their period in six hours or still shorter aliquot parts of a day.

Unfortunately the data relative to the extremes are only roughly, even when at all, known; yet the condition that there shall be but one\* contrary flexure in the curve, together with the possibility of inferring a very approximate value for  $M$ , from the observed temperatures at  $7^h$ ,  $14^h$ , and  $21^h$ , provide the means of remedying these deficiencies to a considerable extent; while it may be justifiable to assume an analogy between the form of the curve sought, and that of those already determined for Bahia Blanca, Buenos Ayres, and Cordoba, to an extent to diminish still further the probable errors of the determinations. In this way very approximate values of the constants of the first three terms may frequently be obtained; and although no one would maintain that an absolute representation of the true daily curve could thus be secured, still we should certainly obtain a curve of which the deviation from the true one is very small, and which intersects it in six nearly equidistant points.

The first, and usually the most troublesome, process to be

\* Notwithstanding indications of secondary maxima and minima have been found in observations made at sundry places, and mentioned by Kämtz and Karsten, while they have been especially discussed by Hellmann, the reality of the phenomenon seems by no means proved. The most conspicuous case of their occurrence is at St. Petersburg, in the new series of observations organized by Dr. Wild with such care that he thinks the error of a single determination cannot exceed  $0.1$  (*Temperatur-Verhältnisse*, p. 29). In the mean of the six years' observations of this series, minima, other than those which he accepts as the true ones, occur at about  $3\frac{1}{2}^h$  in November,  $21\frac{1}{2}^h$  in December, and  $0\frac{1}{2}^h$  in January. Although the amounts of the abnormal fluctuations are petty in themselves, they are far from being unimportant relatively. Indeed for the month of December, for which the maximum mean temperature, as given by the observations, is  $7^{\circ}36'$  at  $13^h 45^m$ , and the minimum  $8^{\circ}28'$  at  $21^h 42^m$ , Wild adopts as the true minimum,  $8^{\circ}03'$  at  $9^h 0^m$ , or only thirty-three minutes earlier than the epoch which he himself gives for the mean temperature of the day.

We cannot but share his opinion that this abnormal fluctuation, by reason of which the observed temperature remains below its minimum for twelve or thirteen hours of the twenty-four, is to be "considered as a remainder from non-periodic perturbations," and likely to disappear from the mean of many years' observations. This is made probable by the fact that the twenty-two years of the old series show no indications of such a phenomenon.

The real existence of any secondary maxima and minima is flatly denied by Wild, and in this we believe him to be correct. But in accounting for their apparent occurrence, he says that it is either simply due to irregular perturbations not equated out, or generated by calculations with Bessel's formula (*Temp. Verhältnisse*, p. 103). Had he said "erroneously inferred from inadequate data," there might have been a show of reason in his statement. But it ought to be needless to mention that errors can only be produced by the use of this general formula in the same way in which they can be caused by the use of a book of logarithms; namely by its misapplication.

effected is the determination of a sufficiently accurate value of  $M$ , the daily mean. When we know the epochs of maximum and minimum or the corresponding temperatures within narrow limits this is easy, for having once attained moderately correct values for  $M$ ,  $m_1$ , and  $m_2$ , tentative processes will soon guide us to congruous values of all three which will accord with the known general form of the daily curve. This can be completely accomplished by numerical methods only, as must be the case with every process of the sort which can claim to be exact; but as is also commonly the case it may be greatly facilitated by graphical methods.

For a first approximate value of  $M$ , the mean of the temperatures observed at 7<sup>h</sup>, 14<sup>h</sup> and 21<sup>h</sup> will usually suffice, although it is always too high, its value being

$$M + 0.16 a \sin(A + 14^h) + 0.24 b \sin(B + 16^h) + 0.80 c \sin(C + 18^h) \\ + 0.67 d \sin(D + 8^h)$$

but when the constants of either the first or second variable term are approximately known, a much closer estimate may be made by taking in the former case  $\frac{1}{2}(T_7 + 2T_{14} + T_{21})$  which gives

$$M + 0.37 a \sin(A + 14^h) + 0.07 b \sin(B + 4^h) + 0.85 c \sin(C + 18^h) \\ + 0.72 d \sin(D + 8^h)$$

or in the latter case  $\frac{1}{5}(2T_7 + T_{14} + 2T_{21})$  which gives

$$M + 0.01 a \sin(A + 2^h) + 0.49 b \sin(B + 16^h) + 0.77 c \sin(C + 18^h) \\ + 0.60 d \sin(D + 8^h)$$

and substituting the numerical values of the known constants. In practice it is almost always possible to estimate the magnitude of the unknown constants with sufficient correctness to deduce a very close approximation to the true values.

But, in the absence of any knowledge of the constants, a very near approach to the true daily mean may be generally obtained for this country from the combination  $\frac{1}{17}(6T_7 + 5T_{14} + 6T_{21})$ ; this being equal to

$$M + 0.11 a \sin(A + 14^h) + 0.32 b \sin(B + 16^h) + 0.79 c \sin(C + 18^h) \\ + 0.65 d \sin(D + 8^h)$$

and the relations between the constants of the different terms being in this region such that the several variable terms in this formula nearly cancel one another.

