

V.—FUNDAMENTAL LOGIC.

AT least three distinct views are possible of the relation between logic and mathematics. Mathematics may be regarded as a special application of logic; or logic may be regarded as a branch of mathematics*; or the two may be regarded as co-ordinate sciences.

I regard the ordinary logic as a co-ordinate science with mathematics: but I further maintain that the ordinary logic on the one hand, and mathematics on the other, are two separate developments of a simpler logic than any which has been usually recognised.

It appears to be admitted by all, that the fundamental relation in mathematics is equality; and it appears to be generally thought that the fundamental relation corresponding to this in the ordinary logic is identity. I dispute this latter position. I maintain that the fundamental relation of the ordinary logic is not identity, but co-existence. But mathematics, or the logic of equality, and the ordinary logic, or the logic of co-existence, both rest on the simplest and most elementary logic, which is that of identity.

John Stuart Mill is the only writer known to me who has clearly seen that the ordinary logic rests not on identity but on co-existence. His system is, in substance, an application of the principles of the ordinary logic to the actual work of discovery and proof; and, seeing that the axioms of identity and contradiction are by themselves able to carry the reasoner but a little way, he proposes as the canon of his logic the axiom that "things which co-exist with the same thing co-exist with each other". His treatment of formal logic is, however, unsatisfactory, or at least incomplete, and I must say a few words in defence of the position that the syllogistic reasoning of the ordinary logic really depends on this axiom.

The relations with which the ordinary logic deals are those of the inclusion of one class in another, and of individuals in classes; and when it is reconstructed by treating propositions as equations, the relations with which it deals are those of the total or partial identity of classes.

For my present purpose it will be best to instance a case of

* Mr. Venn, in his very lucid exposition of Boole's Logical System in MIND No. IV., says (p. 480):—"The prevalent notion about Boole probably is, that he regarded Logic as a branch of Mathematics; that, in fact, he simply applied mathematical rules to logical problems. This is a very natural mistake." If it is a mistake, Boole is himself answerable for it. The full-length title of his great work is *An Investigation of the Laws of Thought, on which are founded the mathematical theories of Logic and Probabilities.*

total identity. In the ordinary logic, as modified by 'quantifying the predicate,' the following would be regarded as a proposition of total identity:—"The things having inertia are the same as the things having gravity." But it may be much better stated as a proposition of co-existence, thus:—"Inertia and gravity always co-exist." I do not lay any stress on the evident truth that the latter mode of expression appears much more natural; but I say that the proposition, though it may with perfect accuracy be stated as one of identity, is essentially and primarily one of co-existence. Inertia is in no sense identical with gravity.

All propositions asserting the inclusion of one class within another, may in like manner be shown to be really propositions asserting co-existence. Thus the proposition, "Chlorine is an imperfect gas," according to the view of the ordinary logic, asserts that "The species chlorine is included in the class of imperfect gases". But if we make no postulate as to the existence of such a class, and state the proposition in its utmost possible simplicity, it becomes the following:—"With the differentia of chlorine (consisting in its colour and its chemical reactions) the (physical) properties of an imperfect gas co-exist."

In Boole's and Jevons's logical systems, propositions are written as mathematical equations, and the co-existence of qualities is symbolised by the combination of terms. If we call inertia x and gravity y , the identity of the things having inertia and those having gravity is asserted by the equation, $x = y$: but if we interpret x and y to mean, not the things having the qualities, but the qualities themselves, then the copula = will mean not identity but co-existence, and the equation will assert the invariable co-existence of the qualities.

In Jevons's notation,* which for its purpose appears absolutely perfect, if x means chlorine and y an imperfect gas, then the equation $x = xy$ asserts that chlorine is an imperfect gas. If, further, z means freely soluble in water, the equation $y = yz$ asserts that imperfect gases are freely soluble in water; and the syllogism whereby, from these two premisses, we infer that chlorine is freely soluble in water, is expressed as follows:— $x = xy$; $y = yz$; therefore, $x = xyz = xz$.

Boole appears to recognise the existence of no simpler logic than that of co-existence, for he begins his system by stating the laws of the combination of terms. He uses 1 as the symbol for "all," and $1 - x$ is consequently his expression for whatever is

* See his *Principles of Science*. Jevons, however, uses the capitals A, B, and C, where I follow Boole in using the small italics x , y , and z . I prefer to make logical equations look as like mathematical ones as possible.

not- x . In logic, as in mathematics, the equation $1x = x$ is thus true of all values of x . He places at the commencement of his system the two following equations, which are his expressions of the laws of identity and contradiction: $x^2 = x$, and $x(1 - x) = 0$. The first of these asserts that, if a term be combined with itself, the result is the same as if it remained uncombined:—thus, “heavy, heavy things” are the same as “heavy things”. The second asserts that a term and its negative cannot be combined:—thus, things which are at once heavy and not heavy cannot exist. These two equations, which in logic are true of all terms whatever, are in mathematics true only of terms having the values of 1 and 0.

