THE

MATHEMATICAL GAZETTE.

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A CHAPTER ON ALGEBRA.

(*Continued from* p. 306.)

VII.

The Exponential Expansion for Commensurable Indices.

If n is positive and commensurable, and $x^{\frac{1}{J}} = 1 + \frac{X}{J}$,

$$x^n \equiv \left(1+\frac{X}{J}\right)^I$$

$$\equiv \sum_0^I \frac{I(I-1)\dots(I-i+1)}{\underline{|i}} \left(\frac{X}{J}\right)^i$$

$$\equiv \sum_0^I \frac{(nX)^i}{\underline{|i}} . 1 \left(1-\frac{1}{I}\right)\left(1-\frac{2}{I}\right)\dots\left(1-\frac{i-1}{I}\right).$$

Consider the coefficient of $\frac{(nX)^i}{\underline{|i}}$.

(1) It is less than unity.

(2) It can be written $1\left(1-\frac{i-1}{I}\right).\left(1-\frac{1}{I}\right)\left(1-\frac{i-2}{I}\right)\dots.$

The sum of each pair of factors is the same, viz. $\left(2-\frac{i-1}{I}\right)$;

$\therefore$ by § I., the product of the outermost pair $1\left(1-\frac{i-1}{I}\right)$ is least;

$\therefore$ the coefficient is greater than $\left(1-\frac{i-1}{I}\right)^{\frac{i}{2}}$,

which, by § III., is greater than $1-\frac{i(i-1)}{2I}$.

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Therefore, *if X is positive,*

$$x^n \text{ lies between } \sum_0^I \frac{n^i X^i}{\lfloor i} \text{ and } \sum_0^I \left\{1 - \frac{i(i-1)}{2I}\right\} \frac{n^i X^i}{\lfloor i}$$

This lower limit $$\equiv \sum_0^I \frac{n^i X^i}{\lfloor i} - \frac{n^2 X^2}{2I} \sum_0^{I-2} \frac{n^i X^i}{\lfloor i},$$

which $$> \left(1 - \frac{n^2X^2}{2I}\right) \sum_0^I \frac{n^i X^i}{\lfloor i}.$$

$$\therefore\ x^n \text{ differs from } \sum_0^I \frac{n^i X^i}{\lfloor i} \text{ by less than } \frac{n^2X^2}{2I} \sum_0^I;$$

but, since $$x^n > \left(1 - \frac{n^2X^2}{2I}\right) \sum_0^I, \quad \therefore\ \sum_0^I < \frac{x^n}{1 - \frac{n^2X^2}{2I}};$$

$$\therefore\ x^n \text{ differs from } \sum_0^I \frac{n^i X^i}{\lfloor i} \text{ by less than } \frac{\frac{n^2X^2}{2I}}{1 - \frac{n^2X^2}{2I}} \cdot x^n.$$

Now let I and J increase indefinitely, their ratio remaining n.

Then X takes the finite value $\log x$ and the above error becomes continuously smaller and ultimately vanishes.

Therefore, *when X is positive, i.e.* when $x > 1$,

$$x^n \equiv \sum_0 \frac{(n \log x)^i}{\lfloor i}.$$

If X is negative, i.e. if $x < 1$, the terms of

$$\sum \frac{(nX)^i}{\lfloor i}\, 1 \left(1 - \frac{1}{I}\right) \dots \left(1 - \frac{i-1}{I}\right)$$

are alternately + and −; but it is still true that the series differs from

$$\sum \frac{n^i X^i}{\lfloor i} \text{ by less than } \sum \frac{i(i-1)}{2I} \frac{(-nX)^i}{\lfloor i}$$

where all the terms are positive;

$$\therefore\ x^n \text{ differs from } \sum_0^I \frac{n^i X^i}{\lfloor i} \text{ by less than } \frac{n^2X^2}{2I} \sum_0^I \frac{n^i(-X)^i}{\lfloor i}$$

Now let $$x'^{\frac{1}{J}} \equiv 1 + \frac{-X}{J};$$

then by the former work,

$$\sum_0^I \frac{n^i(-X)^i}{\lfloor i} < \frac{x'^n}{1 - \frac{n^2X^2}{2I}}.$$

But $x^{\frac{1}{J}} x'^{\frac{1}{J}} \equiv 1 - \frac{X^2}{J^2}$ which < 1;

$$\therefore\ x' < \frac{1}{x};$$

$$\therefore\ \text{the error} < \frac{\frac{n^2X^2}{2I}}{1 - \frac{n^2X^2}{2I}} \frac{1}{x^n}.$$

Hence, for all positive commensurable values of n, whether I, J are finite or not,

$$x^n \text{ differs from } \sum_0^I \frac{n^i X^i}{\lfloor i}, \text{ where } x^{\frac{1}{J}} \equiv 1 + \frac{X}{J},$$

by a quantity less than
$$\frac{\frac{n^2X^2}{2I}}{1-\frac{n^2X^2}{2I}} x^{\pm n},$$

the upper sign being used if $x > 1$ and the lower if $x < 1$.