With the preliminary value of  $M$ , obtained by any of these devices, is to be combined the approximate time  $H$ , of the daily maximum. This epoch, as also that of the minimum, is so variable on account of the large effect exerted upon it by disturbances of the casual class, that a very long series of years

would be needful for determining it with any accuracy. It is furthermore so uncertain, and its determination so insecure, owing to the great extent to which the time of its apparent occurrence is influenced by the inevitable errors of observation, that it may well be questioned how far the minutiae of calculation can advance our real knowledge regarding it, and whether much time has not been uselessly expended in attempts to fix with precision a moment which is in its nature unprecise. But, without entering upon these questions, it may be stated that a considerable error in the assumed epoch will give rise to a relatively small error in the final result. Where no special means of forming an estimate can be made useful, it may be well to begin with the supposition that the highest temperature occurs at 2 P. M. daily.

If we disregard the third variable term in the equation (1) to (4), we have now a means of determining, first, approximate values for the constants  $a$ ,  $A$ ,  $b$ ,  $B$ ; and from these a corresponding curve, which must necessarily intersect the true mean daily curve in the three points given by observation. But, unless the assumed values of both  $M$  and  $H$  have been very near the truth, this very circumstance will cause the general form of the computed curve to be markedly discordant from that of the true one, already approximately known; and very frequently it will be found that double curvatures will present themselves as the only geometrical mode of satisfying the condition that the areas above and below the medial line must be equal.

A second computation is now to be made with a slight change in the assumed value of  $M$ , which a little tact on the part of the computer will usually enable him to bring nearer to the truth; but should a third be found needful, it will be convenient to change the assumption by about the same amount as before, to permit an easy use of the rule of proportion. It will be found that the increase or diminution of the values of the resultant constants depends upon the value assumed for  $M$ ; and a comparison of the corresponding curves will usually permit its true value to be fixed within extremely narrow limits.

Adopting for  $M$  the value thus derived from the first assumption regarding the time of daily maximum,—the question next arises, to what extent any other assumption would bring the form of the curve into nearer analogy with any known one at some station not too diverse either in its geographical or its topographical relations. This may be determined in a similar way by varying the assumptions until a satisfactory result is attained. Should the previous value of  $H$  be found to require any considerable modification, the determination of  $M$  may be revised, although any change in this would be due to the influence of terms of the second order.

It is manifest that the trustworthiness of this method must depend upon a convergence of the successive approximations toward one and only one value of each of the desired constants,  $M$  and  $H$ . This may be tested whenever desired by a similar process, employing an assumed epoch for the daily minimum; or, what amounts essentially to the same, by comparing the epoch resulting from the determinations already made with that which is made probable by other considerations. In the case that observations of the minimum temperature exist, these will afford a more delicate criterion than those of the maximum, since the epoch of minimum is during most of the year far more distant from our hour of morning observation, than is that of maximum from 2 P. M.

It has been already stated that the earlier stages of the computation can be greatly facilitated by graphical processes, but in this place we are only considering the regular numerical routine available under all circumstances.

I append three examples, to illustrate the mode of computation. These are selected from among those places for which hourly observations are on record, because, while they are markedly diverse in their geographical relations, they also belong to the class most difficult to determine, namely that for which we can derive no indication regarding the form of the diurnal curve from a knowledge of that which corresponds to any place in the vicinity or even similarly situated. Thus we have a severe test of the degree of approximation to the true law which is attainable by means of the daily observations at 7<sup>h</sup>, 14<sup>h</sup> and 21<sup>h</sup>, without any other data whatsoever. The discordances, between the hourly temperature thus inferred and those derived from actual observation, represent the sum-total of the various errors due to inaccuracies in determining the constants, and to the disregard of all terms of the general formula beyond that which depends on twice the time. It has been already mentioned that in practice the determinations may be much facilitated by the use of graphical methods for the first approximations; but we will dispense with them in our examples, and likewise with the various indirect processes which may often be advantageously employed; and begin in each case by assuming the mean of the three daily observations for the first approximation to the true daily mean, and 14<sup>h</sup> for the hour of maximum. For readily perceiving the general form of the curve, a rough drawing is of course always desirable.

1. First, let us take the observations made in the month of January during five successive years at the island of St. Helena, and published in degrees of Réaumur's scale by Dove in his second series. Here we have  $T_7 = 13.05$ ,  $T_{14} = 16.02$ ,  $T_{21} = 13.74$ , the mean of which is 14.27.

Putting this mean for  $M$  and  $14^h$  for  $H_1$ , the equations (1) to (4) give  $a=1.09$ ,  $A=225^\circ$ ,  $b=0.71$ ,  $B=41^\circ$ . The resultant curve has a strongly marked secondary minimum at  $21^h$  with its corresponding maximum at about  $1^h$ , showing this value of  $M$  to be inadmissible.

