

A FORMULA OF TRIGONOMETRIC INTERPOLATION.

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Adunanza del 24 agosto 1913.

In a recent paper ¹⁾ the author has discussed the degree of convergence of two formulas of trigonometric interpolation. One is the ordinary formula, giving a trigonometric sum of the n^{th} order which coincides in value with the function to be represented at $2n + 1$ points evenly spaced over an interval of length 2π . This does not always converge, when n increases indefinitely, even if the function is continuous. The second expression does converge in the case of every continuous function, but does not, like the first, take on the same values as the given function at the points used in interpolating ²⁾. It is the purpose of the present note to suggest a formula which converges for every continuous function, and does take on the given values at the assigned points. In combining these properties, to be sure, it sacrifices certain others. As compared with the ordinary interpolating function, it gives a trigonometric sum the order of which is practically the same as the number of points used, instead of being about half as large; and it does not share the advantage in the matter of degree of convergence which was the reason for the introduction of the second formula. That sort of question of degree of convergence will not concern us here at all.

Let $f(x)$ be a function of period 2π , defined for all real values of x . Let t_1, t_2, \dots, t_n be n points ³⁾ situated at successive intervals of $2\pi/n$. The expression to be consi-

¹⁾ D. JACKSON, *On the accuracy of trigonometric interpolation* [Transactions of the American Mathematical Society, Vol. XIV (1913), pp. 453-461]. Cf. also the references in that paper to the earlier work of DE LA VALLÉE POUSSIN and FABER.

²⁾ The justification of the use of the word « interpolation » in this case is that a finite number of given values suffice to determine the approximating function.

³⁾ A more complete notation would attach two subscripts to the t 's, for none of them need remain fixed as n varies, and not more than one can do so. When one of them, as t_1 , is chosen the others are determined. The alteration of t_1 by an integral multiple of $2\pi/n$ has no effect on the approximating function which is to be written down, amounting at most to a rearrangement of the terms under the sign of summation; but there is real arbitrariness in the choice of t_1 in an interval of length $2\pi/n$.

dered is the following:

$$(1) \quad \tau_n(x) = \sum_{i=1}^n \left\{ f(t_i) \frac{\sin^2 \left[\frac{1}{2} n(t_i - x) \right]}{n^2 \sin^2 \left[\frac{1}{2} (t_i - x) \right]} \right\}.$$

It is suggested by FEJÉR's formula for the arithmetic mean of the first partial sums of a FOURIER's series, as the ordinary interpolating expression is suggested by that for the partial sum of the FOURIER's series itself ⁴), and as the other one referred to in the introduction is suggested by the integral with a fourth power which has been studied by the author ⁵) elsewhere.

By substituting $t_i - x$ for u in the identity

$$\frac{\sin^2 \left(\frac{1}{2} n u \right)}{\sin^2 \left(\frac{1}{2} u \right)} = 2 \left[\frac{1}{2} n + (n-1) \cos u + (n-2) \cos 2u + \cdots + \cos (n-1)u \right],$$

expanding the sines and cosines of multiples of $t_i - x$, and performing the summation, we see that $\tau_n(x)$ has the form

$$\tau_n(x) = \frac{1}{2} a_{n0} + a_{n1} \cos x + \cdots + a_{n,n-1} \cos (n-1)x \\ + b_{n1} \sin x + \cdots + b_{n,n-1} \sin (n-1)x,$$

where, for $k = 0, 1, \dots, n-1$,

$$a_{nk} = 2 \frac{n-k}{n^2} \sum_{i=1}^n f(t_i) \cos k t_i, \quad b_{nk} = 2 \frac{n-k}{n^2} \sum_{i=1}^n f(t_i) \sin k t_i.$$

It appears from the expression (1) that when x becomes equal to one of the t 's, say $x = t_j$, the corresponding term in the sum reduces to $f(t_j)$, while all the others vanish, so that

$$\tau_n(t_j) = f(t_j), \quad (j = 1, 2, \dots, n).$$

It remains only to discuss the convergence of $\tau_n(x)$ at intermediate points.

If $f(x) \equiv 1$, it results from a well-known property of the trigonometric functions that all the coefficients a_{nk} and b_{nk} , $k = 1, 2, \dots, n-1$, reduce to zero, and only a_{n0} remains. That is, we have ⁶)

$$(2) \quad 1 = \sum_{i=1}^n \frac{\sin^2 \left[\frac{1}{2} n(t_i - x) \right]}{n^2 \sin^2 \left[\frac{1}{2} (t_i - x) \right]},$$

for all values of n , t_i , and x . Multiplying this identity now by an arbitrary function

⁴) It is to be observed that $\tau_n(x)$ is *not* obtained by taking the arithmetic mean of the first n of the interpolating functions given by the ordinary formula.