Boole (*Laws of Thought*, pp. 49, 50) calls attention to the fact, that these equations, expressing the fundamental laws of thought, are equations of the second degree. This is so surprising a result, that it ought to excite a suspicion, not indeed of the accuracy of Boole’s expression of these laws, but of the truth of the assumption that they are what is simplest and most elementary in logic. I maintain that there is a more elementary logic than Boole’s: a logic in which there are no combined terms, and consequently no equations except those of the first degree; no operations except addition and subtraction; no interpretation of the copula except simple identity; and of which the axioms are true not only in logic but in mathematics.

In what follows I must request the reader to bear in mind that the word identity is used in the sense not only of total but of partial identity, so as to include the relation of a part to the whole.

When expressed in language, the propositions and syllogisms of the logic of identity are similar in form to those of the old logic. The old logic deals chiefly with such cases as the inclusion of class within class; but the same or similar forms will express the inclusion of a part in the whole, or of a constituent in the compound. The following are examples:—“The anther is a part of the flower; the flower is a part of the tree; therefore, the anther is a part of the tree.” “Hydrogen is a constituent of water; water is a constituent of albumen; therefore, hydrogen is a constituent of albumen.” It may be thought that the distinction between propositions of co-existence and of identity is one of interpretation only, and does not belong to formal logic; and in fact this distinction, so far as I am aware, has not been seen till now;—the purpose of this paper is to insist on it. In proof of the really logical nature of the distinction, it is to be observed that, though propositions of co-existence may no doubt be stated as propositions of identity, the converse is not true—propositions of identity cannot be stated as propo-

sitions of co-existence. The two syllogisms last stated have propositions of partial identity for their premisses and their conclusions, and none of these can be stated as propositions of co-existence; and the forms of proposition and syllogism by which, as we have seen, Jevons so admirably expresses the logic of co-existence, cannot, without an unwarrantable strain on their meaning, be made to express the logic of simple identity.

There is another peculiarity of the logic of co-existence which confirms me in the belief that it is fundamentally distinct from that of mere identity. Sir William Hamilton has shown, though I believe he was not the first to discover, the double interpretation, in extension and in comprehension (or intension), which the terms of the ordinary logic admit of. The extension and the comprehension of the meaning of terms, or, in other words, the denotation and the connotation of class-names, vary inversely as each other—that is to say, the number of species included in a class is greater as the number of attributes connoted by the name of the class is less. Thus, if the syllogism above-stated respecting chlorine is interpreted in extension, its meaning will be:—"Chlorine is one of the class of imperfect gases; imperfect gases are part of the class of substances freely soluble in water; therefore, chlorine is one of the class of substances freely soluble in water." But if interpreted in comprehension, its meaning will be:—"The properties of chlorine include those of imperfect gases; the properties of imperfect gases include those of substances freely soluble in water; therefore, the properties of chlorine include those of substances freely soluble in water."

When we interpret terms and propositions in comprehension, we are really treating them as belonging to the logic of co-existence; when we interpret them in extension, we are treating them as belonging to the logic of identity.

Now, in the logic of identity, no interpretation in comprehension is possible; its terms and propositions are interpretable in extension only. This will be made evident by referring to either of the two syllogisms already given as examples of that logic.

Moreover, in propositions asserting the inclusion of class within class, which I regard as really propositions of co-existence, we have seen that the more species a name denotes, the fewer attributes it connotes. But this is reversed in propositions asserting the inclusion of a part in the whole, which I regard as really propositions of mere identity;—the name of the whole connotes more attributes than the name of the part. The tree has a greater variety of attributes than the anther, and the compound than the element.

The distinctness of the logic of co-existence from that of identity seems to be proved by these two closely-connected facts, that propositions of co-existence may be stated as propositions of identity, but not the converse; and that propositions of co-existence may be interpreted either in extension or in comprehension, but propositions of identity can be interpreted in extension only.

It has not, I think, been sufficiently noticed, that propositions are possible respecting a class which do not make any assertion respecting the members of the class. For instance:—Insects are the largest class of animals—Birds are the most sharply defined class of animals.