A second supposition,  $M=14.21$ , gives  $a=1.27$ ,  $A=227^\circ$ ,  $b=0.59$ ,  $B=44^\circ$  and this curve also exhibits a contrary flexure analogous to the former although less pronounced.

A third supposition,  $M=14.15$ , gives  $a=1.46$ ,  $A=229^\circ$ ,  $b=0.47$ ,  $B=48^\circ$ , and since in this curve there is no secondary maximum, the question arises as to the value of  $M$  between  $14.21$  and  $14.15$  for which this disappears. It will quickly be perceived on trial that this value is very near to  $14.18$  for which  $a=1.36$ ,  $A=228^\circ$ ,  $b=0.53$ ,  $B=46^\circ$ .

Inspecting now the corresponding curve and comparing it with straight lines drawn to connect the point for  $14^h$  with those for the other two observations, it becomes evident at a glance that the hour of maximum was assumed too early, and that a better assumption would have been  $14^h 30^m$ . This latter gives the equation

$$T = 14.18 + 1.385 \sin(h + 224^\circ 24') + 0.508 \sin(2h + 26^\circ 41')$$

There is in this curve no indication of any failure to fulfill all the conditions of our problem; and in the total absence of other data, the equation may be regarded as a sufficient representation of the diurnal curve. Some slightly more probable values for the constants might be obtained by new trials, which would fix better limits for  $M$  with  $14^h 30^m$  and better limits for  $H_1$  with  $M=14.18$ ; inasmuch as each of these determinations is quite close enough to render quantities of the second order negligible. Nevertheless any such additional calculations would imply a minuteness scarcely appropriate to the character of the problem. But did we possess any one of the not unfrequently available additional data which are afforded by an approximate knowledge of the time or value of the daily maximum or minimum, we should thus be enabled to fix the constants with much greater precision; while a knowledge of all of them would permit us to deduce with considerable accuracy those of an additional term in the formula.

The hourly values of the temperature which correspond to the curve just deduced are given below, side by side with those actually observed. The grouping of the algebraic signs in the residuals, and the relative magnitude of these, show how large is that part of them due to the omission of the term depending upon  $2h$ .

Hour.	Temperature.		0°-C.	Hour.	Temperature.		0°-C.
	Observed.	Computed.			Observed.	Computed.	
1 <sup>h</sup>	13°·36	13°·41	-0°·05	13 <sup>h</sup>	15°·85	15°·79	+0°·06
2	13°·28	13°·36	-0°·08	14	16°·02	16°·02	0°·00
3	13°·18	13°·25	-0°·07	15	15°·97	16°·01	-0°·04
4	13°·15	13°·12	+0°·03	16	15°·90	15°·80	+0°·10
5	13°·10	13°·00	+0°·10	17	15°·69	15°·42	+0°·27
6	13°·06	12°·96	+0°·10	18	15°·13	14°·94	+0°·19
7	13°·05	13°·05	0°·00	19	14°·48	14°·47	+0°·01
8	13°·26	13°·30	-0°·04	20	13°·95	14°·04	-0°·09
9	13°·68	13°·72	-0°·04	21	13°·74	13°·74	0°·00
10	14°·26	14°·24	+0°·02	22	13°·61	13°·56	+0°·05
11	14°·83	14°·83	0°·00	23	13°·52	13°·47	+0°·05
12	15°·35	15°·38	-0°·03	24	13°·45	13°·44	+0°·01

So far as it can be deduced from the hourly observations, the true formula to the 2<sup>d</sup> variable term inclusive (i. e. with four constants) is

$$T = 14\cdot20 + 1\cdot415 \sin (h + 223^\circ 48') + 0\cdot500 \sin (2h + 20^\circ 50')$$

and the true daily maximum is at 14<sup>h</sup> 38<sup>m</sup>.

2. Let us next consider the observations made at Hobart Town in Tasmania during the month of January in eight successive years, also published by Dove and expressed in degrees of Réaumur. Here  $T_1 = 11^\circ 90$ ,  $T_{14} = 17\cdot29$ ,  $T_{21} = 12\cdot07$ , the mean of these three temperatures being  $13\cdot75$ .

Beginning with this for the value of  $M$  we find the secondary maximum to be very pronounced; and assuming successive values, inferior to this, we find that the highest which gives, for  $H_1 = 14^h$ , a line without contrary flexure is  $13^\circ 45$ , and that none above  $13^\circ 40$  fails to show decided indications of a secondary maximum between  $0^h$  and  $4^h$ . For  $M = 13^\circ 40$  we have  $a = 3\cdot22$ ,  $A = 238^\circ 46$ ,  $b = 0\cdot68$ ,  $B = 32^\circ 56$ , which represent a curve with rounded and flowing outlines. Yet even this not only exhibits a very marked want of symmetry between the portions above and below the medial line, but also shows in other ways the need of further modification. For instance, its minimum occurs before  $2\frac{1}{2}$  A. M., or more than three hours before sunrise, which is totally abnormal for the latitude  $42^\circ$ ; and although the general form of the curve near the maximum,—which is fixed within comparatively narrow limits by the three fundamental observations,—is demonstrably one of rapid rise and fall, yet that portion of the computed curve which corresponds to the night hours is abnormally flat. These considerations agree in indicating that the true value of  $M$  is still smaller.