⁵) See, for example: D. JACKSON, *On approximation by trigonometric sums and polynomials* [Transactions of the American Mathematical Society, Vol. XIII (1912), pp. 491-515].

⁶) It follows at once from this relation that if the absolute value of a function remains always less than or equal to ε , the absolute value of the corresponding $\tau_n(x)$ remains always less than or equal to the same quantity. Consequently, if the quantities $f(t)$ are subject to errors, no one of which exceeds ε in absolute value, these errors can not affect any value of $\tau_n(x)$ by more than ε .

$f(x)$, we may write

$$f(x) = \sum_{i=1}^n \left\{ f(x) \frac{\sin^2 \left[\frac{1}{2} n(t_i - x) \right]}{n^2 \sin^2 \left[\frac{1}{2} (t_i - x) \right]} \right\},$$

and if we form the corresponding $\tau_n(x)$, we obtain the following expression for the error of the approximation:

$$(3) \quad \tau_n(x) - f(x) = \sum_{i=1}^n \left\{ [f(t_i) - f(x)] \frac{\sin^2 \left[\frac{1}{2} n(t_i - x) \right]}{n^2 \sin^2 \left[\frac{1}{2} (t_i - x) \right]} \right\}.$$

Up to this point we have placed no restriction on $f(x)$ except that of periodicity. We assume now that $f(x)$ is everywhere continuous, and proceed to show that in this case $\tau_n(x)$ converges uniformly to the value $f(x)$ as n is indefinitely increased. We denote by M the maximum of the absolute value of $f(x)$. Let ε be a positive quantity assigned at pleasure. Let δ be a positive quantity such that the inequality

$$(4) \quad |f(x'') - f(x')| < \frac{1}{2} \varepsilon$$

is satisfied whenever $|x'' - x'| \leq \delta$. Such a quantity will exist, since $f(x)$ is continuous everywhere.

Inasmuch as each of the numbers t_i may be increased or diminished by any integral multiple of 2π without changing the value of the term in which it occurs, there is no loss of generality in supposing that these numbers all lie in the interval from $x - \pi$ to $x + \pi$. Let the terms of (3) in which $x - \delta \leq t_i \leq x + \delta$ be added to form a partial sum Σ'_n , the sum of the remaining terms being denoted by Σ''_n . The absolute value of Σ'_n will be increased if $\frac{1}{2} \varepsilon$ is written in place of each difference $[f(t_i) - f(x)]$ occurring there. The resulting expression, with the factor $\frac{1}{2} \varepsilon$ removed, contains only a part of the positive terms which make up the right-hand side of (2). Hence

$$|\Sigma'_n| < \frac{1}{2} \varepsilon$$

for all values of n . In Σ''_n we have

$$|\sin \frac{1}{2} (t_i - x)| > \sin \frac{1}{2} \delta;$$

each difference $[f(t_i) - f(x)]$ may be replaced by $2M$, and the number of terms is not greater than n . Consequently

$$|\Sigma''_n| < \frac{2M}{n \sin^2 \frac{1}{2} \delta}.$$

As soon as n is greater than

$$\frac{4M}{\varepsilon \sin^2 \frac{1}{2} \delta},$$

which is independent of x , we shall have $|\Sigma''_n| < \frac{1}{2} \varepsilon$, and

$$|\tau_n(x) - f(x)| < \varepsilon.$$

Thus the uniform convergence is proved.

Suppose that $f(x)$, without being continuous everywhere, is continuous throughout some interval, and that (a, b) is an interval which, with its end points, is interior to an interval of continuity. Suppose further that $f(x)$ remains finite everywhere, and that M is the upper limit of its absolute value. After the choice of any positive ε it will be possible to determine a positive δ so that (4) holds whenever $|x'' - x'| \leq \delta$ and one of the quantities x', x'' is in (a, b) . For if δ is sufficiently small both x' and x'' will necessarily lie in a closed interval where $f(x)$ is continuous. A repetition of the preceding proof now shows that $\tau_n(x)$ converges uniformly throughout (a, b) to the value $f(x)$.