The laws of identity and contradiction are fundamental in logic, and, so far as they can be expressed without combined terms, they may be expressed by the equations $x = x$; and $x - x = 0$. To these it has been usual to add, as a third and co-ordinate law, that of excluded middle, or, to use Jevons's much better phrase, the law of duality. This law, as generally stated, is that every thing must either possess or not possess any given property; but this statement belongs to the logic of co-existence; in the logic of identity its statement is, that any total of which x is a part consists of the sum of x and not- x ; and, 1 being the symbol for "all," it may be expressed by the equation $1 = x + (1 - x)$. When thus stated, it is seen to be not a co-ordinate law with the two preceding, but a corollary from them. This, I think, agrees with Boole's view.

There are, however, two other laws which appear to be co-ordinate axioms with those of identity and contradiction. One is that two negatives form an affirmative or positive:—this law may be expressed by the equation $-(- x) = x$, or what is perhaps a better expression, as not suggesting that a negative term can have any independent meaning, $x - (y - z) = x - y + z$. The other is the law that the order in which addition takes place is indifferent:—it may be expressed by the equation $(x + y) + z = (y + z) + x$. This is the form of the equations of chemical transformation, as will be seen if y is taken to mean oxygen and x and z two oxidisable substances. Such equations really belong to the logic of identity, assuming, however, the physical truths that matter can neither be created nor destroyed, and that every compound may be resolved back into its elements.

Perhaps we ought to enumerate yet another law, to the effect that an equation may be read either way, so that, if $x = y$, it is equally true that $y = x$. It is not unlikely, however, that the statement here made of the laws of the logic of identity may be found to admit of improvement.

It will be observed that all these laws are true, not only in the logic of identity, but also in the logic of co-existence and of equality, that is to say in the ordinary logic and in mathematics.

It is worth while to show that a complete though very simple symbolic method is possible in the logic of identity, without any combination of terms, and with no operations except addition and subtraction.

I propose to express the proposition "all x is y ," or " x is a part of y ," by the equation $x = y - p$, p being so much of y as is not x :—and the parallel expression for "no x is y " is $x = (1 - y) - p = 1 - y - p$.

We will speak first of conversion. The problem of logical conversion may be thus stated in its utmost possible generality:—Having described x in terms of y , to describe y in terms of x . The affirmative proposition "all x is y ," or $x = y - p$, is converted by simply transposing p , when it becomes $x + p = y$. The negative proposition, "no x is y ," or $x = 1 - y - p$, is converted by subtracting both sides of the equation from unity and transposing p , when we get $1 - x - p = y$.

The forms of syllogism may be expressed with equal facility. An ordinary syllogism will read thus: $x = y - p$; $y = z - q$; therefore, $x = z - q - p$: or, by transposing p and q , $x + p = y$; $y + q = z$; therefore, $x + p + q = z$.

If we assign to these symbols the same meaning that we assigned when speaking of interpretation in comprehension, this syllogism will mean, "Chlorine is one of the class of imperfect gases; imperfect gases are part of the class of substances freely soluble in water; therefore, chlorine is one of the class of substances freely soluble in water":—

Chlorine $= x = z$

Imperfect gases $= y = x + p$.

Substances freely soluble in water $= z = x + p + q$.

But if we interpret the same syllogism in comprehension, and use Jevons's notation accordingly, as explained above, then

Chlorine $= x = xyz$

Imperfect gases $= y = yz$

Substances freely soluble in water $= z = z$

The increasing number of letters in the one notation shows the increased magnitude of the classes, while the decreasing number of letters in the other shows the decreased number of attributes in their description:—thus, we may almost say, showing to the eye how extension and comprehension vary inversely as each other.

We have now to see how the transition is made from the logic of identity to the ordinary logic and to mathematics.

A glance at the algebraic form of syllogism given above for the logic of identity, will show its canon to be that things identical with the same thing are identical with each other: or, in other words, that identical terms may be substituted for each other. This is not a distinct axiom, but an immediate corollary of the principle of identity. The axioms that things which are equal to the same thing are equal to each other, and that things which co-exist with the same thing co-exist with each other, are also corollaries from the same. In order to make this clear, we have to state the following definitions:—(1) *Similar*s are things concerning which the same predication can be made; in other words, *similar*s are things whereof the symbols may be substituted for each other.* (2) *Equality* is similarity of magnitude. (3) *Co-existence* is identity of position either in space or in time.

From these definitions, the truth of the reasoning $x = y$; $y = z$; therefore, $x = z$, follows without any other axiom being needed than that of identity; and this is equally true, whether the copula $=$ is taken to mean identity, co-existence, or equality. The only distinction between the subject-matter of logic and that of mathematics appears to be that the copula, which in mathematics means equality, in logic means either identity or co-existence.