If we assume  $M = 13^\circ 35$  we have  $a = 3\cdot37$ ,  $A = 239^\circ 30'$ ,  $b = 0\cdot59$ ,  $B = 39^\circ 12'$  these representing a curve which manifests the same peculiarity of a too early minimum. But by vary-

ing the epoch of the maximum, it becomes evident that the assumption of an earlier maximum will remedy this defect,—and chiefly by increasing the importance and modifying the epoch of the second variable term. Thus putting  $H_1=13^h 44^m$  we find

$a=3.369$ ,  $A=240^\circ 42'$ ,  $b=0.61$ ,  $B=53^\circ 57'$  and the minimum at  $3^h 36^m$ ;

while for  $H_1=13^h 36^m$  we find

$a=3.368$ ,  $A=240^\circ 56'$ ,  $b=0.63$ ,  $B=54^\circ 46'$  and the minimum at  $3^h 56^m$ .

Plotting these curves we select the latter as possessing the more probable form of the two; thus giving, together with an absolute representation of the three fundamental observations, a curve of which the general form is normal and the epoch of minimum not improbable. These two conditions cease to exist when the assumed values of  $M$  and  $H_1$  are much changed, and they fulfill approximately the function of that additional observation which would permit the constants to be rigorously determined.

Referring now to the series of hourly observations, we find these to be as given below. At their side are the temperatures which correspond to our approximate formula; and a glance at the column 0—C suffices to show how largely the residuals are due to the neglected term depending upon  $3h$ .

Hour.	Temperature.		0—C.	Hour.	Temperature.		0—C.
	Observed.	Computed.			Observed.	Computed.	
1 <sup>h</sup>	10°·79	10°·41	+ 0°·38	13 <sup>h</sup>	17°·20	17°·25	—0°·05
2	10·49	10·23	+ 0·26	14	17·29	17·29	0·00
3	10·29	10·18	+ 0·11	15	16·98	16·96	+ 0·02
4	10·18	10·27	—0·09	16	16·58	16·33	+ 0·25
5	10·05	10·68	—0·63	17	15·95	15·50	+ 0·45
6	10·76	11·12	—0·36	18	14·76	14·56	+ 0·20
7	11·90	11·90	0·00	19	13·51	13·64	—0·13
8	13·11	12·89	+ 0·22	20	12·56	12·79	—0·23
9	14·26	13·99	+ 0·27	21	12·07	12·07	0·00
10	15·30	15·10	+ 0·20	22	11·67	11·50	+ 0·17
11	16·13	16·09	+ 0·04	23	11·33	11·03	+ 0·30
11	16·89	16·84	+ 0·05	24	11·04	10·68	+ 0·36

The true formula as deduced from the hourly observations is

$$T=13.38+3.536 \sin (h+239^\circ 33') + 0.645 \sin (2h+66^\circ 1') + 0.365 \sin (3h+34^\circ 31')$$

the corresponding maximum occurring at  $13^h 31^m$ , and the minimum at  $3^h 26^m$ . The mean error of the approximate formula is  $+0^\circ.25$ .

3. As a third and last example we will take the important series of observations made at Upsala in Sweden during the ten years 1869–1879, and consider the mean diurnal variation during the entire year.

Here the three observations are  $T_7=3.25$ ,  $T_{14}=7.48$ ,  $T_{21}=4.07$ , in degrees of the centigrade scale. Using for  $M$  the mean of these three temperatures we have a secondary maximum in the curve; and decreasing the assumed daily mean until the tendency to abnormal flexure disappears, we furthermore perceive that the epoch of maximum is probably later than  $14^h$ .

Supposing then this epoch to be  $14^h 36^m$ , we find for  $M=4.70$ ,  $\alpha=2.29$ ,  $A=228^\circ$ ,  $b=0.52$ ,  $B=20^\circ$ , with an evident tendency to a secondary maximum and a very flattened curve for the night-hours. Even for  $M=4.65$  the same characteristics are recognizable. But for  $M=4.60$ , for which the curve appears satisfactory in all other respects, we find the improbable epoch  $2^h 42^m$  for the minimum.

It is furthermore readily seen that either an increased value of  $M$ , or an earlier value of  $H$ , will give a later epoch of minimum. Thus we have for

$M=4.65$	$H_1=14^h 36^m$	$\alpha=2.45$	$A=228^\circ 55'$	$b=0.43$	$B=18^\circ 17'$	$H_2=3^h 24^m$
4.60	14 36	2.60	229 43	0.33	17 1	2 42
4.60	14 24	2.60	230 41	0.32	31 28	3 15
4.60	14 12	2.58	231 34	0.33	45 19	3 24
4.58	14 24	2.66	230 56	0.28	32 40	3 20

Plotting these curves, it is clear that either of the last three will give satisfactory results, although the general form of the last but one seems the more probable.