If J is increased indefinitely, this error vanishes, and, for all positive values of x,
$$* \; x^n = \sum \frac{(n \log x)^i}{\lfloor i} \text{ when } n \text{ is positive.}$$

The case of a negative index presents no difficulty;

for
$$x^{-n} = \left(\frac{1}{x}\right)^n = \sum \frac{\left(n \log \frac{1}{x}\right)^i}{\lfloor i} \text{ which } = \sum \frac{(-n \log x)^i}{\lfloor i}.$$

VIII.

The Binomial Expansion.

Let B_n stand for $x^n - \sum_0^\kappa (n)_i (x-1)^i$, where n has any value, and $(n)_i$ represents
$$\frac{n(n-1)\dots(n-i+1)}{1.2\dots i}.$$

The terms $(x-1)^i$ can be expanded in powers of x:
Assume that we so get
$$B_n \equiv x^n - [N_0 + N_1 x + N_2 x^2 \dots + N_\kappa x^\kappa],$$
where N_0, $N_1 \dots$ are algebraic functions of n of dimensions not greater than κ.

(1) If n is a positive integer less than κ (ν say), $B_n = 0$;
$$\therefore \; N_0, N_1 \dots \text{ all vanish when } n = \nu, \text{ except } N_\nu \text{ which} = 1.$$

Hence
$$N_\nu = \frac{n(n-1)\dots(n-\kappa)}{n-\nu} . c_\nu,$$
where c_ν contains no n because the dimensions of the other factor are κ;

To find c_ν we have $\quad N_\nu = 1$ when $n = \nu$;
$$\therefore \; c_\nu = \frac{1}{\{\nu . \nu - 1 \dots 1\}.\{(-)^{\kappa-\nu}. 1 . 2 \dots \kappa - \nu\}} \text{ or } \frac{(-)^{\kappa-\nu}}{\lfloor \kappa} (\kappa)_\nu.$$

(2) Again, B_n vanishes for all values of n if $x = 1$;
$$\therefore \; 0 = 1 - [N_0 + N_1 + \dots + N_\kappa].$$

Hence
$$B_n \equiv N_0(x^n - 1) + N_1(x^n - x) \dots + N_\kappa(x^n - x^\kappa),$$
which by the above may be written $n(n-1)\dots(n-\kappa) \sum_{i=0}^{i=\kappa} c_i . \frac{x^n - x^i}{n-i}$.

* This expansion might have been more simply established by first confining ourselves to the case $n=1$ (*i.e.* $I=J$ in the above work), and afterwards using the relation $\log x^n = n \log x$. The less simple proof has been given for the sake of the result when I and J are not indefinitely increased.

(3) To obtain limits for the value of $\sum_{i=0}^{i=\kappa} c_i \frac{x^n - x^i}{n-i}$, we use the Exponential Expansion established in § VII.

We have $$x^n \equiv 1 + \frac{nX}{\lfloor 1} + \frac{n^2X^2}{\lfloor 2} + \ldots$$

$$\equiv 1 + nX\left[1 + \tfrac{1}{2} \cdot \frac{nX}{\lfloor 1} + \tfrac{1}{3} \cdot \frac{n^2X^2}{\lfloor 2} + \ldots\right].$$

Now by § V., $\frac{1}{i} = \underset{\theta=0}{\overset{\theta=1}{\mathrm{Av.}}} \mid \theta^{i-1} \mid$;

$$\therefore\ x^n \equiv 1 + nX . \overset{1}{\underset{0}{\mathrm{Av.}}}\left[1 + \theta nX + \theta^2 \frac{n^2X^2}{\lfloor 2} + \ldots\right]$$

$$\equiv 1 + nX \overset{1}{\underset{0}{\mathrm{Av.}}} (x^{n\theta});$$

$$\therefore\ \frac{x^n - x^i}{n-i} \equiv x^n . \frac{x^{i-n} - 1}{i-n} \equiv x^n . X . \overset{1}{\underset{0}{\mathrm{Av.}}} \left| x^{(i-n)\theta} \right|;$$

$$\therefore\ B_n = \frac{n \ldots (n-\kappa)}{\lfloor \kappa} . x^n X \overset{1}{\underset{0}{\mathrm{Av.}}} \left| x^{-n\theta} \sum_0^{\kappa} (-)^{\kappa-i} (\kappa)_i x^{i\theta} \right|$$

$$\equiv n . (n-1)_\kappa X \overset{1}{\underset{0}{\mathrm{Av.}}} \left| x^{n\theta'} (x^\theta - 1)^\kappa \right|, \text{ where } \theta' \equiv 1 - \theta.$$

