T′ or of T; that is, for a given position of P we have $(2\lambda-4)\lambda'$ positions of T. The correspondence between T, P is thus a $[2\lambda'(\lambda-2),\ 2(\lambda-1)(\lambda-2)]$ correspondence, and the number of united tangents is $2(\lambda+\lambda'-1)(\lambda-2)$; or the curves R, Δ meet in $2(\lambda+\lambda'-1)(\lambda-2)$ points.

Reckoning the contacts twice, the total number of intersections of R, Δ is $2\lambda'+4\lambda-4+2(\lambda+\lambda'-1)(\lambda-2),\ =(\lambda+\lambda')(2\lambda-2)$, as it should be.

In the particular case $\lambda=\lambda'=2$, the curves Δ, Δ′ are conics, and the curve R is a quartic curve touching each of the conics 4 times; this is at once verified, since the equations here are $ac-b^2=0$, $a'c'-b'^2=0$, $4(ac-b^2)(a'c'-b'^2)-(ac'+a'c-2bb')^2=0$.

On the Cartesian Equation of the Circle which cuts three given Circles at given Angles. By JOHN GRIFFITHS, M.A.

[*Read March* 12*th*, 1874.]

The equation of a circle cutting three given ones at known angles may be at once obtained from the following proposition:

Let P be any point on a circle U cutting a given circle S_1, whose centre is C, at the angle θ_1; Q the point on U diametrically opposite to P; p the length of the perpendicular from Q upon the polar of P with respect to S_1; then

$$p\times(\text{distance CP})=-\,2Rr_1\cos\theta_1,$$

R and r_1 being the radii of U and S_1 respectively.

This property admits of an easy proof by ordinary geometry, and may also be deduced in the following manner from the Cartesian equations of the two circles.

If we write

$$S=x^2+y^2-r^2=0,\quad U=(x-a)^2+(y-\beta)^2-R^2=0,\quad \text{where } a^2+\beta^2=\delta^2,$$

perpendicular from (x, y) on polar of (ξ, η) in regard to S is

$$=\frac{\xi x+\eta y-r^2}{\sqrt{\xi^2+\eta^2}};$$

i. e., perpendicular from $(\xi_1, \eta_1)=\dfrac{\xi\xi_1+\eta\eta_1-r^2}{\sqrt{\xi^2+\eta^2}}$;

and if (ξ_1, η_1) is opposite to (ξ, η), each being on circle U,

$$(\xi-a)^2+(\eta-\beta)^2-R^2=0,\quad \xi_1=2a-\xi,\quad \eta_1=2\beta-\eta,$$

$$p\sqrt{\xi^2+\eta^2}=\xi(2a-\xi)+\eta(2\beta-\eta)-r^2=\delta^2-R^2-r^2=-2Rr\cos\theta.$$

Putting, then, the equations of the given circles in the homogeneous forms

$$S_1 = x^2+y^2+2g_1xz+2f_1yz+c_1z^2 = 0,$$
$$S_2 = x^2+y^2+2g_2xz+2f_2yz+c_2z^2 = 0,$$
$$S_3 = x^2+y^2+2g_3xz+2f_3yz+c_3z^2 = 0,$$

and supposing that (x, y), (x', y') are two diametrically opposite points on the circle which cuts S_1, S_2, S_3 at the angles θ_1, θ_2, θ_3, respectively, we get immediately from the above,

$$x'\frac{dS_1}{dx}+y'\frac{dS_1}{dy}+\frac{dS_1}{dz} = -4Rr_1\cos\theta_1,$$
$$x'\frac{dS_2}{dx}+y'\frac{dS_2}{dy}+\frac{dS_2}{dz} = -4Rr_2\cos\theta_2,$$
$$x'\frac{dS_3}{dx}+y'\frac{dS_3}{dy}+\frac{dS_3}{dz} = -4Rr_3\cos\theta_3;$$

hence, by eliminating x', y', the equation of the required circle U may be written

$$\begin{vmatrix} \frac{dS_1}{dx}, & \frac{dS_1}{dy}, & \frac{dS_1}{dz}+4Rr_1\cos\theta_1 \\ \frac{dS_2}{dx}, & \frac{dS_2}{dy}, & \frac{dS_2}{dz}+4Rr_2\cos\theta_2 \\ \frac{dS_3}{dx}, & \frac{dS_3}{dy}, & \frac{dS_3}{dz}+4Rr_3\cos\theta_3 \end{vmatrix} = 0;$$

viz., this is

$$\Omega+4R\begin{vmatrix} \frac{dS_1}{dx}, & \frac{dS_1}{dy}, & r_1\cos\theta_1 \\ \frac{dS_2}{dx}, & \frac{dS_2}{dy}, & r_2\cos\theta_2 \\ \frac{dS_3}{dx}, & \frac{dS_3}{dy}, & r_3\cos\theta_3 \end{vmatrix} = 0,$$

where $\Omega = 0$ is the equation of the orthotomic circle.