Comparing the observed values with those deduced from this system of constants, we find

Hour.	Temperature.		0—C.	Hour.	Temperature.		0—C.
	Observed.	Computed.			Observed.	Computed.	
1 <sup>h</sup>	2°·55	2°·55	0°·00	13 <sup>h</sup>	7°·28	7°·29	—0°·01
2	2·30	2·36	—0°·06	14	7·48	7·43	0·00
3	2·10	2·27	—0°·17	15	7·45	7·41	+0°·04
4	2·02	2·28	—0°·26	16	7·20	7·08	+0°·12
5	2·19	2·43	—0°·24	17	6·78	6·59	+0°·19
6	2·60	2·75	—0°·15	18	6·15	5·97	+0°·18
7	3·25	3·25	0°·00	19	5·46	5·31	+0°·15
8	4·01	3·90	+0°·11	20	4·75	4·66	+0°·09
9	4·84	4·65	+0°·19	21	4·07	4·07	0·00
10	5·64	5·47	+0°·17	22	3·54	3·57	—0°·03
11	6·36	6·23	+0°·13	23	3·13	3·15	—0°·02
12	6·91	6·86	+0°·05	24	2·82	2·82	0·00

The true value of  $M$ , as determined by the hourly observations, is 4.62, the maximum being at  $14^h 10^m$  and the minimum



at  $3^h 22^m$ . The mean discordance  $0^{\circ}-C$ . deduced from the table of comparison is  $\pm 0^{\circ} \cdot 13$  which would be reduced to  $\pm 0^{\circ} \cdot 10$  by adding a third term,  $0^{\circ} \cdot 10 \sin (3^h + 45^{\circ})$ .

These examples will show how by a rough use of tentative processes a tolerable approximation to the true form of the diurnal curve may be obtained from observations made three times daily. The labor, though often considerable, is by no means so great in actual practice as might be inferred from the detailed description. A very little experience and tact are sufficient to indicate to the computer those hypothetical values which will guide him most readily to the desired end, this being the recognition of those limits for the unknown quantities within which they are not contraindicated and the selection of a probable curve within these, generally narrow, limits.

I do not fear that any unbiased person will misapprehend the claims here made in behalf of this mode of inquiry. It is of course only approximate; but in regions where the climatic relations have previously been unknown, it is certainly of high importance to obtain, from such data as may be accessible, determinations which are demonstrably not far from the truth; even though they may be destitute of that exactness which can only be obtained from the mean of many daily observations through an extended series of years.

But so soon as we are in possession of knowledge of other fundamental data to be combined with the three daily observations,—when for example we may infer from analogy the probable form of the true daily curve, or know the epoch or the temperature for the maximum or minimum, within moderate limits,—we may not only infer with confidence the constants of the second term, but may often deduce very fair values for the third.

Under ordinary circumstances this is not difficult, and it may therefore be well to consider the present state of our knowledge regarding the epochs of daily maximum and minimum. We will endeavor to avoid misplaced refinements of calculation, since attempts at minuteness in determining these epochs are for the most part justified neither by the nature of the phenomenon nor by the character of the observations; but will take into consideration only the unmistakable fact of the case.

The fact that the epochs of the daily extremes are to a high degree vague and uncertain cannot escape the most inattentive observer. It is of course the duty of the meteorologist to fix their typical values and the limits of their normal variation with such degree of precision as the nature of the case permits; still all endeavors must prove vain which aim at determining them with the minuteness which belongs to the epoch either of a sharply defined or of a tolerably regular phenome-

non. Cases are by no means infrequent in which the highest temperature of the day occurs as early as 8 A. M., or as late as 8 P. M. Sudden changes of wind, or in the amount of cloudiness exert so great an influence that great care is often needful for obtaining values for the epochs which shall be even approximately near to the normal ones. And allusion has already been made to the very great length of the period which would be requisite for determining these values with any precision. Two successive periods of ten years each might easily give average times for the daily maximum, differing by many minutes, whether the interval considered be one of 5, 7, 10, or 30 days, or the entire year. The next following would, in its turn, generally differ from both the previous ones; indeed the duration of the observations needed for obtaining a value, which should be essentially unchangeable by their longer continuance, is such that it has certainly been obtained for few, if indeed for any, places on the earth's surface. This certainly does not absolve us from the duty of investigating and remedying the sources of constant errors which would not probably disappear from the mean of a larger number of observations; but none the less is it futile to attempt to fix the epochs within limits which nature has not prescribed. To these considerations is to be added the independent one that, in many places, the diurnal curve of temperature varies so slightly at certain seasons in the vicinity of its extremes that a point of maximum or minimum value has scarcely more than a theoretical existence; so that small, and often inevitable, errors of observation would suffice to change the epochs by an amount relatively very large.