Now $$(x^\theta - 1)^\kappa < \begin{matrix} (x-1)^\kappa \\ 0 \end{matrix};$$

$$\therefore\ B_n < n(n-1)_\kappa \left| \begin{matrix} (x-1)^\kappa \\ 0 \end{matrix} \right| . X \overset{1}{\underset{0}{\mathrm{Av.}}} . x^{n\theta},$$

and $$X \overset{1}{\underset{0}{\mathrm{Av.}}} . x^{n\theta} \equiv \frac{x^n - 1}{n}, \text{ as before.}$$

Hence $$x^n \equiv \sum_0^{\kappa} (n)_i (x-1)^i + \text{an error } (n-1)_\kappa (x-1)^\kappa \left| \begin{matrix} x^n - 1 \\ 0 \end{matrix} \right|.$$

This error vanishes as κ increases if $(x-1) < 1$ numerically.

In the special case of $x - 1 = 1$ or $x = 2$, the error vanishes ultimately if $(n-1)_\kappa$ vanishes.

Now the part of $(n-1)_\kappa$ after the signs of the factors have all become negative varies as $\left(1 - \frac{n}{\nu}\right)\left(1 - \frac{n}{\nu+1}\right) \ldots \left(1 - \frac{n}{\kappa}\right)$, which, by § III., if $n > 0$

$$< \left(1 + \frac{1}{\nu}\right)^{-n} \left(1 + \frac{1}{\nu+1}\right)^{-n} \ldots \left(1 + \frac{1}{\kappa}\right)^{-n} < \left(\frac{\nu}{\kappa+1}\right)^n;$$

and if $n = -f$, where $f < 1$,

$$\left(1 + \frac{f}{\nu}\right) \ldots \left(1 + \frac{f}{\kappa}\right) > \left(1 + \frac{1}{\nu}\right)^f \ldots \left(1 + \frac{1}{\kappa}\right)^f > \left(\frac{\kappa+1}{\nu}\right)^f;$$

$\therefore$ when $n > 0$ the error vanishes ultimately, but when $n < 0$ it increases indefinitely.

For the special case of $x - 1 = -1$ or $x = 0$ we must return to the beginning of the article.

We need only consider the case of $n > 0$, for otherwise x^n is not a finite quantity.

Thus, in this case, $B_n \equiv -N_0 = \frac{(n-1) \ldots (n-\kappa)}{\lfloor \kappa} (-)^{\kappa+1}$, since $(\kappa)_0 = 1$.

This vanishes ultimately, as before, if $n > 0$.

IX.

On Incommensurable Indices.

The condition of obedience to the law $a^m \times a^n = a^{m+n}$ establishes the meaning of a^n when n is any commensurable, but leaves it undefined when n is incommensurable. Indicating the incommensurable by a Greek letter, we now seek to give a meaning to x^ν. Our freedom in giving such a meaning is limited only by the condition of continuity, viz. that if ν lies between two commensurables n and n', a^ν must be so defined that it lies between a^n and $a^{n'}$, whose meaning is already fixed.

Now we may suppose n to be $\frac{I}{J}$ and n' to be $\frac{I+1}{J}$.

Then as in § IV., $x^n = \left(1+\frac{nX}{I}\right)^I$ and $x^{n'} = \left(1+\frac{n'X}{I+1}\right)^{I+1}$,

where X is given by $x^{\frac{1}{J}} = 1+\frac{X}{J}$.

Now consider the expression $\left(1+\frac{\nu X}{I}\right)^I$.

(1) If X is positive,

$$\text{since } \nu > n, \text{ this} > \left(1+\frac{nX}{I}\right)^I \text{ and } \gtrless \left(1+\frac{n'X}{I+1}\right)^{I+1}$$

according as
$$\left(1+\frac{\nu X}{I}\right)^{\frac{I}{I+1}} \gtrless 1+\frac{n'X}{I+1}.$$

But by § III.,

$$\left(1+\frac{\nu X}{I}\right)^{\frac{I}{I+1}} < 1+\frac{\nu X}{I+1} \text{ and therefore} < 1+\frac{n'X}{I+1} \text{ since } \nu < n'.$$

Therefore when $x > 1$, $\left(1+\frac{\nu X}{I}\right)^I$ is a sound definition of x^ν, provided I and J are considered indefinitely great, and therefore X is $\log x$.