The algebraical solution of the problem will, therefore, be complete if we can determine the value of R, the radius of U.

This can be done by using Prof. Cayley's equation connecting the mutual distances of four points in a plane. The result is a quadratic in R; viz.,

$$(a+b+c+2f+2g+2h)\,R^2$$
$$-2\,\{(a+h+g)\,r_1\cos\theta_1+(h+b+f)\,r_2\cos\theta_2+(g+f+c)\,r_3\cos\theta_3\}\,R$$
$$+A'B'C'+2F'G'H'-A'F'^2-B'G'^2-C'H'^2 = 0;$$

where a, b, c, &c., are the invariant quantities given in a paper by the present writer, which was published in Vol. III., No. 37, of the Proceedings of the Mathematical Society.

It may be observed that some of the properties of the circle U, given by Plücker in his solution of the present problem, may be readily deduced from the foregoing equation. For instance, the radical axis of U and J, the Jacobian of the circles S_1, S_2, S_3, is the line

$$\left(\frac{dS_2}{dx}\frac{dS_3}{dy}-\frac{dS_2}{dy}\frac{dS_3}{dx}\right) r_1\cos\theta_1 + \ldots\ldots = 0;$$

or, if we use areal coordinates x, y, z, and take the triangle formed by joining the centres of the given circles for a triangle of reference,

$$r_1\cos\theta_1 x + r_2\cos\theta_2 y + r_3\cos\theta_3 z = 0;$$

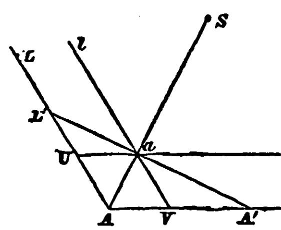
i.e., the radical axis in question coincides with an axis of similitude of three circles of radii $r_1\cos\theta_1$, $r_2\cos\theta_2$, $r_3\cos\theta_3$, and concentric with S_1, S_2, S_3, respectively. The above method will also give us the equation of a sphere which cuts four given spheres at given angles.

On another System of Poristic Equations.

By Prof. WOLSTENHOLME, M.A.

[*Read March* 12*th*, 1874.]

If there be two similar and similarly situated polygons of n sides, ABC ... KL, $abc \ldots kl$, and if through a we draw a straight line L′A′ terminated by LA, AB, and then draw A′bB′, B′cC′, ... K′lL″, terminated by BC, CD, ... KL, then to every position of L′ there is one and only one position of L″, and to every position of L″ one and only one of L′, so that if $AL' = x'$ and $AL'' = x''$, x', x'' must be connected by an equation of the form $Ax'x'' + Bx' + Cx'' + D = 0$. Hence in general there will be two positions of L′ for which L″ will coincide with it, *i.e.*, in general there are two and only two polygons inscribed in the given polygon ABC ... KL, and circumscribed to the polygon $abc \ldots kl$. It may, however, under certain circumstances, happen that L″ always coincides with L′, and the conditions for this I proceed to investigate.



Let $\frac{AA'}{AB} = x_1$, $\frac{AL'}{AL} = 1 - x_n$; produce ba, la to meet AL, AB in U, V; then since the polygons are similar and similarly situated, $\frac{AU}{AL} = \frac{AV}{AB} = k$ suppose. We shall then have for L′A′ passing through a,

$$\frac{k}{x_1} + \frac{k}{1-x_n} = 1, \text{ or } \frac{1}{x_1} + \frac{1}{1-x_n} = \frac{1}{k};$$

and for an in- and circum-scribed polygon the system of equations

$$\frac{1}{x_1} + \frac{1}{1-x_n} = \frac{1}{x_2} + \frac{1}{1-x_1} = \frac{1}{x_3} + \frac{1}{1-x_3} = \ldots = \frac{1}{x_n} + \frac{1}{1-x_{n-1}} = \frac{1}{k}.$$