If determinations of the average times of the epochs are to be made within narrow limits of error and in such a way as to be serviceable for scientific ends, they must either be separately investigated for clear weather and cloudy, high barometer and low, north wind and south, dry air and moist, hot days and cool ones,—or else, what is far less practicable, observations must be employed which extend over a period sufficiently long for the effects of all such varying conditions to be eliminated from their mean results. Moreover, exact values, even though attainable, would contribute comparatively little to the progress of discovery, unless accompanied by determinations of the extent of their normal variation, and of the amount to which they are modified by influences analogous to those just mentioned.

The distinguished physicist, to whose criticisms and strong denunciations of the employment of the general mathematical formula we have alluded, has repeatedly asserted that errors in the times of daily maximum and minimum are produced by the employment of Bessel's formula; and that this places the

minima too early. Indeed he attempts to demonstrate these supposed facts by numerical illustrations derived from observations made at Katharinenburg in April for 18 years, and at Tiflis in May for 10 years. The deserved influence of Dr. Wild in all that relates to meteorological investigation gives to his opinion and counsel an importance which calls for a careful disproof of his mistakes in this respect, lest the progress of research be seriously impeded by the proposed disregard of algebraic generalization.

That a general mathematical formula, which absolutely represents all the fundamental observations, must afford more correct results than can be obtained from the same observations in any other way, would seem too nearly a truism to require mention. If graphical determinations, based on the same data as the numerical ones, differ from these, they must be erroneous; yet it is asserted (*Temp. Verhältn.*, p. 61) that even with 12 terms of the formula (24 constants) the epoch of minimum for Tiflis as deduced from hourly mean temperatures, is not correctly given. It may be admitted that Dr. Wild's tabular view might at the first glance suggest such interpretation; and, furthermore, conceded that the small numerical errors,\* which have apparently given him the impression that the formula failed to represent the observations, do not essentially affect the gradual advance of the epoch of minimum, as the several terms of the formula are successively incorporated in the computation. In this particular case it would appear that nearly the whole twelve terms are requisite for giving the correct epoch of mean minimum; if so, we must of course accept the fact, and investigate the influences to which it is due. These we cannot but believe to be exceptional, and to arise either from local peculiarities or errors of observation, — aggravated perhaps by the employment of an inadequate number of years.

Well founded exceptions might be taken to the way in which these results have been obtained, since they are not deduced from the pure observations, but from data which have been modified by an empirical correction, for the purpose of eliminating the so-called influence of annual variation. The course pursued has been practically equivalent to dividing the difference of the normal mean temperature for the first and last days of the month by the number of intermediate hours; and then by means of the hourly difference, thus obtained, modifying twenty-three out of the twenty-four daily observations in order to refer them to the epoch of the remaining one by applying the supposed amount of annual variation during the interval. The various statistical tabulations of the daily, monthly,

\* These, although insignificant in themselves, are in fact sufficient to change the corresponding epoch of maximum by thirteen minutes, or from 14<sup>h</sup> 15<sup>m</sup> to 14<sup>h</sup> 28<sup>m</sup>.

and yearly results are thus made numerically accordant; but at a sacrifice of real accuracy. The corrections thus applied have the effect of belating the minimum in the spring half of the year and anticipating it in autumn. For it cannot seriously be maintained that the influence of an increase or decrease of the sun's declination continues through the night-hours as through the day. This influence is necessarily intermittent; in dealing with annual variations, we cannot use hours as the independent variables; although here also the numerical changes are small, and although their influence disappears from the daily means and from determinations based upon them, yet whatever effect they may have goes to produce distortion and error in the resultant curve of diurnal variation. In this very case of May at Tiflis, the observations themselves give  $4^h 47^m \cdot 3$  for the mean epoch of minimum, instead of  $4^h 50^m \cdot 5$ , which results after they have been modified by this reduction for annual variation, or of  $5^h 0^m$  as deduced graphically by Dr. Wild. This modification has a still greater effect upon the time of mean maximum, for it changes this epoch from its true value  $14^h 18^m \cdot 8$  to  $14^h 28^m$  or to  $14^h 32^m$  if we accept the graphical result.

But, leaving this relatively unimportant question, the most striking fact to be noted in the example now considered is that the form of the diurnal curve, in the vicinity of its maximum, is such that, while the minimum temperature— $4^\circ 60$ , as given by only three variable terms, corresponds to an epoch  $4^h 17^m \cdot 6$ , yet the analogous value— $4^\circ 76$ , deduced from the full series of twelve terms, and only  $0^\circ 16$  lower, corresponds to the epoch  $4^h 50^m \cdot 5$ , more than half an hour later. That obtained graphically and given as the correct one by Dr. Wild, is, as above mentioned,  $5^h 0^m$ .