(2) If X is negative,

$$\left(1+\frac{\nu X}{I}\right)^I < \left(1+\frac{nX}{I}\right)^I$$

and $$\left(1+\frac{\nu X}{I}\right)^I > \left(1+\frac{n'X}{I+1}\right)^{I+1}, \text{if } 1+\frac{\nu X}{I} > \left(1+\frac{n'X}{I+1}\right)\left(1+\frac{n'X}{I+1}\right)^{\frac{1}{I}};$$

now $$\left(1+\frac{n'X}{I+1}\right)^{\frac{1}{I}} < 1+\frac{n'X}{I(I+1)};$$

$$\therefore \text{ the last condition is satisfied if } 1+\frac{\nu X}{I} > \left(1+\frac{n'X}{I+1}\right)\left(1+\frac{n'X}{I(I+1)}\right).$$

Remembering that X is negative,

this becomes $$\nu < n' + \frac{n'^2 X}{(I+1)^2}, \quad i.e. \quad < n' + \frac{X}{J^2},$$

which is true if J is large enough.

Hence in this case also, the definition $x^\nu \equiv \left(1+\frac{\nu X}{I}\right)^I$, where I and J are taken indefinitely great, is valid.

Hence, reasoning as in § VII., we may say that

$$x^n \equiv 1 + n\log x + \frac{(n\log x)^2}{\underline{|2}} + \dots$$

for all values of n, commensurable or not.

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It is easy to show that x^ν as so defined, satisfies the laws of indices.

$$\text{Thus } x^\mu . x^\nu \equiv \left(1+\frac{\mu X}{I}\right)^I \left(1+\frac{\nu X}{I}\right)^I \equiv \left(1+\frac{\overline{\mu+\nu}.X}{I}\right)^I \left[1+\frac{\mu\nu X^2}{I^2+\overline{\mu+\nu}.X_I}\right]^I$$

$$= x^{\mu+\nu}\left[1+\frac{a}{I^2+bI}\right]^I \text{ say;}$$

$$\text{and, by § III.,} \quad \left(1+\frac{a}{I^2+bI}\right)^I > 1+\frac{a}{I+b} \text{ and} < \frac{1}{1-\frac{a}{I+b}};$$

and $\therefore$ (I being indefinitely great) $=1$.

$$\text{Again, } \log x^\mu = \text{Lt}\, I\left[(x^\mu)^{\frac{1}{I}}-1\right] = \text{Lt}\, I\left[\left(1+\frac{\mu X}{I}\right)-1\right] = \text{Lt}\, \mu X = \mu \log x;$$

$$\therefore \ (x^\mu)^\nu \equiv \left(1+\frac{\nu \log x^\mu}{I}\right)^I \equiv \left(1+\frac{\mu\nu \log x}{I}\right)^I \equiv x^{\mu\nu} \text{ by definition.}$$

Thus in the Exponential Expansion the number e can now be introduced: and we can prove $x = e^{\log x}$.

X.

On the Same Theorems for a Vector Variable.

The work in § VII. is applicable to vector values of x. The logarithm being defined exactly as before, takes the form $(\log a + \sqrt{-1}\,.\,\alpha)$, where a, α are the length and inclination of x. In the expression for the error, X and x are replaced by their moduli, and the error vanishes ultimately. The Binomial Theorem follows by § VIII. (provided that x is within two right angles of the line denoted by 1), the words 'lies between' at the end assuming a suitable meaning for vectors. The logarithmic expansion can be deduced from the exponential by expanding n^i in the finite algebraic form $\sum_{j=1}^{j=i} \left|(n)_j . i_j\right|$, whence $(x-1)^j \equiv \sum_{i=j}^{i=\kappa}\left|\frac{X^i}{\lfloor i} . i_j\right|$ with an error vanishing ultimately as κ is increased. Certain simple properties of the coefficient i_j establish the logarithmic expansion and other less simple expansions for higher powers of $\log x$.

HARROW, *July*, 1903. W. N. ROSEVEARE.

THE SLIDE RULE AND ITS USE IN TEACHING LOGARITHMS.

1. I should like at the outset to disclaim any intention of laying down the law dogmatically. The opinions expressed are stated definitely, but they are submitted with deference for your consideration.

My object is to argue two points:

(1) That the construction of a simple form of slide rule furnishes beginners with a good mode of approaching the subject of logarithms.

(2) That the use of the slide rule at any earlier stage than has been customary deserves every encouragement.

2. It is a commonplace remark that a principle may be presented to a pupil in such an abstract form that his mind fails to assimilate it. This