For convergence at a single point, it is sufficient that $f(x)$ be continuous at that point and everywhere finite. The choice of δ will be made now so that (4) is true when $|x'' - x'| \leq \delta$ and one of the points x', x'' coincides with the point in question. With this change and the omission of all reference to uniform convergence the proof remains as before.

Let us suppose now that one of the quantities t_i , for instance t_1 , is held fast as n varies. Then the only values of $f(x)$ which enter into the definition of $\tau_n(x)$ are those formed for values of the argument which differ from t_1 by a rational multiple of 2π . If we replace the requirement of continuity at a point $x = a$ by the requirement that

$$\lim_{x=a} f(x) = f(a)$$

when x approaches a *passing only through values belonging to the set just specified*, and the requirement of finiteness everywhere by that of finiteness for all points belonging to the set, the preceding theorems of convergence remain true, and their proofs remain unchanged. It is possible for a function $f(x)$ which is everywhere discontinuous and which does not remain finite in any interval to give rise to an interpolating function $\tau_n(x)$ which converges uniformly everywhere, though of course not everywhere to the value $f(x)$; to this end it is necessary only that for values of x belonging to the selected set $f(x)$ take on the values of a function which is everywhere continuous. As might naturally be expected, nothing corresponding to a condition of integrability is needed here in the theory of interpolation. On the other hand, the requirement that $f(x)$ shall remain finite, for the values of x actually used, can not be replaced by any condition of the nature of mere absolute integrability. What can be said in this connection is, that if a good approximation is not demanded in the immediate neighborhood of points of infinite discontinuity, the function may be redefined in a number of intervals, as small as may be desired, including these points, and convergence to the value of the original function at points of continuity elsewhere will be assured.

The expression which we have used can be transformed so as to give a formula for polynomial interpolation at a set of points *unequally* spaced, in the manner pointed out at the close of the author's earlier paper on interpolation already referred to. The given function being $f(y)$, defined in the interval $-1 \leq y \leq 1$, we set $y = \cos x$, and approximate to $f(\cos x)$ by the corresponding $\tau_n(x)$. If the points t_i , in addition

to being evenly spaced, are symmetrically situated with respect to the origin, $\tau_n(x)$ will be an even function ⁷⁾ of x , and hence a polynomial in $\cos x$. In this way, a polynomial of degree $n - 1$ in y may be obtained which is equal to $f(y)$ at the $\frac{1}{2}n$ points

$$y = \cos \frac{(2i - 1)\pi}{n}, \quad i = 1, 2, \dots, \frac{1}{2}n, \quad n \text{ even,}$$

or at the $\frac{1}{2}(n + 1)$ points

$$y = \cos \frac{2i\pi}{n}, \quad i = 0, 1, \dots, \frac{1}{2}(n - 1), \quad n \text{ odd.}$$

If $f(y)$ is assumed continuous, this polynomial will converge uniformly to $f(y)$ as n becomes infinite, by our previous work.

The usual formula for polynomial interpolation would give a polynomial of the $(n - 1)^{\text{th}}$ degree coinciding with $f(y)$ in value at n points. It was pointed out by RUNGE ⁸⁾ that while this expression, formed for equidistant points, does not necessarily converge even for an *analytic* function, it is possible to secure convergence for every such function by a suitably modified arrangement of points. That this is the case, may be seen by transforming the ordinary trigonometric interpolating function into a polynomial by the device which we have just applied to $\tau_n(x)$. The distribution of points thus obtained is essentially that suggested by RUNGE, and the resulting expression converges, not only for all analytic functions, but for functions of a far more general class, by the convergence properties of the trigonometric interpolation ⁹⁾. But it remains apparently an open question, whether it is possible to assign to each positive integral value of n a set of n points in the interval from -1 to $+1$, such that the polynomial of the $(n - 1)^{\text{th}}$ degree which takes on the values of a given function $f(x)$ at these points shall converge to $f(x)$ throughout the interval as n becomes infinite, whenever $f(x)$ is continuous there. The search for such an arrangement of points is not made more hopeful by the facts in the trigonometric case, where there is no apparent reason why the uniform distribution should not be as good as any.

Harvard University, Cambridge, Mass., August 3, 1913.

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⁷⁾ This is seen by replacing x by $-x$ and at the same time t_i by $-t_i$.

⁸⁾ C. RUNGE, *Theorie und Praxis der Reihen* (Leipzig, G. J. Göschen, 1904), pp. 137-142.

⁹⁾ See the papers of DE LA VALLÉE POUSSIN, FABER, and the present writer, referred to at the beginning of this note. The first of these authors also deals with polynomial interpolation.