The accomplished physicist, whose views we reluctantly oppose, maintains that his method gives the epoch with accuracy to within  $2^m$ . With reference to this we will only remark that careful trials, plotting the observations with the utmost care upon the scale employed by Dr. Wild, show that the graphical method could give any value from  $4^h 48^m$  to  $5^h 12^m$  according to the taste or fancy of the draughtsman. Tracing the curve as it results from the formula, its true form may be recognized. If graphical processes are to be trusted in such cases, they must be based upon more frequent observations:—half-hourly, at least, in case the result is to represent the actual mean epoch within eight or ten minutes. How inordinately the resultant epoch would be affected by the, certainly not improbable, error of a few hundredths of a degree in the mean observed temperature, either for  $4^h$  or for  $5^h$ , is manifest. It may not be amiss to add that the mean discordance between the observed and

calculated temperatures, using only three variable terms of the formula, is less than the tenth of a degree. Using five variable terms, it is less than four hundredths of a degree.

In case the preceding month, April, had been selected for the illustration instead of this exceptional one of May, we should have found the time of minimum when obtained by the use of only six variable terms to be one minute *later* than if deduced from the complete series.

Or if the next following month, June, had been taken, the formula with only five variable terms would have given the epoch of minimum not only within a single minute of that resulting from the full series, but also *later* than any of those resulting from the successive incorporation of the four following terms.

When we examine the results obtained for Katharinenburg, the case is analogous, but even more favorable for illustrating the facts to which we call attention. Here the formula with but five variable terms represents the observations with such completeness that the mean discordance amounts to only  $0^{\circ}02$ . With eight variable terms the discordance does not exceed one one-hundredth of a degree at any hour from 2 to 7 A. M.; and the same minute results for the epoch of minimum as when the complete formula is used. Indeed, with only six variable terms, the difference does not exceed five minutes. The value obtained graphically by Dr. Wild is five minutes later than that which results from the formula using the same data; and careful trials, employing the scale used by him, have convinced me that curves may be so drawn as to seem equally plausible, while their times of minima vary by some fifteen minutes. The experiment can easily be tried by any one for himself; and it will be found that the errors, inevitably committed in the plotting and reading off, are far less than those resulting from the sketching of the curve.

The extreme variability and uncertainty of these mean epochs of diurnal maxima and minima find an excellent illustration in the values corresponding to successive years in the admirable series of hourly observations made at Upsala under the direction of Dr. Hildebrandson. During the eight years 1869–1876, the mean epoch of daily minimum, as deduced from all the observations of the year, fluctuated between the limits  $3^h 32^m$ , and  $4^h 12^m$ , and that of maximum from  $13^h 51^m$ , to  $14^h 42^m$ . Dr. Wild himself acknowledges that the mean epoch of daily minimum as deduced from a whole year's observations was "tolerably constant," when it varied by  $50^m$  during the years from 1857 to 1862. [Temp. Verh., p. 54].

It has already been stated that in the trials for obtaining the daily mean, the time of maximum may, in the absence of any

knowledge, be at first advantageously assumed as 2 P. M.; and also show how a rough approximate knowledge of the epochs may be made to contribute essentially to that of the formula.

To aid the attainment of this let us consider the facts at our disposal regarding the values of these epochs in general.

Very extensive materials are available for this purpose, but it is safer to restrict ourselves to a limited number of points for which the observations have been scrutinized, and the inferences deduced, with special care, than to base our researches upon a larger number of less trustworthy results. For these reasons the values adopted by Dr. Wild in the work already cited, have generally been preferred to others; and those for a number of places, such as Upsala, Berne, Leipzig, Greenwich, St. Helena, Lisbon, have been specially determined here. All depend upon hourly observations, with the single exception of Schwerin; but at Santiago de Chile, and Valparaiso, the observations were made during only a few days in each month, in different years. At Santiago the total number of days' of observation was 130, at Valparaiso, 60. To facilitate the examination of the results here given, and the deduction of analogous ones for other places, it may be convenient for those who are not astronomers, to have ready access to general tables of the equation of time, the sun's declination, and the times of sunrise and sunset in different latitudes. Those which follow\* represent the mean values; and although not strictly accurate in every year, they may be used without hesitation for meteorological purposes. The equation of time as here given is to be applied to the mean time, to obtain the apparent; or with reversed sign to the apparent if the mean is desired.

The first of these tables gives the mean values for each decade during the year, corresponding to our arrangement, according to which all the days subsequent to the 20th of each month are made to form the third decade. The second table gives the apparent time of sunrise if the latitude be North, and of sunset if it be South, for the middle of each month.

Our data relative to the mean epochs of the diurnal extremes are deduced from the observations in each month separately during the period considered, and also from the annual means; the number of years of observation employed being mentioned in each case. The values are given in apparent time; those of the observed maxima being counted from apparent noon, and those of the minima being given by the number of minutes before sunrise, for which moment the arrival of the sun's center at the horizon is used, without any correction for refraction.

\* The tables mentioned are given in the original article, but are necessarily omitted here.—EDS.