In the notation which I have proposed for the logic of identity, we have seen that there are no operations except addition and subtraction, and these have exactly the same meaning as in mathematics. But in the logic of co-existence there is another operation on the symbols, namely combination, symbolising the co-existence of qualities, to which there is nothing in mathematics precisely analogous. This appears to support the view that the logic of identity is the fundamental logic.

The following are the principal points which I have endeavoured to bring out in this paper.

The ordinary logic is not primarily a logic of identity, but of co-existence; but the logic of co-existence and mathematics, which is the logic of equality, rest on a more elementary logic of identity.

In this logic there is no combination of terms, and no operation except addition and subtraction.

The axioms of this logic are true also in the logic of co-existence and in mathematics. The fundamental axioms of Boole's logic of co-existence, $x^2 = x$, and $x(1-x) = 0$, are on

* See Jevons's *Substitution of Similar*s. He states the definition, however, as an axiom, that "what is true of a thing is true of its like".

the contrary inapplicable to the logic of identity, and are not generally true in mathematics.

Propositions of co-existence may be reduced to the form of propositions of identity, but the converse is not true.

The terms and propositions of the logic of co-existence may be interpreted in either extension or comprehension, but those of the logic of identity in extension only.

I have, in conclusion, to make a few remarks on the "logic of relatives". This will probably be found to be an extension of the logic of co-existence. The combination of logical terms, symbolising co-existence, is analogous, though not closely so, to the combination of mathematical terms, symbolising multiplication; at least such an analogy is implied throughout Boole's system. It will probably be found that the relation of x to y in logic may be appropriately symbolised by $\frac{x}{y}$; and that relation in logic is to ratio in mathematics, what co-existence in logic is to multiplication in mathematics.

We have seen that in Boole's system 1 is the symbol for "all," or "universe"; so that the equation $1x = x$ is true in logic, as in mathematics, for all values of x . The equation $\frac{x}{1} = x$ is also true in mathematics for all values of x . Is it so in logic? and if so, what is its interpretation? I venture to suggest that it is true in logic, and that it is the logical expression of the truth of the relativity of knowledge—that is to say, as I understand it, the truth that only relations can be the objects of knowledge. If relation in logic is analogous to ratio in mathematics, the expression $\frac{x}{1}$ means the relation of x to the universe, and the equation in question means that, for all purposes of knowledge, a thing is identical with its relation to the universe; including, as part of the universe, the mind which knows the relation.

Another indication of the same or a kindred truth is afforded by the fact, that the same symbol may either be interpreted in comprehension to mean a quality, or in extension to mean the things having the quality. This may be regarded as an expression of the truth, that for all purposes of knowledge a thing is identical with the sum total of its qualities.

I make these suggestions with much diffidence, and the more so because I am inclined to dread mixing up metaphysics with logic; nevertheless, I think them worth making.

It will be perceived that I adhere to the doctrine of the "quantification of the predicate"; and I have to add, that I

regard the science of logic as primarily conversant neither with names nor with concepts, but with things. This view of the subject of logical science is the justification I offer for what will to some appear an illegitimate treatment of the inclusion of a part in the whole as a similar though not identical case to the inclusion of a species in the class.

It is in my opinion a profound error to think that logic depends on psychology. It is a misleading expression to call the laws of logic the laws of thought. No doubt they are so, but only in the same sense in which any truths whereof the contrary is unthinkable may be called laws of thought. The laws of logic, unlike the laws of the association of ideas, do not depend on the structure of the mind—they are laws of thought because they are laws of the universe.

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VI.—LORD AMBERLEY'S METAPHYSICS.*

THE only portion of the late Lord Amberley's *Analysis of Religious Belief* which is of special interest to the student of philosophy, is the Second Book, which treats of "The Religious Sentiment Itself". This occupies little more than a hundred pages of the thousand or so of which the work is composed; and all that is of peculiar value in it might have been compressed within narrower limits. A few pages will be sufficient to show what it amounts to, and what is its significance for us at the present time. I do not express any opinion upon the value of his collection of data. It is sufficiently complete to supply a basis for the analysis of the religious sentiment into its "ultimate elements," though it may be that it was scarcely needed for that purpose. The "ultimate elements" which Lord Amberley finds are the components of the religious sentiment may be discovered by every individual for himself, if he will only question his consciousness when turned upon religion.

Lord Amberley, as the result of his elaborate investigations, finds that all religions have certain features in common. They are all concerned with consecrated actions and consecrated places, and nearly all have to do with consecrated persons and a consecrated class. These are assumed to be the means, or media, through which man communicates with God. But as religions also imply that God addresses man, there are means

* *An Analysis of Religious Belief*, by Viscount AMBERLEY, 2 vols., 1876. Trübner & Co.