The data thus collected afford some opportunity for generalization; but this we reserve for some future occasion; simply calling attention to the relatively small extent of deviation of the average epoch of maximum from 2 P. M., and to the large amount by which the average times of minimum, as deduced from the whole number of observations, differ from the mean of those deduced for the several months. The general tendency of the interval, between the minima and true sunrise, to increase with the latitude is also readily perceptible. This is of course due to the sun's near approach to the horizon long before its actual arrival there.

Desiring a rough test both of the average epoch of maximum, as deduced from a short period of observation, and of its ordinary extent of variation, I wrote in April, 1880, to several of the gentlemen, at different places, to whom we are indebted for our meteorological observations, asking them to note the temperature half-hourly for a period of some hours which should include the epochs of maximum during ten or twelve successive days.

These observations referred to apparent time give the following results:

Place.	Approx. Lat.	Period.	Mean Epoch of Maximum.	Uncertainty of Determinat'n.	Variation.
Bahia Blanca -----	38° 45'	May 12-23	2 <sup>h</sup> 13 <sup>m</sup> P. M.	5 $\frac{1}{2}$ <sup>m</sup>	1 <sup>h</sup> 27 <sup>m</sup> - 2 <sup>h</sup> 48 <sup>m</sup>
Buenos Aires -----	34 16	Apr. 21-30	2 22	4 $\frac{1}{2}$	1 2 -2 56
S. Antonio de Areco	34 13	Apr. 21-30	2 19 $\frac{1}{4}$	3	1 38 -2 44
Cordoba -----	31 25	May 1-10	2 14	3 $\frac{1}{2}$	1 27 -3 51
		May 11-20	2 11	6	11 28 -3 40
Villa Hernandarias	31 15	May 1-14	2 45 $\frac{1}{2}$	3	1 40 -3 28
Goya -----	29 9	May 1-10	1 55 $\frac{1}{2}$	5	12 48 -2 48
Salta -----	24 50	Apr. 23-30	2 38	4	1 14 -2 57

The column entitled "Uncertainty of Determination," contains the sum of the various estimated possible errors in plotting the observations, drawing the curves, and reading of the results. The limits of probable, or possible, errors of the last two classes depend of course, principally, upon the form of the curve itself. The column entitled "Variation," contains the difference between the extreme values on different days during the continuance of the series.

It will be seen that these results confirm our previous inferences.

That the later terms of the formula should become more important in high latitudes during the winter season is a manifest consequence of the inequality of the two portions into which the minimum divides the interval between the normal epochs of maximum. When sunrise occurs so early as to render this

inequality comparatively slight, the first variable term, the period of which is twelve hours, acquires an overwhelming significance; and the residuals which it leaves are disposed of with comparative ease by the other terms of long period. But when the sun rises later the reverse must be the case. By means of the data above collected, provisional assumptions of the form of the diurnal curve may often be so made as to afford important assistance in deducing its true form, with great approximation, from a small number of observations.

On the other hand, no argument is needed to show how unjustifiable is that frequent misuse of the general formula by which the values deduced for certain terms from observations made during the day time or at inadequate intervals, are employed to represent the nocturnal portion of the curve. Indeed it is usually the most important terms which are thus least accurately determined. Bi-hourly observations will give the constants as far as the 6th term; hourly ones, will give them as far as the 12th; but these same arguments, which call for hourly observations for accurate determination of the epochs, apply with equal force to semi-hourly ones, for a more exact determination still. How far the game is worth the candle, is an entirely distinct question.

Summing up the several conclusions at which we have arrived we find as follows:

1. The formula, depending upon the sines and cosines of multiples of the time since an arbitrary epoch, is a very simple, —probably the simplest,—form of expressing the law, of a cyclical variation of this class. It is not a mere formula of interpolation; nor can it, when legitimately used, give rise to any systematic error whatsoever.

2. If in such variation there be a general law, this will be made manifest by the relative inappreciableness of the higher terms; and the chief function of these will be to equate out the errors of observation.

3. Although the results of hourly observations may be absolutely expressed by the formula with twenty-four constants, yet the values given by such a formula would probably be in general less correct than those resulting from the use of fewer terms, first, because it would imply an absence of law in the variation; secondly, because it would necessarily include all errors of observation to their full extent, instead of equating them out.

4. Three daily observations, tolerably well distributed, together with an approximate value of the epoch of maximum and the condition of only one point of contrary flexure in the curve, give a close approximation to the true formula for the daily variation, so far as this can be expressed without includ-



ing those terms which complete their period in less than eight hours.

5. An approximate knowledge of the epoch of the maximum or minimum temperature, will, for each of them, enable us to add an additional constant to the formula with a good degree of approach to the truth.

6. The mean epochs of maximum or minimum can seldom be determined with precision even by employing the highest refinements of observation and calculation known to science. Indeed it appears unlikely that they ever have been ascertained for any points of the earth's surface with less uncertainty than several minutes. It is moreover questionable whether there are any such epochs sufficiently marked to permit determinations without the introduction of various conditions; unless by the employment of observations extending over a series of years sufficiently long to eliminate all the variety of conditions. Even if this be possible it will not be within the attainment of the present generation or their near posterity.